A NOTE ON THE TOPOLOGY OF POLYHEDRAL HYPERSURFACES AND COMPLETE INTERSECTIONS

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ABSTRACT. We discuss mixed faces of Minkowski sums of polytopes, and show that any stable complete intersection of pointed hypersurfaces is homotopy Cohen-Macaulay, generalizing a result of Hacking, and answers a question of Markwig and Yu. In particular, it has the homotopy type of a wedge of spheres of the same dimension.

1. INTRODUCTION

A **pointed hypersurface** X in $\mathbb{R}^d$ is a $(d-1)$-dimensional polyhedral complex that decomposes $\mathbb{R}^d$ into pointed polyhedra. A **complete intersection (of codimension** n**)** is the intersection of n pointed hypersurfaces. We prove

Theorem 1.1. *The complete intersection $X_1 \cap \cdots \cap X_n$ of n pointed hypersurfaces in $\mathbb{R}^d$ is $(d-n-1)$-connected.*

Unfortunately, the naive intersection is dissatisfying on occasion, and within (tropical) intersection theory, a different form of intersections is more interesting.

1.1. Stable complete intersections.

A pointed hypersurface is **regular** if it is obtained as the domains of linearity of a piecewise linear convex function.

Let $X_1, \ldots, X_n$ be pointed hypersurfaces in $\mathbb{R}^d$. The stable intersection is a subcomplex of pure dimension $d-n$ in $X_1 \cap \cdots \cap X_n$, which can be realized as a Hausdorff limit of transverse intersections of translates of $X_1, \ldots, X_n$ by small displacement vectors. This is not well-defined in general, and motivates our restriction to regular pointed hypersurfaces, in which case it is [MS15, §3.6].

Theorem 1.2. *The stable complete intersection $X_1 \cap_{st} \cdots \cap_{st} X_n$ of n regular pointed hypersurfaces in $\mathbb{R}^d$ is homotopy Cohen-Macaulay of dimension $d-n$.*

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The regularity is only used to make the limit property work; it is still true under milder conditions however that the stable intersection is well-defined, and in those conditions imposed on pointed hypersurfaces[1], the above theorem applies.

Note that, if $X_1 \ldots, X_n$ meet properly, meaning that their intersection has dimension $d-n$, then the stable intersection is equal to the naive intersection.

The theorem solves a special case of a question of Markwig and Yu [MY09], who asked whether tropical complete intersections are shellable (which is a combinatorial property stronger than homotopy Cohen-Macaulay). This question remains open.

The result also generalizes a result of Hacking [Hac08], who proved the same for rational cohomology (i.e. Cohen-Macaulay with respect to rational homology) under a mild general position condition.

Note that the poset of faces of a hypersurface fan X_i is dual to the poset of faces in the boundary of a full-dimensional polytope $P_i \subset \mathbb{R}^d$, and the poset of faces of $X_1 \cap_{st} \cdots \cap_{st} X_n$ is dual to the poset of *mixed faces* of the Minkowski sum $P_1 + \cdots + P_n$, meaning those faces $F_1 + \cdots + F_n$, with F_i a face of P_i, such $\sum_{i \in I} F_i$ has dimension at least $\#I$, for every subset $I \subset \{1, \ldots, n\}$. Counting mixed faces was discussed in [AS16]

Corollary 1.3. *Let $P_1, \ldots, P_n$ be d-dimensional polytopes in $\mathbb{R}^d$. Then the poset of mixed faces of the Minkowski sum $P_1 + \cdots + P_n$ is homotopy Cohen-Macaulay of rank $d-n$.*

1.2. **Polyhedral Hodge theory.**

Following Mikhalkin (see for instance [IKMZ16]) , we recall the p-groups $\mathcal{F}_p(\Sigma)$ of a fan Σ in $\mathbb{R}^d$ is the subgroup of $\bigwedge^p \mathbb{R}^d$ generated by the exterior products of vertices in a common face.

We consider now a polyhedron P, and a subset Q of the faces of ∂P on its boundary closed under reverse inclusion. We call this a **tropical domain**. It is called **projective** if there is a point v outside P such that $\partial P \setminus Q$ is induced by the facets of ∂P invisible from the point (i.e. they cannot be connected to v by lines that do not intersect P).

