How Dull Are Monotonic Mechanisms

Sergiu Hart

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Joint work with

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and

Noam Nisan

The Hebrew University of Jerusalem

Paper

- **Ran Ben Moshe, Sergiu Hart, and Noam Nisan**
  "Monotonic Mechanisms for Selling Multiple Goods"
  (2022)

  www.ma.huji.ac.il/hart/abs/mech-monot.html

A Simple Problem

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- 1 SELLER

A Simple Problem

- 1 **SELLER**
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- k **GOODS (ITEMS)**

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OBJECTIVE:

MAXIMIZE the **REVENUE** of the **SELLER**

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    (good 1 and good 2 = $X_1 + X_2$)

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SELLER :

- **no value** and **no cost** for the GOODS

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REV(X) := optimal revenue from valuation X

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One Good: Solution

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Myerson 1981

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$$\text{REV}(X) = \max_p p \cdot (1 - F(p))$$

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$$\text{REV}(X) = 11 \qquad p = 22$$

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Hart and Nisan 2013/2019
(Briest, Chawla, Kleinberg, Weinberg 2010/2015 for $k \geq 3$)

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For example:

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For example:

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Multiple Goods ($k \geq 2$)

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For example:

- selling separately
- selling bundled
- all deterministic mechanisms
- mechanisms with bounded "menus" (at most m choices, for finite m)

Multiple Goods ($k \geq 2$)

An extremely complex problem

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"CONCEPTUAL COMPLEXITY"

Monotonicity

Monotonicity of Revenue

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BUYER's willingness to pay increases

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BUYER's willingness to pay increases

$\Longrightarrow$ SELLER's revenue increases

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- **correct** for one good

Non-Monotonicity of Revenue

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- **FALSE** for multiple goods !

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$\Rightarrow$ SELLER's revenue increases

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- **FALSE** for multiple goods !

Hart and Reny 2014

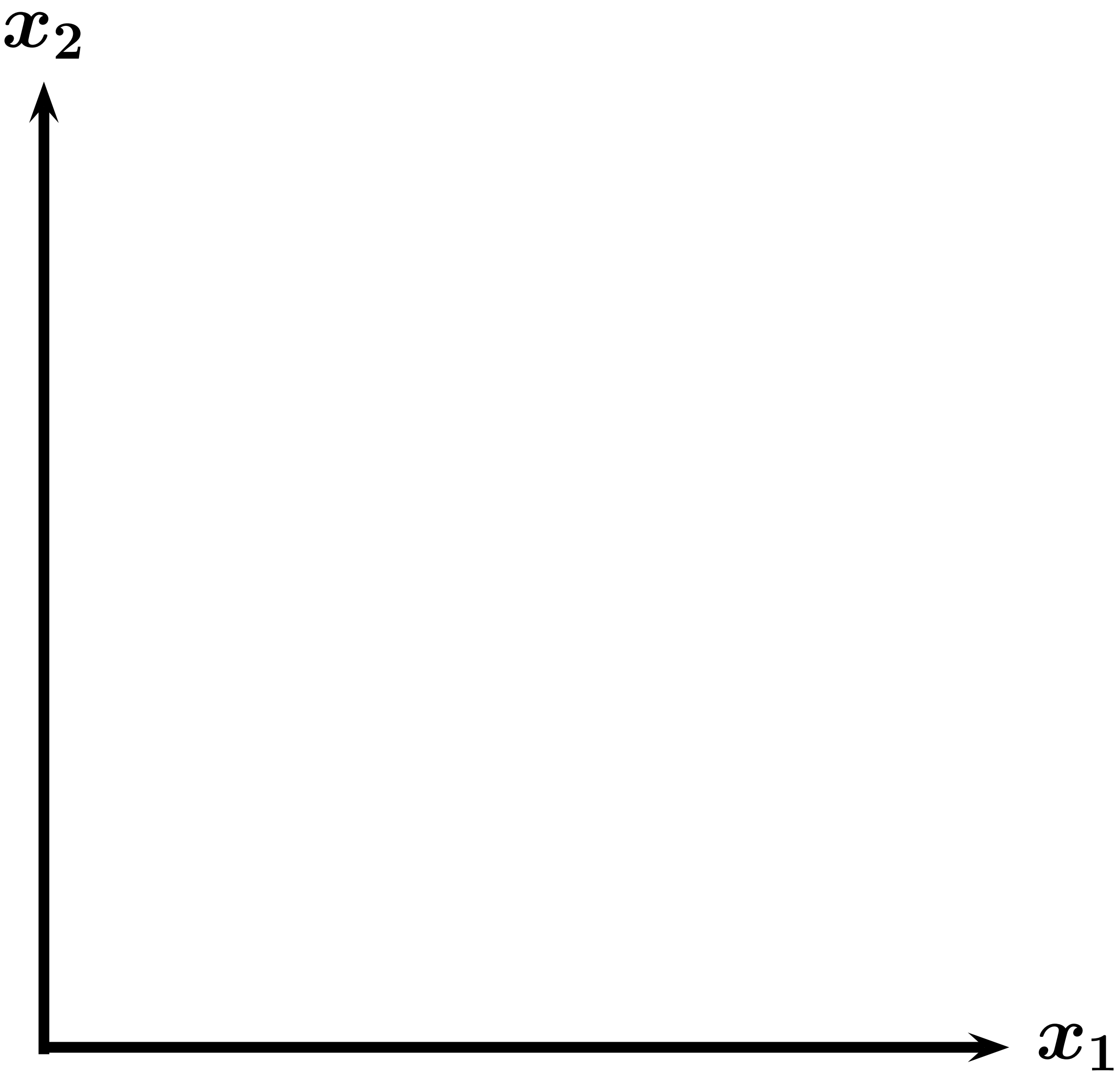
Non-Monotonic Mechanism

Non-Monotonic Mechanism

Menu

good 1	$ 10
good 2	$ 20

Non-Monotonic Mechanism



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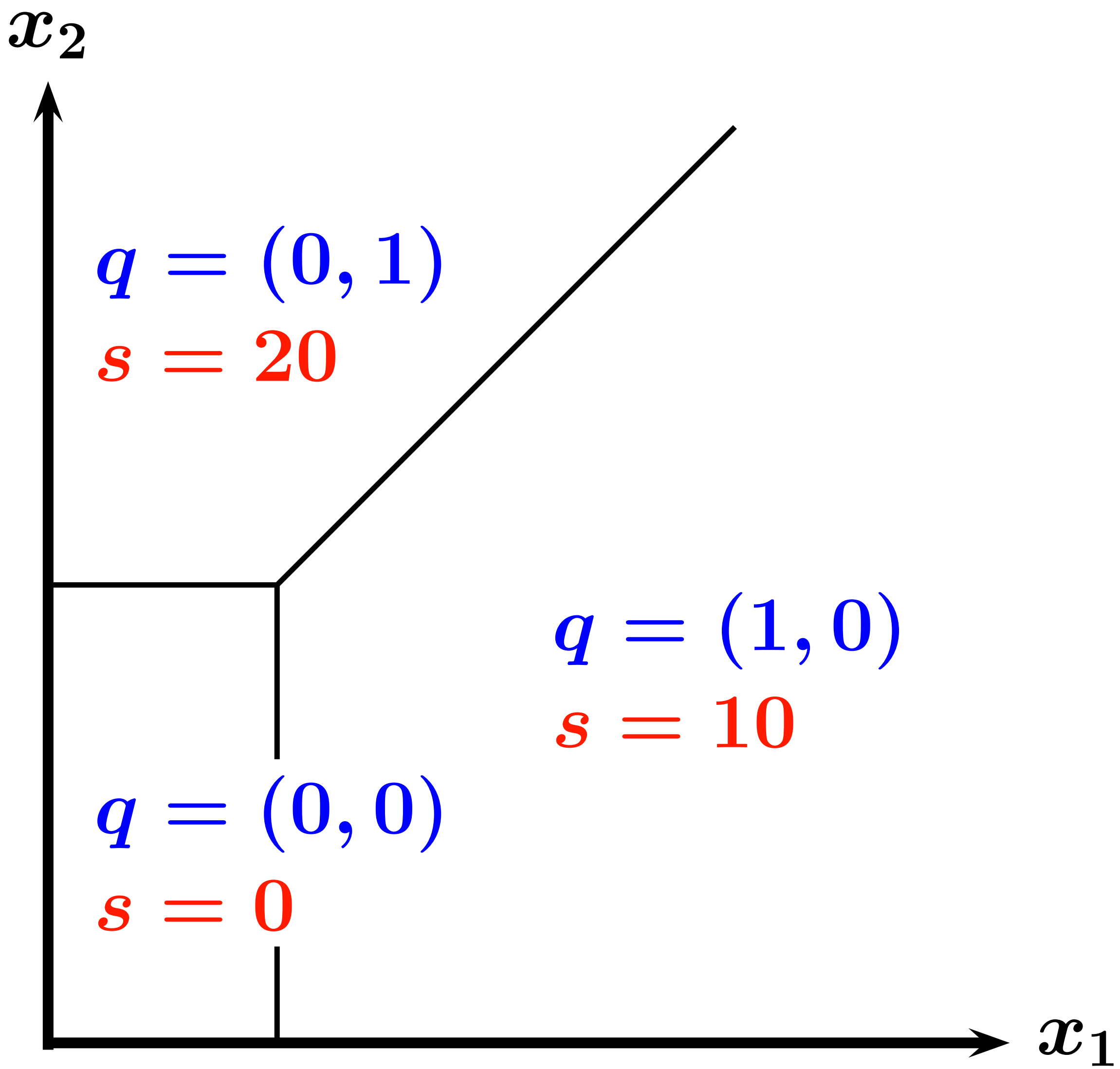

