Single-Peakedness and Strict Concavity

Sergiu Hart*

March 1, 2017

Abstract

In Hart, Kremer, and Perry (2017) we use the assumption that a collection of functions is *single-peaked.* It is shown here that this condition is essentially equivalent to applying a monotonic transformation to a collection of strictly concave functions.

A real function $f : \mathbb{R} \to \mathbb{R}$ is *single-peaked* if there is a point v such that f is increasing for $x \leq v$ and decreasing for $x \geq v$; thus, f has a unique local maximum—which is therefore its global maximum—at v. The (slightly stronger, but more convenient) differentiable version requires that there is v such that

$$f'(v) = 0 \quad \text{and} \quad f''(v) < 0, \tag{1}$$

because then $f'(x) > 0$ (and so f is increasing) for $x < v$, and $f'(x) < 0$ (and so f is decreasing) for $x > v$; in particular, it follows that v is the unique point where f' vanishes.[1]

From now on we assume that all the functions are twice continuously differentiable, i.e., C^2, and use the differentiable version (1) of single-peakedness.

A finite set $(h_t)_{t \in T}$ of real functions (where T is a finite index set) satisfies the *single-peakedness condition (SP)* if all weighted averages of the functions h_t, i.e., $h_q := \sum_{t \in T} q_t h_t$ for all[2] $q \in \Delta(T)$, are single-peaked functions; see Hart, Kremer, and Perry (2017).

*The Hebrew University of Jerusalem (Center for the Study of Rationality, Institute of Mathematics, and Department of Economics). *E-mail*: `hart@huji.ac.il` *Web site*: `http://www.ma.huji.ac.il/hart`

[1]An equivalent way to state the condition is that $f'(x) = 0$ implies $f''(x) < 0$, and there is some x with $f'(x) = 0$ (the uniqueness of such x then follows since between any two local maxima $x \neq y$ there must be a local minimum z, where $f'(z) = 0$ and $f''(z) \geq 0$).

[2]$\Delta(T) = \{q = (q_t)_{t \in T} \in \mathbb{R}^T_+ : \sum_{t \in T} q_t = 1\}$ denotes the unit simplex on T, i.e., the set of probability distributions on T.

Remarks (cf. Hart, Kremer, and Perry 2017, Section III.A and Appendix C.3)**.**

(a) All the functions h_t being single-peaked does *not* imply that their averages must be single-peaked as well, and so is not sufficient for (SP).

(b) All the functions h_t being strictly concave with a finite maximum implies that so are all their averages, and so (SP) holds.

(c) The condition (SP) is invariant to applying a strictly increasing transformation to the variable x: If $(h_t)_{t\in T}$ satisfies (SP) then so does $(h_t \circ \psi)_{t\in T}$, for any $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi'(x) > 0$ for all x. Indeed, $(h_t(\psi(x))' = h_t'(\psi(x))\psi'(x)$ and $(h_t(\psi(x))'' = h_t''(\psi(x))(\psi'(x))^2 + h_t'(\psi(x))\psi''(x)$, and therefore $(h_t(\psi(x))' = 0$ if and only if $h_t'(\psi(x)) = 0$, in which case $(h_t(\psi(x))'' = h_t''(\psi(x))(\psi'(x))^2 < 0$.

(d) One may allow the peak to be infinite (i.e., $v = +\infty$ or $v = -\infty$, which means that the function is monotonic), in which case everything (including the result below) goes through, except that the functions no longer need to have a finite maximum.

Combining (b) and (c) yields a sufficient condition for (SP). We now show that this condition is also necessary.

Theorem 1 *The finite set of functions $(h_t)_{t\in T}$ satisfies (SP) if and only if there exists a monotonic transformation $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi'(x) > 0$ for all x such that $h_t \circ \psi$ is a strictly concave function with a finite maximum for every $t \in T$.*

Proof. We will construct the inverse $\varphi = \psi^{-1}$ of ψ. Assuming the conclusion holds, we have $g_t(\varphi(x)) = h_t(x)$ and $\varphi'(x) > 0$ for all $x \in \mathbb{R}$ and $t \in T$, and so $g_t'(\varphi(x)) = h_t'(x)/\varphi'(x)$ and $g_t''(\varphi(x)) = [h_t''(x)\varphi'(x) - h_t'(x)\varphi''(x)]/[\varphi'(x)]^3$. To get $g_t''(y) < 0$ for all y we need $h_t''(x)\varphi'(x) - h_t'(x)\varphi''(x) < 0$ for all t and all x, i.e.,

$$h_t''(x) < h_t'(x)\frac{\varphi''(x)}{\varphi'(x)} \tag{2}$$

for all t and all x.

We will show that for every x there is a finite scalar $\lambda(x)$ such that for all t we have

$$h_t''(x) < h_t'(x)\lambda(x). \tag{3}$$

Putting $\varphi''(x)/\varphi'(x) = (\ln\varphi'(x))' = \lambda(x)$ yields $\ln\varphi'(x) = \int_0^x \lambda(z)\mathrm{d}z$, hence $\varphi'(x) = \exp\left(\int_0^x \lambda(z)\mathrm{d}z\right) > 0$, and $\varphi(x) = \int_0^x \exp\left(\int_0^y \lambda(z)\mathrm{d}z\right)\mathrm{d}y$; then (2) holds, which gives $g_t''(y) < 0$ for all y (and we have $g_t'(\varphi(v_t)) = 0$ at that v_t with $h_t'(v_t) = 0$).

To prove the claim of the previous paragraph, fix x. Let

$$C := \operatorname{conv}\{(h'_t(x), h''_t(x)) : t \in T\} = \{(h'_q(x), h''_q(x)) : q \in \Delta(T)\} \subset \mathbb{R}^2,$$

then the compact convex polygon C is disjoint from the closed ray $D := \{(0, w) : w \geq 0\}$ (because h_q is single-peaked and so $h'_q(x) = 0$ implies $h''_q(x) < 0$). Therefore one can separate them strictly: there is $p = (p_1, p_2) \in \mathbb{R}^2$ such that

$$\max_{t \in T} p_1 h'_t(x) + p_2 h''_t(x) = \max_{c \in C} p \cdot c < \inf_{d \in D} p \cdot d = \inf_{w \geq 0} p_2 w.$$

Therefore $p_2 \geq 0$ (otherwise the right-hand side equals $-\infty$), the right-hand side equals 0, and we can take $p_2 > 0$ (slightly increasing p_2 preserves the strict inequality, because C is compact). Then $\lambda(x) := -p_1/p_2$ satisfies (3). ■

References

Hart, S., I. Kremer, and M. Perry (2017), "Evidence Games: Truth and Commitment," *American Economic Review* (forthcoming).