Consider now a polyhedral complex X in P. We call it tropical in (P, Q) if faces of X intersect Q orthogonally and transversally.

The p-group at a point of x of X is the p-group of its tangent fan, and the tangent fan at a cell σ of Q is the tangent fan of σ intersected with X. For faces F, G of X with G included in F, this induces a natural map of p-groups from the p-group at F to the p-group at G, with the map at the boundary of P given by orthogonal projection. Notice that this allows us to define homology with p-group coefficients.

[1]For instance, when the hypersurface admits a balancing with strictly positive weights

Finally, let us fix the notation for a pointed hypersurface in P. We say X is a **pointed hypersurface** in P if X divides P into polyhedra, that is, closed regions combinatorially equivalent to pointed polyhedra.

We say it is **stably** pointed if there exists a polyhedron P' containing P in its interior, with a pointed hypersurface X' such that the restriction to P induces a combinatorial isomorphism to X. This is to ensure stable intersection is well-defined in the case of regularity.

We call it **regular** if it lifts to a convex function as usual, and **ample** if the restriction to any open component of the complement in $P^\circ \cup Q$ is combinatorially an orthant.

We prove:

Theorem 1.4. *Consider (P, Q) a projective tropical domain of dimension d. The stable complete intersection $X_1 \cap_{st} \cdots \cap_{st} X_n$ of n regular ample stably pointed hypersurfaces in (P, Q) has vanishing cohomology $H_q(X, \mathcal{F}_p))$ provided $p + q < d - n$. Moreover, it is homotopy Cohen-Macaulay.*

1.3. **Convex geometry.** We should say what the notion of pointedness is important, as it is clearly necessary for the theorems to hold (indeed, observe that the theorem is wrong for general hypersurfaces, for instance, a collection of parallel hyperplanes.) The key observation lies in the following basic fact from convex geometry:

Consider the space of convex functions $\mathcal{C}(P, Q)$ on a polyhedron P that vanish on the boundary of P and are constant on a polyhedron $Q \subset \mathrm{int}P$ with the same recession cone as P. We call such Q coherent with P, and allow Q to be empty.

Lemma 1.5. *If P is pointed, then the subset $\mathcal{C}_s(P, Q)$ of functions that are strictly convex in the interior $\mathrm{int}P \setminus Q$ is dense in $\mathcal{C}(PQ)$. In fact, it is comeagre in the uniform topology.*

This follows a basic construction of Klee [Kle59].

Proof. Consider a point v in $\mathrm{int}P \setminus Q$, and the recession cone C of P. Consider then the minimal convex function $\mathbf{1}_v$ on $\mathcal{C}(P)$ that evaluates to 1 on $v + C$ and Q. Notice that it is strictly convex at v.

Given any function f in $\mathcal{C}(P, Q)$, consider the function $f + \varepsilon \mathbf{1}_v$. It is strictly convex at v, and so is every function in a small neighborhood of it. This shows that the set of non-strictly convex functions is of the first Baire category. □

We call smooth elements $\mathcal{C}_s(P,Q)$ the (P,Q)-coherent functions. As smooth functions are comeagre (again a result of Klee in the same paper), the (P,Q)-coherent functions are comeagre.

2. Naive intersections of hypersurfaces

We first address the case of pointed polyhedra and their naive intersection, that is, Theorem 1.1.

Lemma 2.1. *Consider a pointed polyhedron P of dimension d in $\mathbb{R}^d$, a complete intersection X of codimension n, and a polyhedron Q coherent with P.*

Then $(P \setminus Q) \cap X, \partial P \cap X)$ is $(d-n-1)$-connected.

To clean up the induction, we clarify: Having established this statement for parameters up to n and d will be abbreviated by $C_{n,d}$.

We prove this in conjunction with Theorem 1.1. Having proven this statement for parameters up to n and d will be abbreviated by $P_{n,d}$.

2.1. **Tandem proof.**

We shall prove the following two implications:

- $P_{n,d-1} \implies C_{n,d}$.
- $P_{n-1,d} \wedge C_{n-1,d} \implies P_{n,d}$

This leaves us with proving the statement when n or d is 0, which is trivial.