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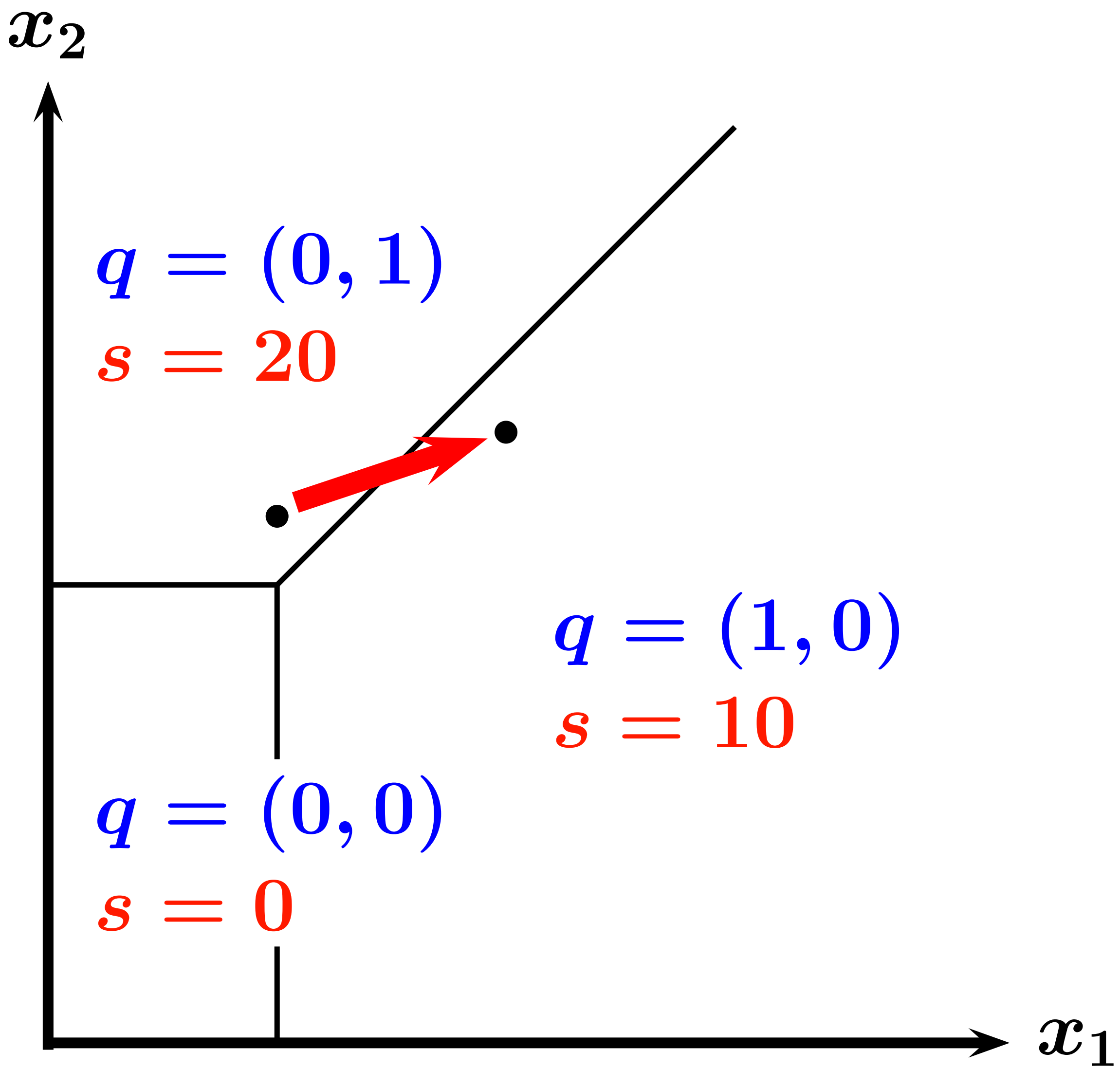

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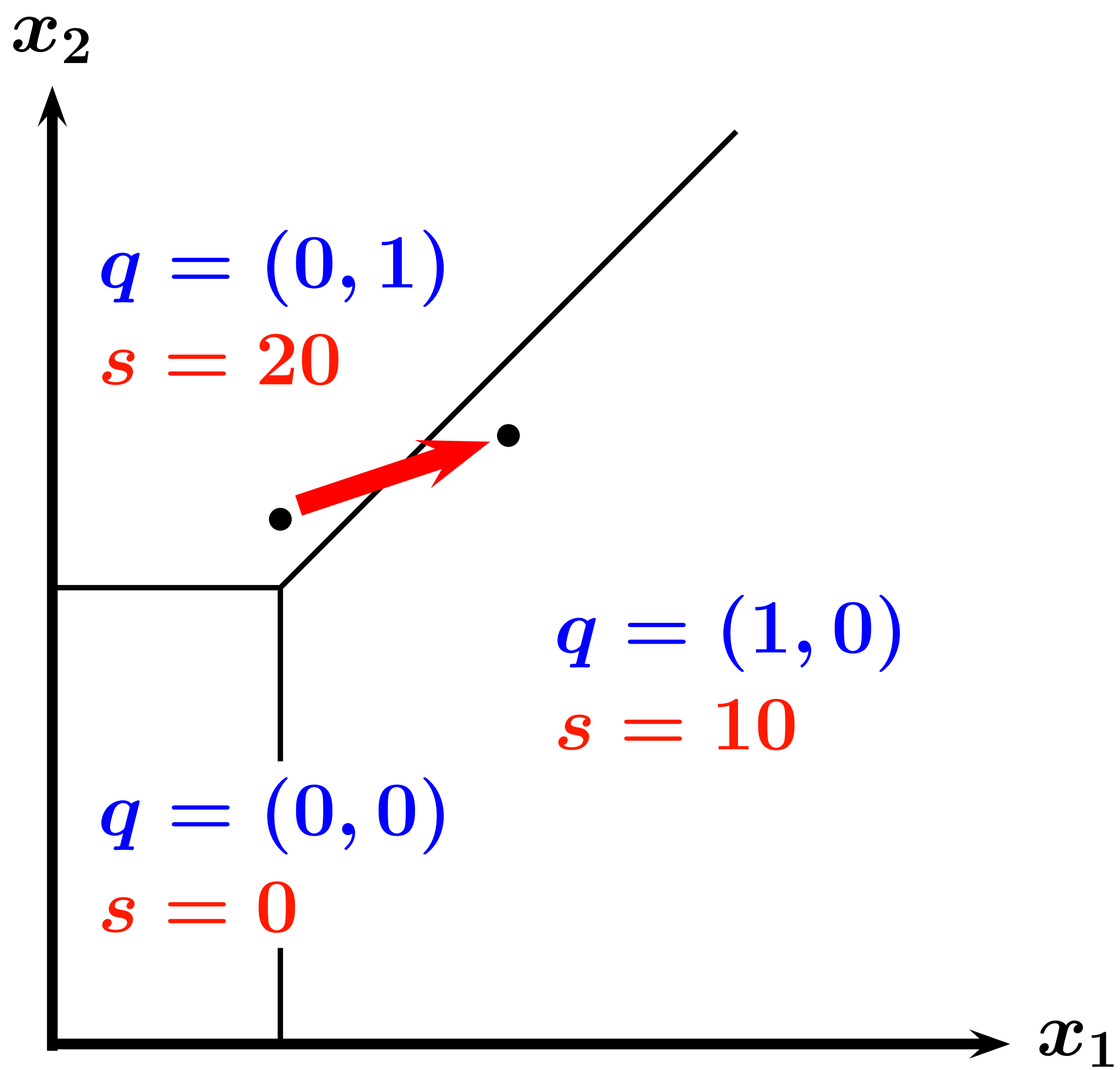