We begin by proving

$$P_{n,d-1} \implies C_{n,d}.$$

Consider a generic (P,Q)-coherent function f, so that f is a stratified Morse function on X, in the sense [GM88]. Explicitly, assume that f takes a maximum on cells of X (i.e. relative interiors of faces) at distinct times. We can now trace the change in topology of $(P \setminus P_t) \cap X$ as t decreases.

At every change in topology, that is, at every critical point, the Morse data is given by a codimension n complete intersection in $\mathbb{R}^{d-1}$.

We can see this elementarily as follows: consider the critical point x, that is, a maximum of f in a cell of X. Assume for the start that x is a vertex of X, and encountered at time t. Then $(P \setminus P_{t-\varepsilon}) \cap X$ is obtained from $(P \setminus P_{t+\varepsilon}) \cap X$, up to homotopy equivalence, by attaching $\mathrm{T}_x X \cap \mathrm{T}_x f^{-1}[0,t]$, where T_x denotes the tangent fan at a point. The latter component, $\mathrm{T}_x f^{-1}[0,t]$, is a halfspace, say with boundary H.

Consider now a parallel hyperplane H' to H in the interior of that halfspace. Then $\mathrm{T}_x X \cap \mathrm{T}_x f^{-1}[0,t]$ is the cone over $H' \cap \mathrm{T}_x X$, which in turn is a codimension n complete intersection in H'. This is $(d-n-2)$-connected by $P_{n,d-1}$. The claim follows by the exact sequence of relative homotopy groups.

If x lies in the relative interior of a face σ of X instead (say of dimension k), and encountered at time t, then $\mathrm{T}_x X \ \cap \mathrm{T}_x f^{-1}[0,t]$ splits into two parts, the so called normal and tangential Morse data: Consider the orthogonal complement N to $\mathrm{T}_x\sigma$. Then $\mathrm{T}_x X \ \cap \mathrm{T}_x f^{-1}[0,t]$ is homotopy equivalent to the free join of $\partial\sigma$ with

$$\mathrm{T}_x X \ \cap \mathrm{T}_x f^{-1}[0,t] \cap N.$$

The latter is the cone over a $(d-n-1-k)$-dimensional set, the former is $(k-2)$-connected. Hence, their free join is $(d-n-1)$-connected, as desired. This proves the first implication. □

We now prove

$$P_{n-1,d} \ \wedge \ C_{n-1,d} \Longrightarrow \ P_{n,d}$$

Consider then a codimension $n-1$ complete intersection X', and another pointed hypersurface X. By Lemma 2.1, the stable intersection of X' and X is obtained from X' by attaching cells[2] of dimension $d-n+1$. This can introduce nontrivial homotopy groups of dimension $d-n$, but not below. Hence, the stable intersection of X' and X is $(d-n-1)$-connected, because X' is. □

3. Stable intersections

We now prove Theorem 1.2. Once again, it is enough to prove that the stable complete intersection of codimension is $(d-n-1)$-connected; the links of nonempty faces are analyzed as before. Once again, we prove two statements in tandem:

Lemma 3.1. *Let X be a stable intersection of n regular pointed hypersurfaces in $\mathbb{R}^d$. Let P be a pointed polyhedron of dimension d in $\mathbb{R}^d$, let $v \in \mathbb{R}^d$, and let $P_s = P + sv$. Let Q be coherent in P. Then $\lim_{s\to 0}(P_s \setminus Q) \cap X$ is obtained from $\lim_{s\to 0}(\partial P_s \cap X)$ by attaching cells of dimension $(d-n)$.*

To clean up the induction, we clarify once again: Having established this statement for parameters up to n and d will be abbreviated by $C'_{n,d}$.

Notice in particular:

[2]When speaking of attaching cells, we generally understand this up to homotopy equivalence

Corollary 3.2. *In the situation above, i-spheres mapped to $\lim_{s\to 0}(P_s \setminus Q) \cap X$ homotope to $\lim_{s\to 0}(\partial P_s \cap X)$ provided that $i \le d-n-1$. In particular, every relative $(i+1)$-disk $(D, \partial D)$ mapped $(X \cap Q, X \cap \partial Q)$ is a restriction of a relative disk mapped to the pair*

$$(\lim_{s\to 0}(P_s \setminus Q) \cap X, \lim_{s\to 0}(\partial P_s \cap X)),$$

provided that $i \le d - n - 1$.

The implication of this corollary is direct for the same parameters, so we shall not assign it another symbol. Finally, we note that instead of proving Theorem 1.2, it is enough to prove the following proposition. We shall cleanly argue so in the next section.

Proposition 3.3. *A stable complete intersection of codimension n in $\mathbb{R}^d$ is $(d-n-1)$-connected.*

Having proven this statement for parameters up to n and d will be abbreviated by $P'_{n,d}$.

3.1. **Proof of Theorem 1.2.** Once again, we need the following implications:

- $P'_{n,d-1} \implies C'_{n,d}$.
- $P'_{n-1,d} \wedge C'_{n-1,d} \implies P'_{n,d}$

Once again, the cases when n or d equals 0 are trivial.

The implications work as before, with the only nontrivial word to be said for the implication $P'_{n,d-1} \implies C'_{n,d}$ when $v \neq 0$.

Indeed, if we begin by proving

$$P'_{n,d-1} \implies C'_{n,d}$$

for $v = 0$, then once again, we start by considering a generic (P, Q)-coherent function f, so that f is a stratified Morse function on X, in the sense [GM88]. Explicitly, assume that f takes a maximum on cells of X (i.e. relative interiors of faces) at distinct times. We can now trace the change in topology of $(P \setminus P_t) \cap X$ as t decreases, and analyze the Morse data as before.

We consider this now the case $v \neq 0$, treating the limit of the approximations of P by its translates $P_s = P + sv$. Note that the face poset of $P_s \cap X$ is independent of s, for sufficiently small s. Since we have proved the first implication for $v = 0$, we may assume that $P_s \setminus Q \cap X$ is obtained from $P_s \cap X$ by attaching $(d-n)$-cells. As X itself is $(d-n)$-dimensional, this is equivalent the homotopy groups of the pair $(P_s \setminus Q \cap X, \partial P_s \cap X)$ vanishing up to dimension $i = d - n - 1$. Equivalently, if D^i is an i-disc, with $i \le d-n-1$, then any map of pairs $(D^i, \partial D^i) \to (P_s \setminus Q \cap X, \partial P_s \cap X)$ is homotopic relative to ∂D^i to a map into $\partial P_s \cap X$.

Notice first that $\lim_{s\to 0}(P_s \cap X) = \lim_{s\to 0}(P \cap P_s \cap X)$. Let us therefore abbreviate $P'_s := P \cap P_s$, and $LX := \lim_{s\to 0} P'_s \setminus Q \cap X$ and $\partial LX := \lim_{s\to 0}(\partial P'_s) \cap X$. It remains to show that $(LX, \partial LX)$ is $(d-n-1)$-connected. By the case $v = 0$, applied to case when the enveloping polyhedron is P'_s and Q is a close inner approximation of P, the inclusion map $LX \to P'_s \cap X$ induces a surjection of relative homotopy groups

$$\pi_i(P'_s \setminus Q \cap X, (\partial P'_s) \cap X) \twoheadrightarrow \pi_i(LX, \partial LX)$$

for $i \leq d-n$. In fact, every relative i-disk mapped to $(LX, \partial LX)$, $i \leq d-n$ is the restriction of a relative i-disk mapped to $(P'_s \setminus Q \cap X, (\partial P'_s) \cap X)$, see Corollary 3.2.

We must show that any map of pairs of pairs $(D^i, \partial D^i) \to (\lim_{s\to 0}(LX, \partial LX)$ is homotopic relative to ∂D^i to a map into $\lim_{s\to 0}(\partial P_S \cap X)$. Now, this disk $(D^i, \partial D^i)$ is the restriction of a disk $(D^i_s, \partial D^i_s)$ in $(P'_s \setminus Q \cap X, (\partial P'_s) \cap X)$.