Menu

good 1	\$ 10
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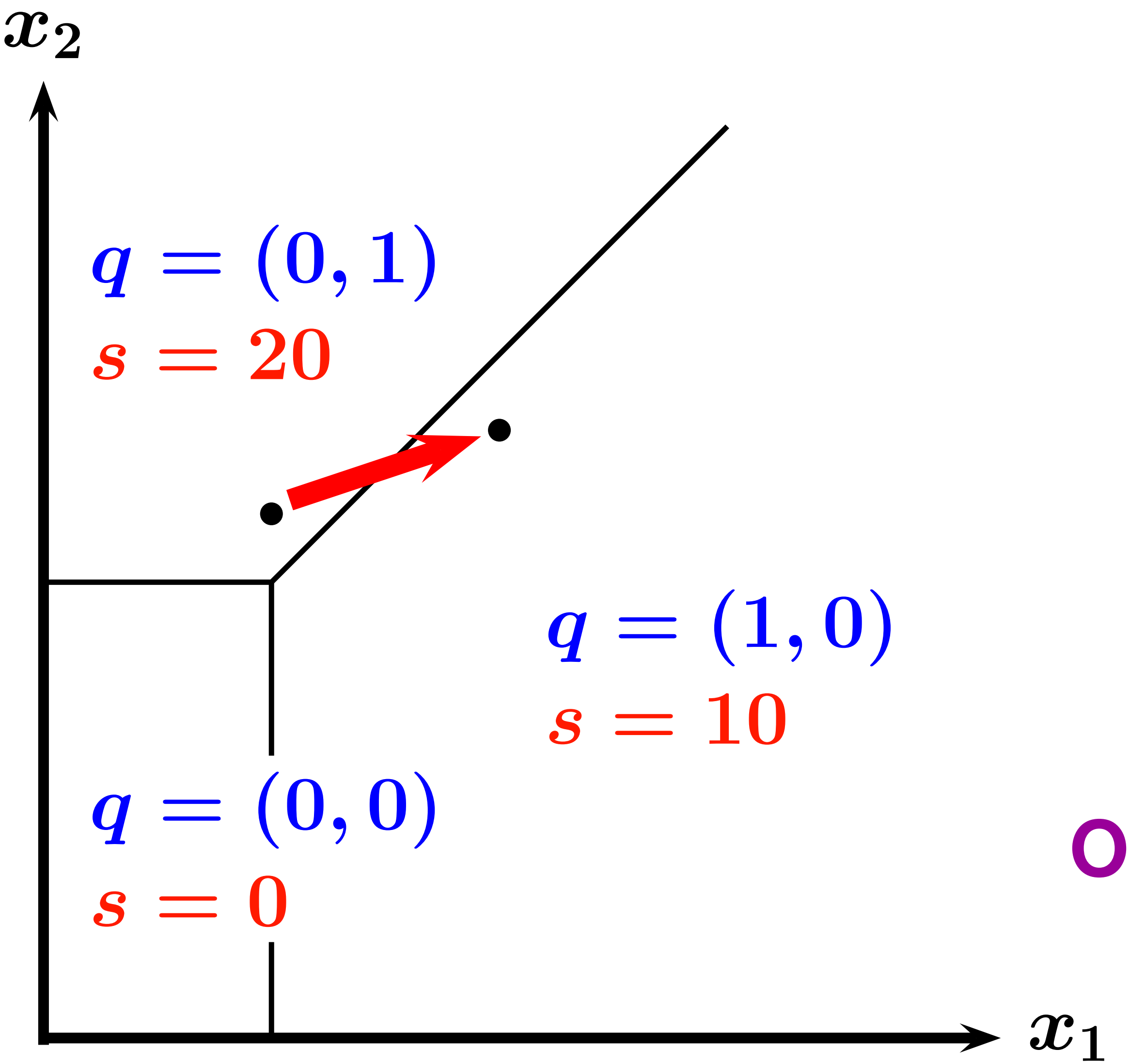
$(10, 23)$ pays \$ 20

$(20, 27)$ pays \$ 10

Non-Monotonic Mechanism



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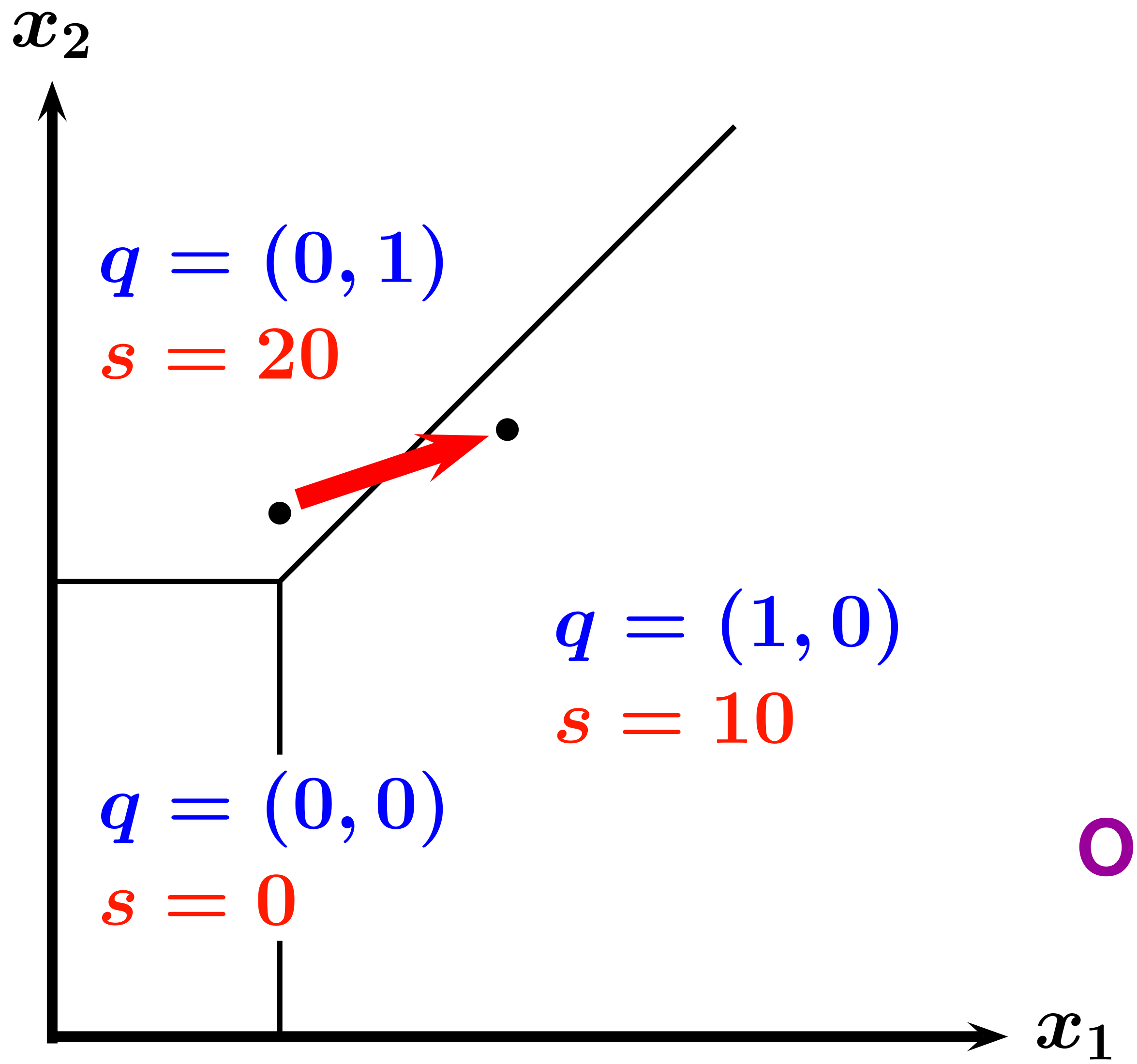
Optimal for some X?