As the combinatorial type of $P_s \cap X$ remains unchanged for s small, we can assume that $\widetilde{D}^i_s$ has combinatorially invariant image, (as the combinatorial type, and therefore the underlying CW complex, remain unchanged) that is, we can take $\widetilde{D}^i_s$ as a combinatorially invariant polyhedral i-disk $\widetilde{D}^i$ that maps cobinatorially to $\partial(P_s) \cap X$, such that the maps φ_s are invariant as maps of posets.

For the reader unfamiliar with combinatorial topology, we can construct $\widetilde{D}^i$ as follows. We take a map $\widetilde{D}^i_s$ to the i-skeleton of $\partial P_s \cap X$ that is locally injective on every i-cell of the latter. As the i-skeleton generates the i-th homotopy group (this well known fact is a basic application of the exact sequence of relative homotopy groups for the skeleta of a cell complex), we can ensure that these conditions are satisfied. Now, pull back the cell decomposition to $\widetilde{D}^i_s$, which gives the desired disk $\widetilde{D}^i$.

Now, if P (and therefore P_s) is compact, we are done: each of these faces have limits as compact polyhedra, and hence also the image of $\widetilde{D}^i_s$ has a limit. This is easy: Remember that a cellular map can be realized, up to homotopy equivalence, as a facewise linear map on the barycentric subdivision (in our case of $\widetilde{D}^i_s$) as it is realized by a simplicial map [Bry02]. But the space of facewise maps on a fixed simplicial complex to a compact set is itself compact (as the finite power of a compact space is compact, so that up to passing to a subsequence, the limit exists. In fact, it is not necessary to pass to a subsequence: since we chose the map to be combinatorially invariant, and each face of $\partial P_s \cap X$ has a compact limit face, the limit exists as a simplicial map.[3]

[3] In fact, we can simply take the honest barycenters, and use the following simple but cute corollary of the integral formula for the volume of polytopes as given by Lawrence [Law91]: If σ_s is a convergent sequence of normally equivalent polytopes, then the limit of the barycenters is the barycenter of the limit. Note that normal equivalence is critical here, and the same is not true for arbitrary limits.

If P is not compact, then we observe that any polyhedral complex whose faces are pointed polyhedra deformation retracts onto its subcomplex of bounded faces (which we call the bounded part for short).

The bounded part is compact, as a finite union over compact sets, so that D_s^i lie in the compact bounded part of $X \cap \partial P_s$.

It remains to see that this sequence of bounded parts is uniformly bounded as s tends to 0. For this, observe first that if σ is a bounded face of X, then any intersection of it is bounded, so it remains to understand the intersection of $\sigma \cap \partial P_s$ for unbounded σ. Here, the situation is simple: if the sequence is unbounded as s tends to 0, then a face of ∂P_s and σ share a ray in the recession cone by the aforementioned stability property. Hence, for s small enough, $\sigma \cap \partial P_s$ is unbounded, and hence not in the bounded part. This proves that the limit of bounded parts exists and is bounded, and so therefore we can guarantee the limit of D_s^i to exist in it.

This finishes the proof of the first implication.

Next we prove the second implication:

$$P'_{n-1,d} \ \wedge \ C'_{n-1,d} \Longrightarrow \ P'_{n,d}$$

Suppose both propositions hold for $(d, n-1)$. Recall that, as discussed in Section 3, if X' is a stable intersection of hypersurfaces in $\mathbb{R}^d$, and X is a hypersurface in $\mathbb{R}^d$, then

$$X' \cap_{st} X = \lim_{\epsilon \to 0} X' \cap (X + \epsilon v)$$

for a generic displacement vector v.

We must show that a stable intersection of n pointed hypersurfaces in $\mathbb{R}^d$ is $(d-n-1)$-connected.

Let $X' = X_1 \cap_{st} \cdots \cap_{st} X_{n-1}$. We consider the stable intersection $\lim_{\epsilon\to 0} X' \cap X_n + \epsilon v$, for a generic displacement vector v.