Non-Monotonic Mechanism



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$(10, 23)$ pays \$ 20

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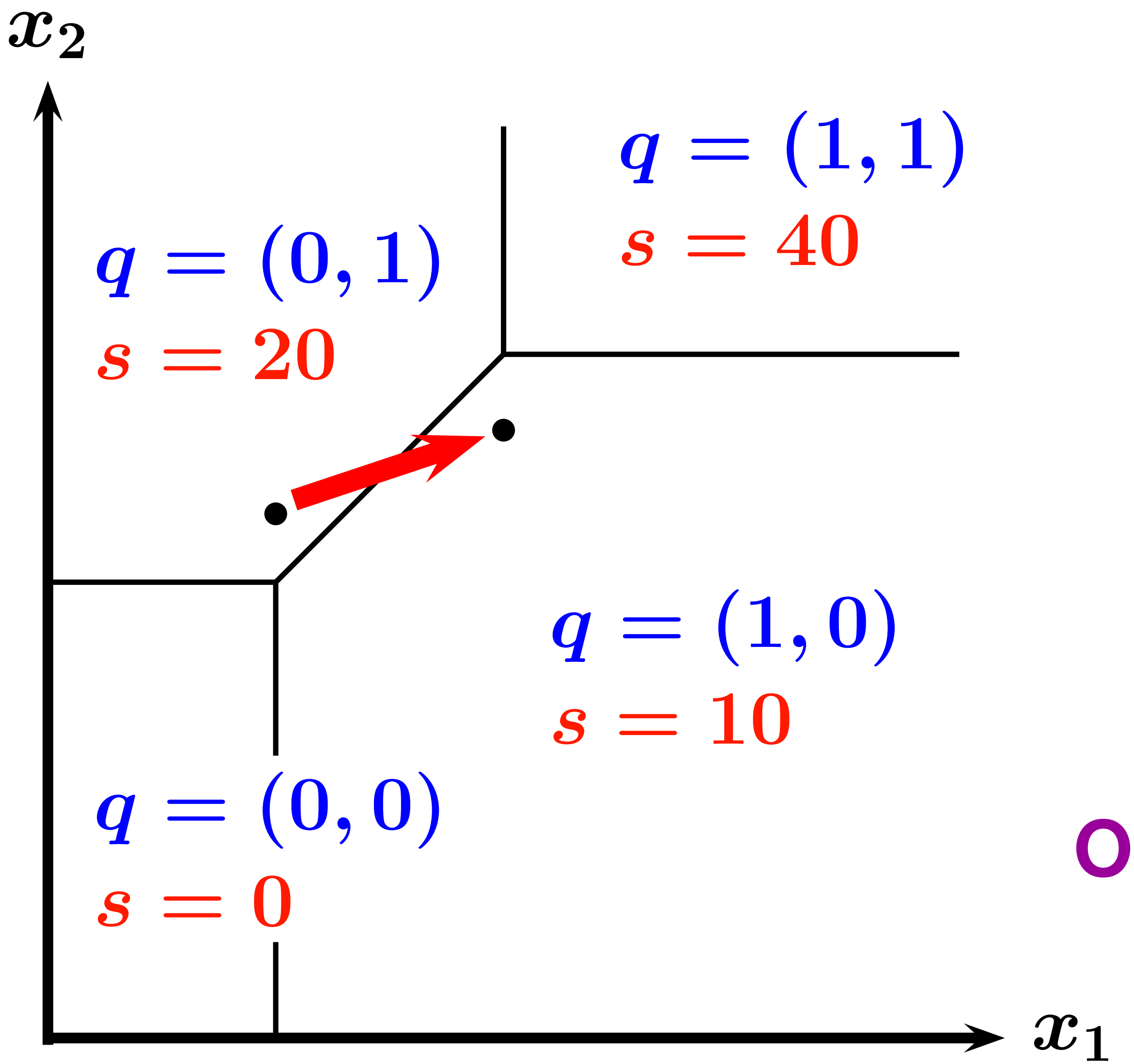
Optimal for some X?

No!

Non-Monotonic Mechanism



Menu	
good 1	\$ 10
good 2	\$ 20
both	\$ 40

$(10, 23)$ pays \$ 20

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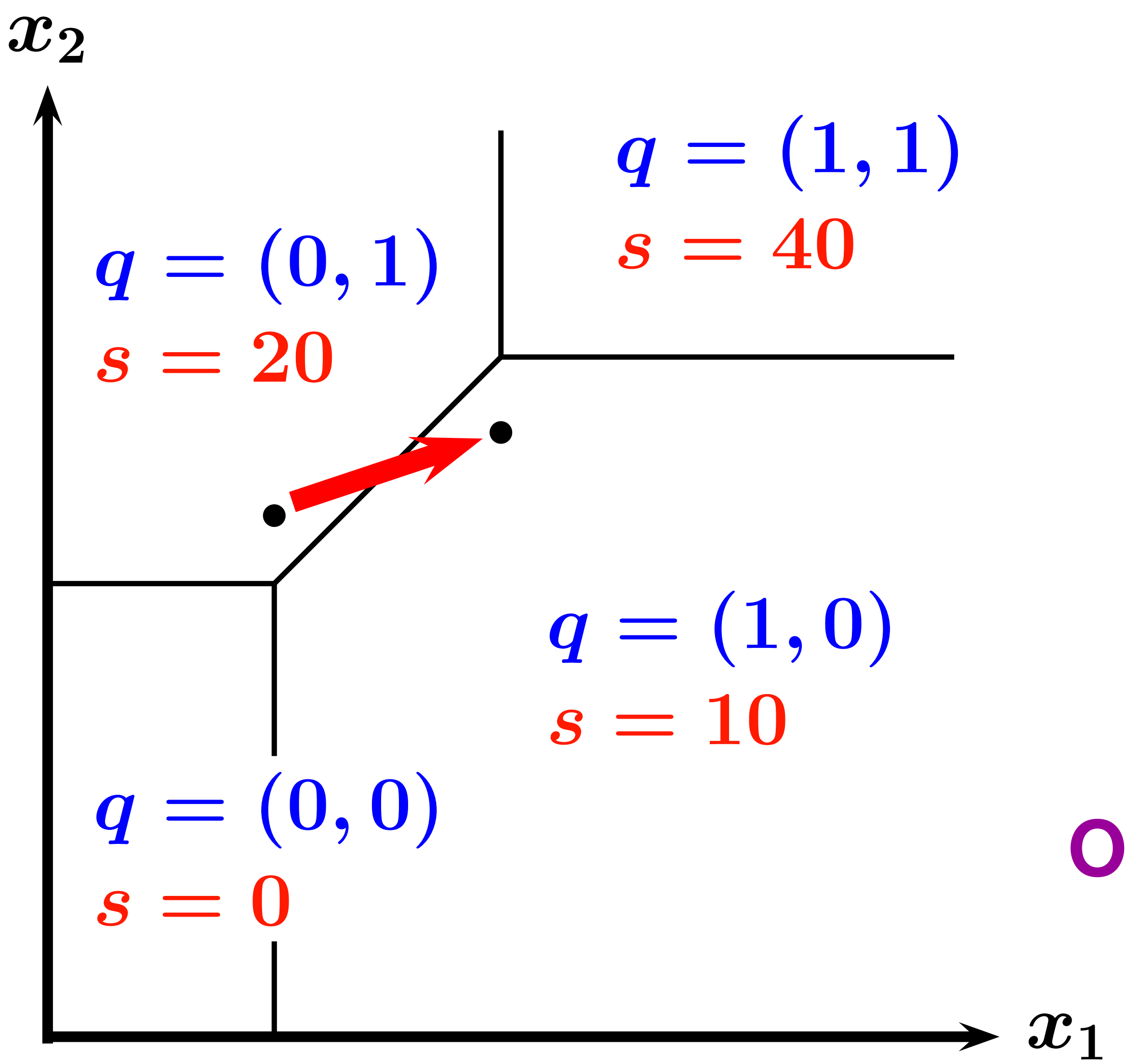
Optimal for some X?

Non-Monotonic Mechanism



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Non-Monotonicity

Non-Monotonicity

- There are simple 2-good valuations X for which the above **NON-MONOTONIC** mechanism **MAXIMIZES REVENUE**

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  - moreover: unique maximizer; robust

Non-Monotonicity

- There are simple 2-good valuations X for which the above **NON-MONOTONIC** mechanism **MAXIMIZES REVENUE**
  - moreover: unique maximizer; robust
- There are simple 2-good valuations X, X' such that

$$X' \geq X \text{ but } \textsf{REV}(X') < \textsf{REV}(X)$$

Non-Monotonic Mechanisms

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- **Conclusion**: **NON-MONOTONIC** mechanisms are **needed** in order to obtain the maximal revenue

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  - **Answer 1**: a ***non-negligible*** amount

Non-Monotonic Mechanisms

- **Conclusion**: **NON-MONOTONIC** mechanisms are **needed** in order to obtain the maximal revenue
- **Question**: How much additional revenue can one gain by using **NON-MONOTONIC** mechanisms?
  - **Answer 1**: a ***non-negligible*** amount
  - **Answer 2**: ***most*** of the revenue !

The Setup

Mechanisms

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(Direct) ***mechanism*** $\mu = (q, s)$:

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$$q = (q_1, q_2, ..., q_k) : \mathbb{R}_+^k \to [0, 1]^k$$

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(Direct) ***mechanism*** $\mu = (q, s)$:

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$$s : \mathbb{R}_+^k \longrightarrow \mathbb{R}$$

Mechanisms

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  - $q_i(x)$ = probability of getting good i for **BUYER** with valuation x

- ***Payment*** function

$$s : \mathbb{R}_+^k \rightarrow \mathbb{R}$$

  - $s(x)$ = payment from **BUYER** with valuation x to **SELLER**

Mechanism

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- **BUYER** payoff function

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$$b(x) = q(x) \cdot x - s(x)$$

Mechanism

- **BUYER** payoff function

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- **INDIVIDUAL RATIONALITY (IR)**

$$b(x) \geq 0 \text{ for all } x$$

Mechanism

- **BUYER** payoff function

$$b(x) = q(x) \cdot x - s(x)$$

- **INDIVIDUAL RATIONALITY (IR)**

$$b(x) \geq 0 \text{ for all } x$$

- **INCENTIVE COMPATIBILITY (IC)**

$$q(y) \cdot x - s(y) \leq b(x) \text{ for all } x, y$$

Optimal Revenue

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- Payoff of **SELLER** from mechanism $\mu = (q, s)$

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Optimal Revenue

Given a random valuation $\boldsymbol{X}$

- Payoff of **SELLER** from mechanism $\boldsymbol{\mu} = (\boldsymbol{q}, \boldsymbol{s})$

$$\boldsymbol{R}(\boldsymbol{\mu}; \boldsymbol{X}) := \mathbb{E}[\boldsymbol{s}(\boldsymbol{X})]$$

- Optimal **REVENUE**

$$\textbf{REV}(\boldsymbol{X}) := \sup_{\boldsymbol{\mu}} \boldsymbol{R}(\boldsymbol{\mu}; \boldsymbol{X})$$

  - supremum is taken over all (IR and IC) mechanisms $\boldsymbol{\mu}$

Monotonic Mechanisms

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- A mechanism $\mu = (q, s)$ is **MONOTONIC** if its payment function s is nondecreasing:

$$x \geq y \;\text{ implies }\; s(x) \geq s(y)$$

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- A mechanism $\mu = (q, s)$ is **MONOTONIC** if its payment function s is nondecreasing:

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- **MonRev**$(X) :=$ maximal revenue obtainable by **MONOTONIC** mechanisms

Monotonicity of Revenue

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Claim. If $X \geq Y$ (more generally: if X first order stochastically dominates Y) then

$$\textsc{MonRev}(X) \geq \textsc{MonRev}(Y)$$

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Proof. For every monotonic mechanism μ:

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$$\begin{aligned} X \geq Y &\Rightarrow s(X) \geq s(Y) \\ &\Rightarrow \mathbb{E}[s(X)] \geq \mathbb{E}[s(Y)] \\ &\qquad R(\mu; X) \geq R(\mu; Y) \end{aligned}$$

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$$\Rightarrow \sup_\mu R(\mu; X) \geq \sup_\mu R(\mu; Y)$$

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For $k = 1$:

$$\text{Rev}(X) \geq \text{Rev}(Y)$$

Monotonicity of Revenue

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For $k = 1$:

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Proof 1. Every one-good (IC) mechanism is monotonic, and so $\textsc{Rev} = \textsc{MonRev}$

Proof 2. For every price p

$$p \cdot \mathbb{P}[X > p] \geq p \cdot \mathbb{P}[Y > p]$$

Monotonic Revenue

Monotonic vs. Bundled

Monotonic vs. Bundled

Theorem. For every k-good valuation X

$$\text{MonRev}(X) \le k \cdot \text{BRev}(X)$$

Monotonic vs. Bundled

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Proof. Put $X^{max} \;:=\; max_{1 \le i \le k} X_i$.

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$$\begin{aligned} & \textsc{MonRev}(X_1, ..., X_k) \\ \le\; & \textsc{MonRev}(X^{max}, ..., X^{max}) \\ \le\; & \textsc{Rev}(X^{max}, ..., X^{max}) \end{aligned}$$

Monotonic vs. Bundled

Theorem. For every k-good valuation X

$$\text{MonRev}(X) \le k \cdot \text{BRev}(X)$$

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Monotonic Revenue

Monotonic Revenue

Corollary. Let $k \geq 2$.

Monotonic Revenue Is Low

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- There are k-good valuations X such that

$$\textsc{MonRev}(X) = 1 \quad \text{and} \quad \textsc{Rev}(X) = \infty$$

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$$\textsf{MonRev}(X) < \varepsilon \cdot \textsf{Rev}(X)$$

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Proof.

$$\frac{\textsc{MonRev}}{\textsc{Rev}} \leq k \cdot \frac{\textsc{BRev}}{\textsc{Rev}}$$

Use Hart and Nisan 2013/2019 (Briest et al 2010/2015 for $k \geq 3$) for BREV

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Proof. Use Hart and Nisan 2013/2019

Monotonic vs. Separate

Monotonic vs. Separate

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$$\mathbb{P}[X^{max} > p] \le \quad \mathbb{P}[X_1 > p] + ... + \mathbb{P}[X_k > p]$$

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$$p\cdot\mathbb{P}[X^{max} > p] \leq p\cdot(\mathbb{P}[X_1 > p] + ... + \mathbb{P}[X_k > p])$$

Monotonic vs. Bundled/Separate

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Theorem. For every k-good valuation X

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- **Tight?**

Monotonic vs. Bundled/Separate

Theorem. For every k-good valuation X

$$\text{MonRev}(X) \leq k \cdot \min\{\text{BRev}(X), \text{SRev}(X)\}$$

- **Tight?**
  - **BRev: Yes**

Monotonic vs. Bundled/Separate

Theorem. For every k-good valuation X

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- **Tight?**
  - **BREV: Yes**

There are k i.i.d. goods s.t.

$$\textsc{SRev}(X) > (k-\varepsilon)\textsc{BRev}(X)$$

(Hart and Nisan 2012/2017)

Monotonic vs. Bundled/Separate

Theorem. For every k-good valuation X
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$\text{MonRev}(X) \ge \text{SRev}(X) > (k-\varepsilon)\text{BRev}(X)$

(Hart and Nisan 2012/2017)

Monotonic vs. Bundled/Separate

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  - **BRev: Yes**
  - **SRev: ??**

Monotonic vs. Bundled/Separate

Theorem. For every k-good valuation X

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  - **SRev: ??**

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$$\text{BRev}(X) \geq \Omega(\log k) \cdot \text{SRev}(X)$$

(Hart and Nisan 2012/2017)

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- **Tight?**
  - **BRev: Yes**
  - **SRev: ??** [**between** $\Omega(\log k)$ **and** k]

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Monotonic vs. Bundled/Separate