Lemma 3.1 for $(d, n-1)$ implies that

$$\pi_i(X', \lim_{\epsilon \to 0} X' \cap (X_n + \epsilon v))$$

which, as group, is the direct sum over

$$\lim_{\epsilon \to 0} \pi_i(X' \cap (P + \epsilon v), \lim_{\epsilon \to 0} X' \cap (\partial P + \epsilon v))$$

ranging over the regions of P of the complement of X_n, is concentrated in dimension $d - n + 1$. In particular, X' is obtained from $\lim_{\epsilon\to 0} X' \cap (X_n + \epsilon v)$ by attaching cells of dimension $d-n+1$. But this can only affect homotopy groups in dimension $d-n$ and $d-n+1$, and hence preserves $(d-n-1)$-connectivity.

3.2. **The spherical case.** Let us connect this to Theorem 1.2, and to Cohen-Macaulayness, by examining tangent spaces and spherical hypersurfaces. Indeed, it remains to discuss the links of nonempty faces to finish the proof of the theorem.

A pointed hypersurface X inside the sphere S^d is a polyhedron that divides S^d convex polyhedra whose closure contain no antipodal points. In other words, the cones over the faces of X induce a pointed hypersurface fan in $\mathbb{R}^{d+1}$.

With this viewpoint as fans, we can extend the notion of stable intersection to pointed hypersurfaces in the sphere, provided the decomposition is regular. We have the following two facts.

Proposition 3.4. *The stable intersection of n pointed hypersurfaces in S^d is of dimension $d-n$, and $(d-n-1)$-connected. This is still true when restricted to an open hemisphere.*

The second claim is clear, by central projection and Theorem 3.3. For the first claim, we use the second claim and the following corollary of Lemma 2.1.

Lemma 3.5. *Consider a stable complete intersection X of codimension n in S^d. Let H denote a closed hemisphere of S^d. Then $X \cap H$ is obtained from $X \cap \partial H$ by attaching cells of dimension $d-n$.*

Proof. Pick a spherical polyhedron P in the interior of H which contains (in its interior) all faces of X in the interior of H, for instance by taking the convex hull of those faces, and enlarging the result slightly. Then $X \cap H \setminus P$ deformation retracts to $X \cap \partial H$, so it suffices to prove that $X \cap P$ is obtained from $X \cap \partial P$ by attaching cells of dimension $d-n$. The claim follows by central projection and Lemma 3.1. □

This finishes the proof of Proposition 3.4, and in particular also the proof of Theorem 1.2.

3.3. **The case of Hodge groups.** We now prove Theorem 1.4. The homotopy case works as before, the same proof going through without modification.

For the case of tropical Hodge groups, we make some basic observations. First, let us observe that a stable reguler complete intersection if nowhere acyclic:

We call a polyhedral fan Σ *nowhere acyclic* if it is pure and, for any point p of Σ, the tangent fan to Σ in p does not lie in a closed halfspace with linear boundary unless it lies within that boundary. A polyhedral complex is nowhere acyclic if the tangent fans are nowhere acyclic.

As stable complete intersections of regular pointed hypersurfaces is balanced with positive weights. The nowhere acyclic property follows immediately (see the proof of Lemma 10.09 in [AB14]). We now need also the lemma immediately following this in the same source, telling us we can push local coefficient systems through critical points.

Lemma 3.6. *Consider a nowhere acyclic fan Σ of dimension $n \geq 2$ in $\mathbb{R}^d$ and let H^+ denote a closed general position halfspace whose boundary contains the origin.*

Then $(\mathcal{F}_1\Sigma)_{|\mathbf{0}}$ is generated by $(\mathcal{F}_1(\Sigma \cap H^+))_{|\mathbf{0}}$.

We obtain results as follows:

(1) We observe first that (P, Q) satisfies Theorem 1.4 with zero hypersurfaces. This follows at once by induction on the number of facets in Q, and using the fact that Q is shellable.
(2) Next, argue by tandem induction to prove that (C") given a stable intersection X' of codimension $n-1$, and another hypersurface X, the inclusion of $X' \cap_{st} X$ into X' induces isomorphisms for (p, q)-homology in dimension $p+q \leq d-n$, and a surjection in dimension $p+q = d-n+1$, and (P") that the homology of X' vanishes up to $p+q \leq d-n$. Both are proved in tandem as in the previous cases. Theorem 1.4 follows.

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