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  - **BRev: Yes**
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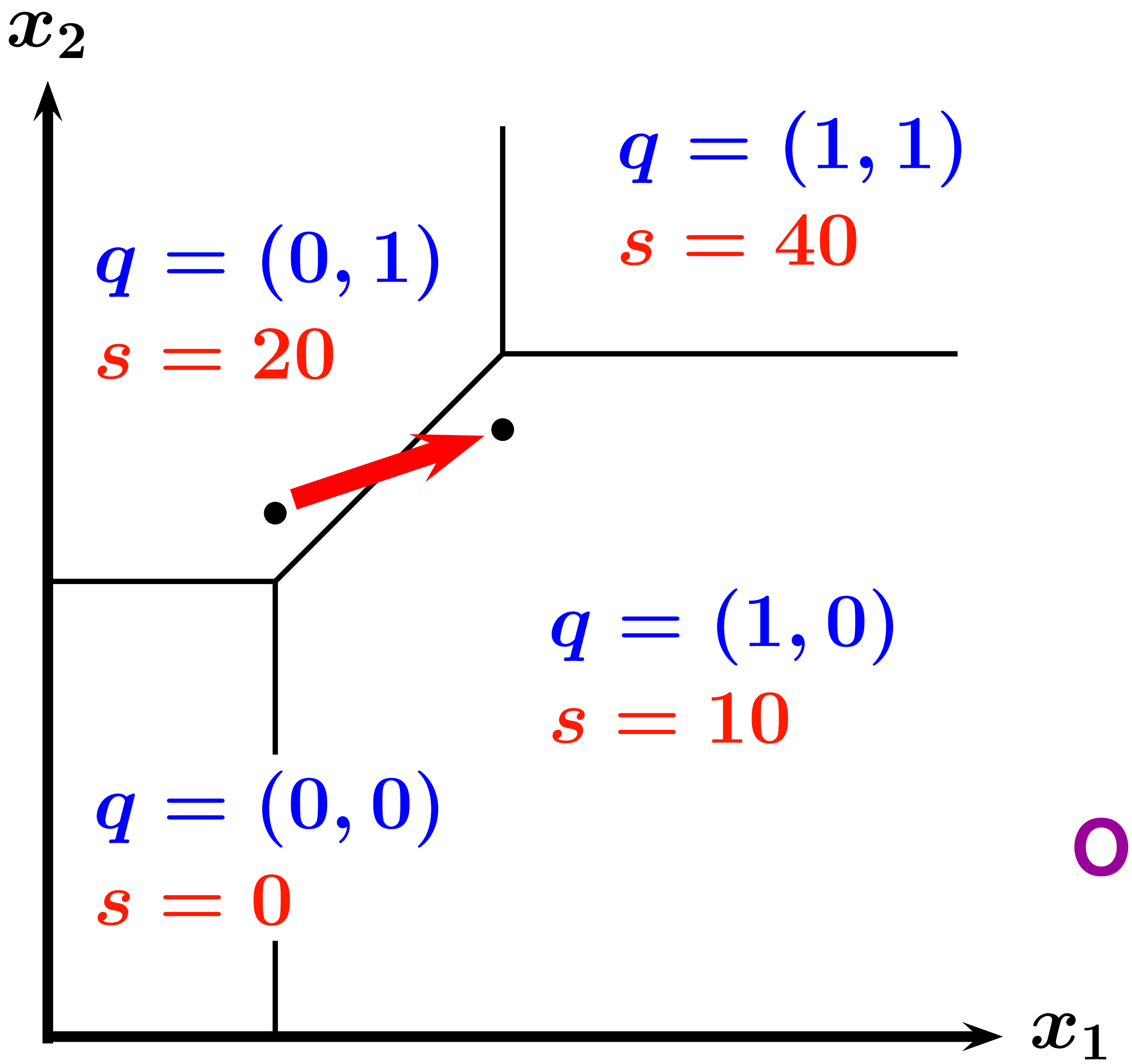
Monotonic Mechanisms

Non-Monotonic Mechanism



Menu	
good 1	\$10
good 2	\$20
both	\$40

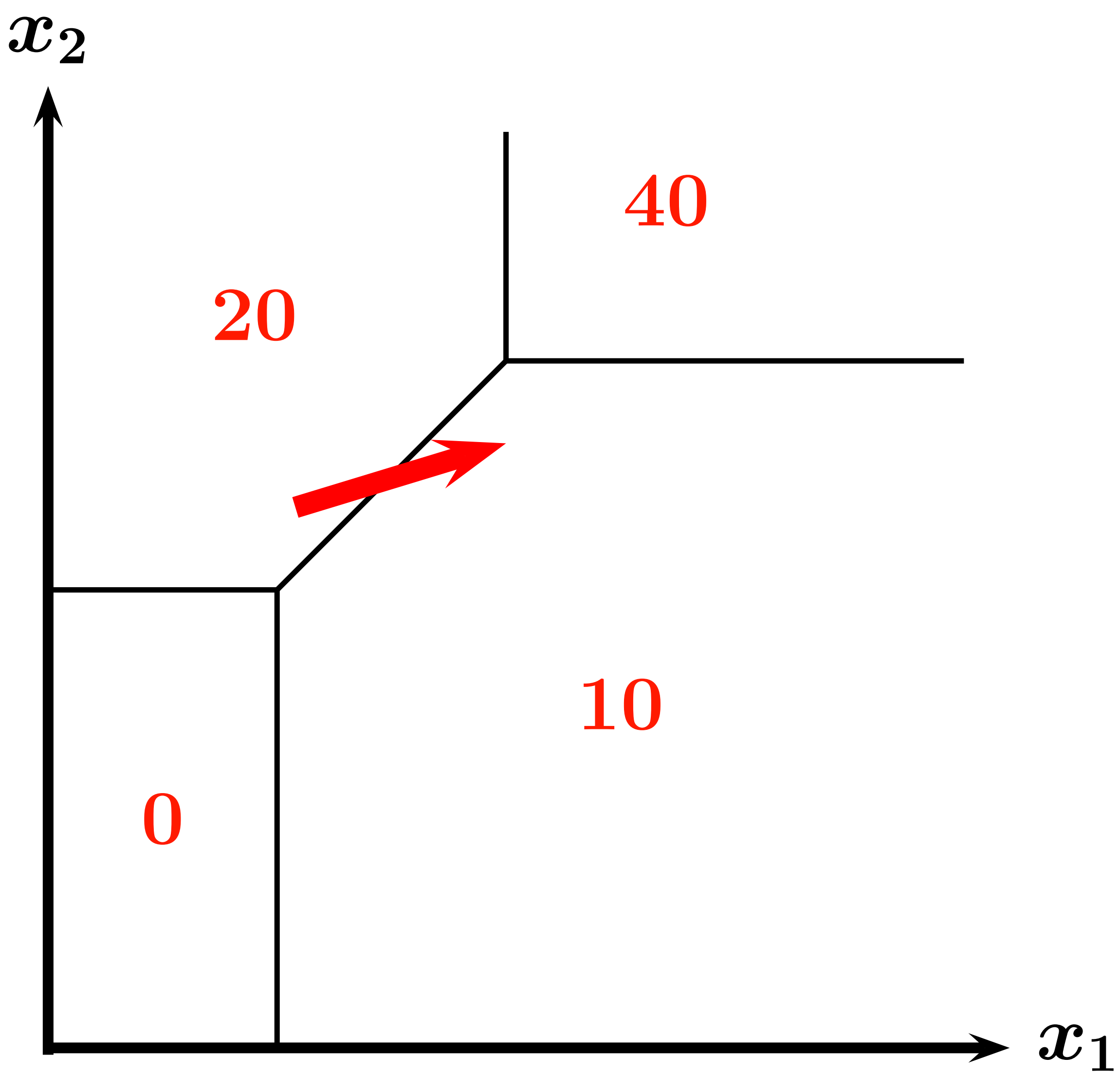
$(10, 23)$ pays \$20

$(20, 27)$ pays \$10

Optimal for some X?

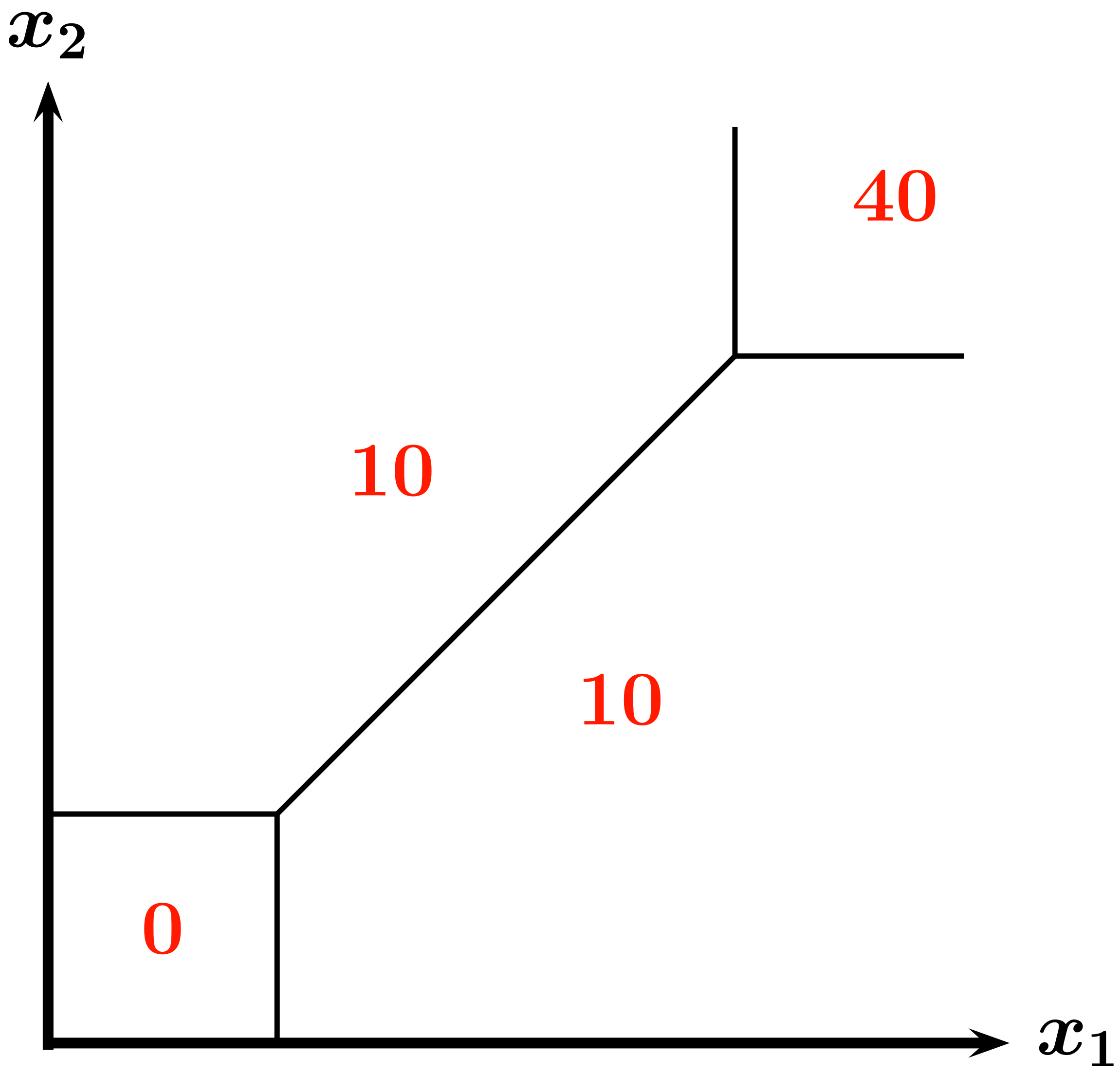
YES!

Non-Monotonic Mechanism



Menu	
good 1	\$ 10
good 2	\$ 20
both	\$ 40

Monotonic Mechanism



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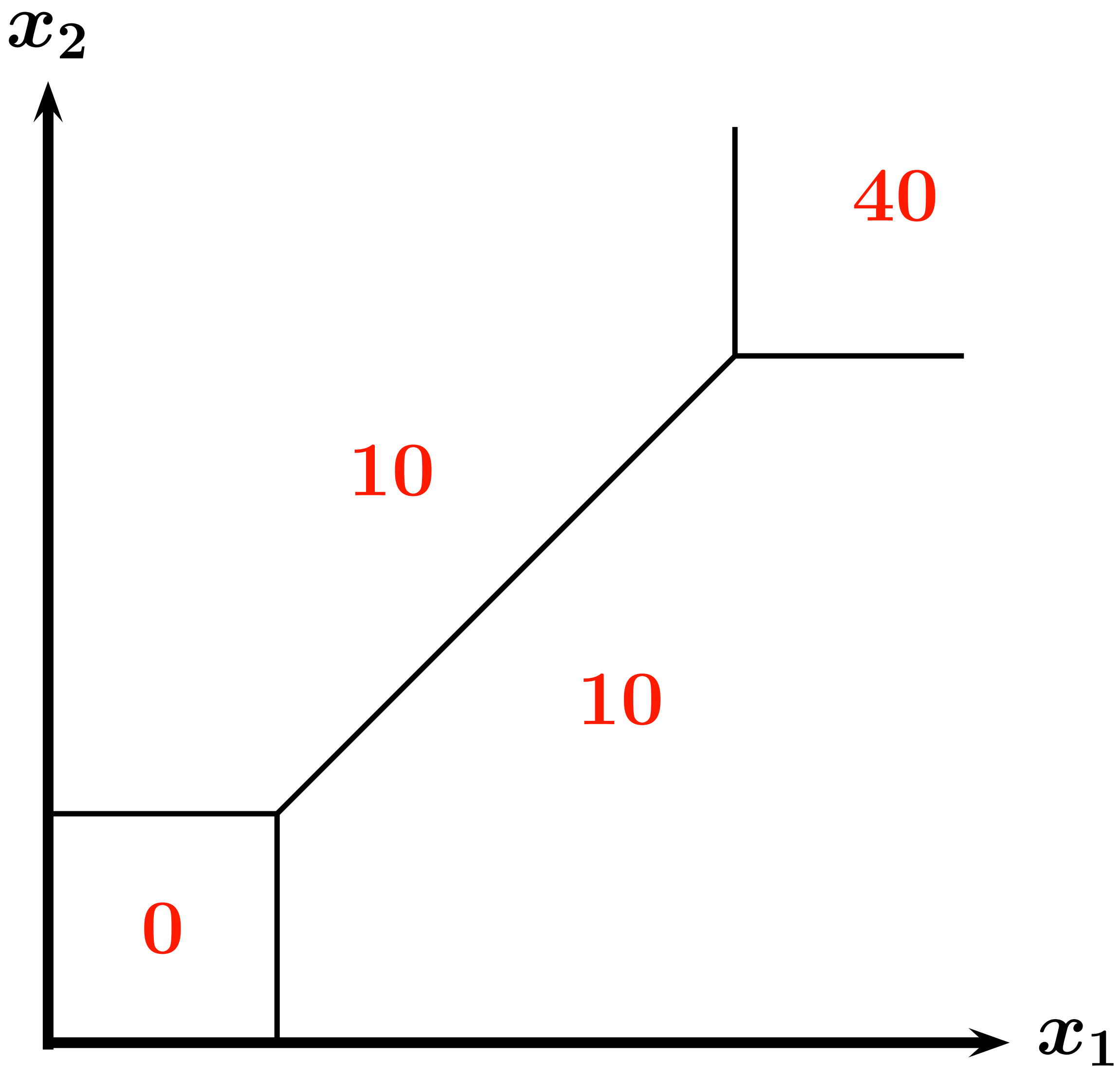

Menu	
good 1	$ 10
good 2	$ 10
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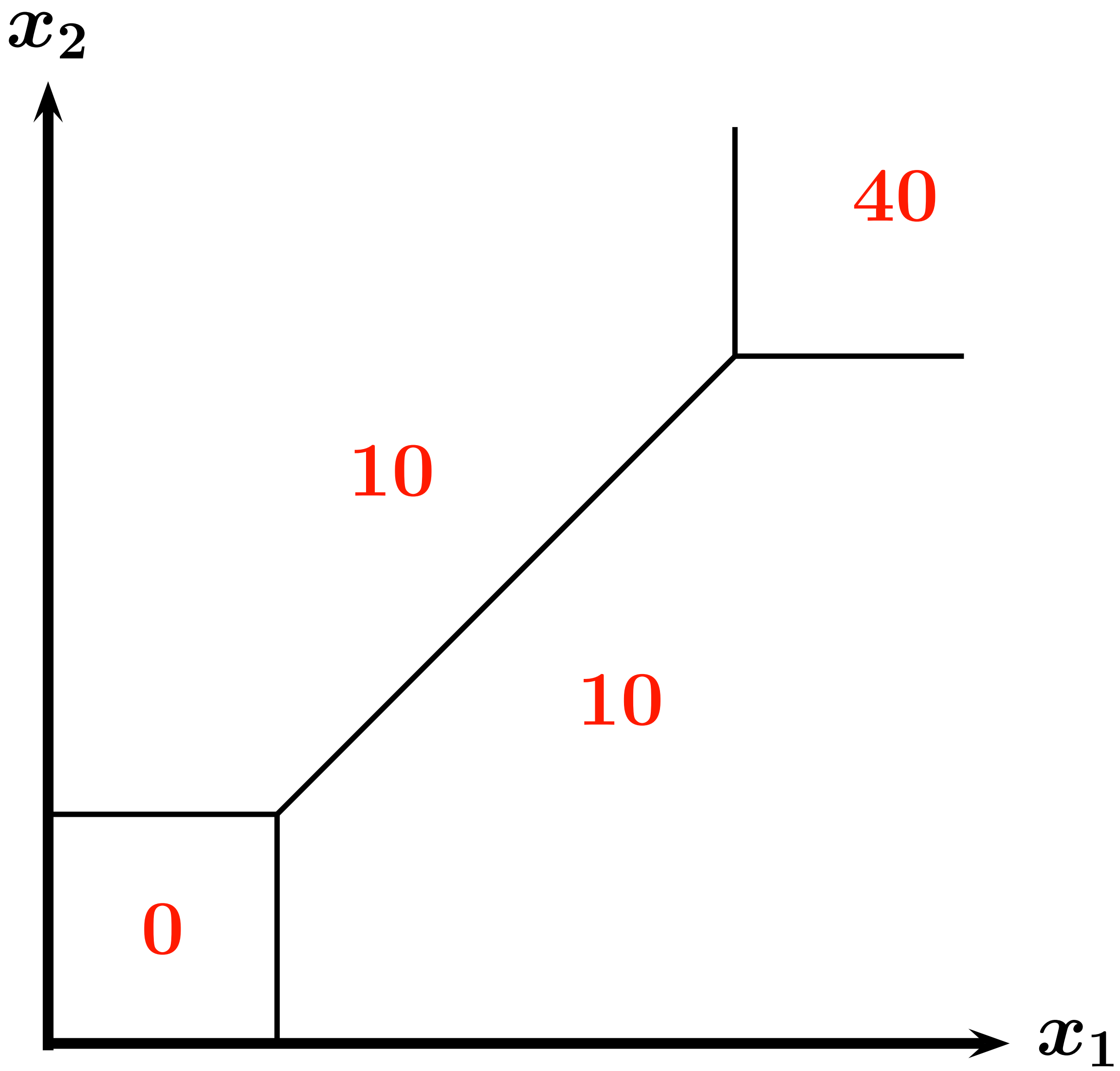
MONOTONIC

Monotonic Mechanism



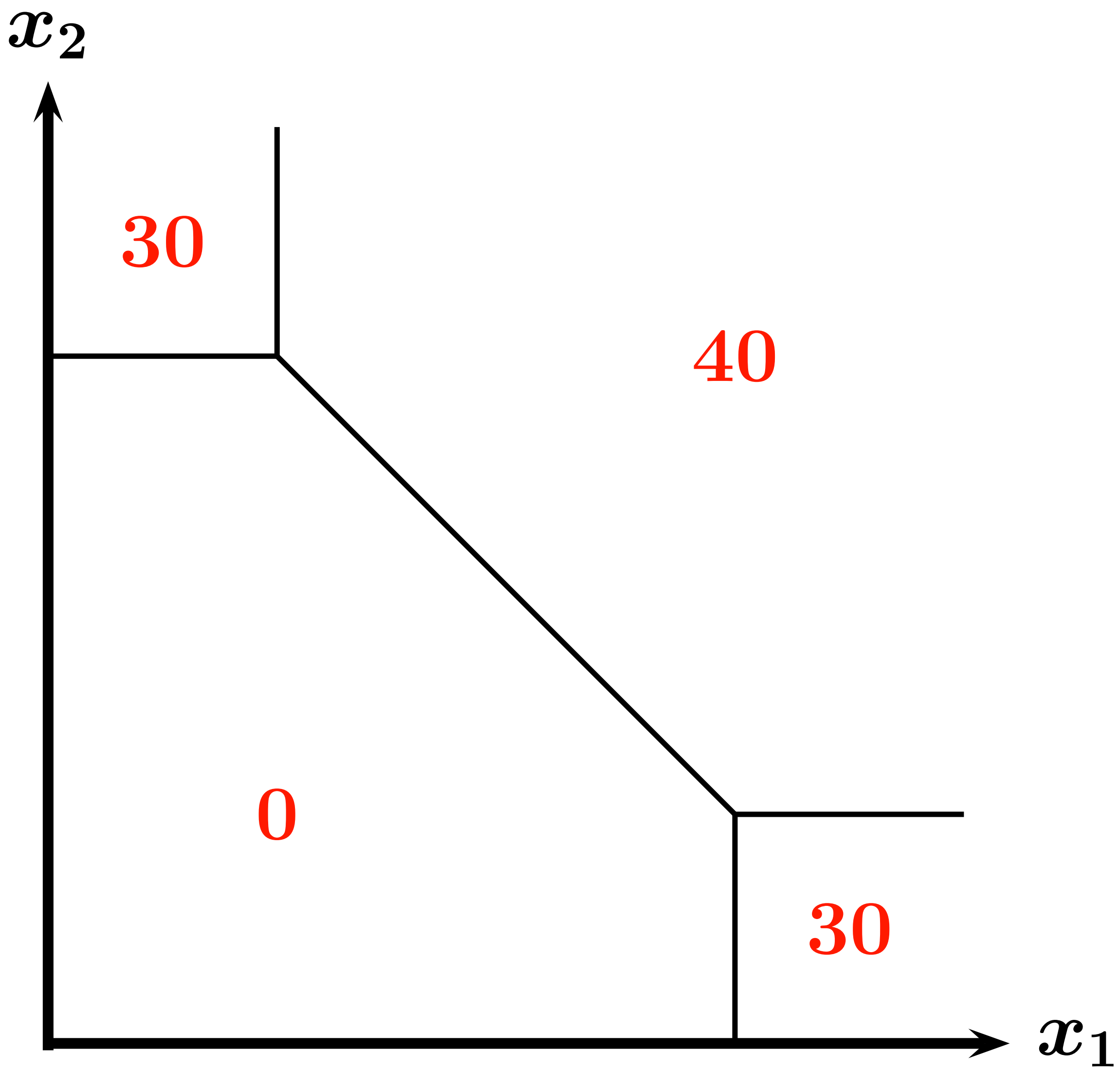
Menu	
good 1	$ 10
good 2	$ 10
both	$ 40

MONOTONIC

Symmetric

Deterministic

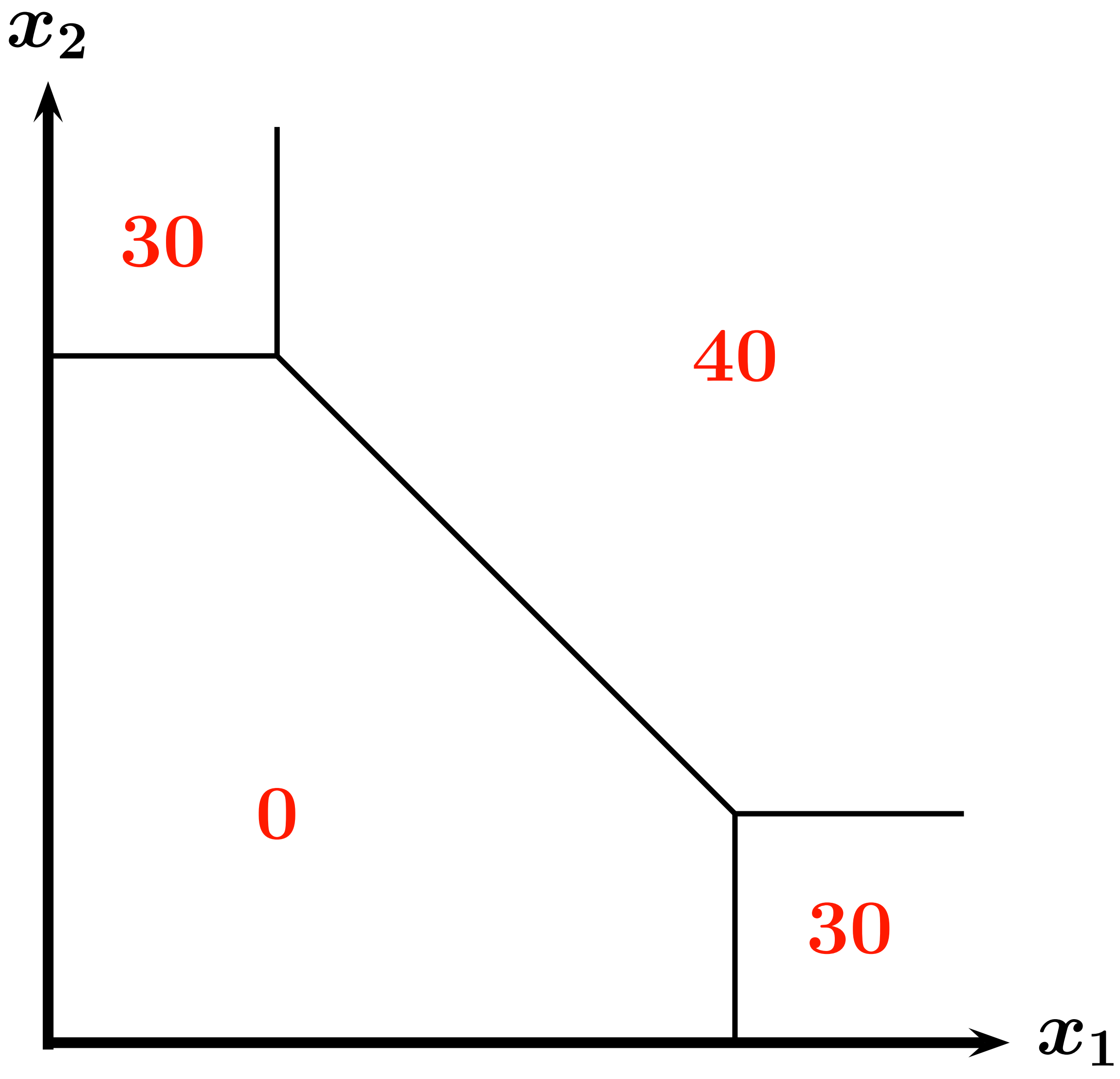
Monotonic Mechanism



Menu

good 1	\$ 30
good 2	\$ 30
both	\$ 40

Monotonic Mechanism



Menu	
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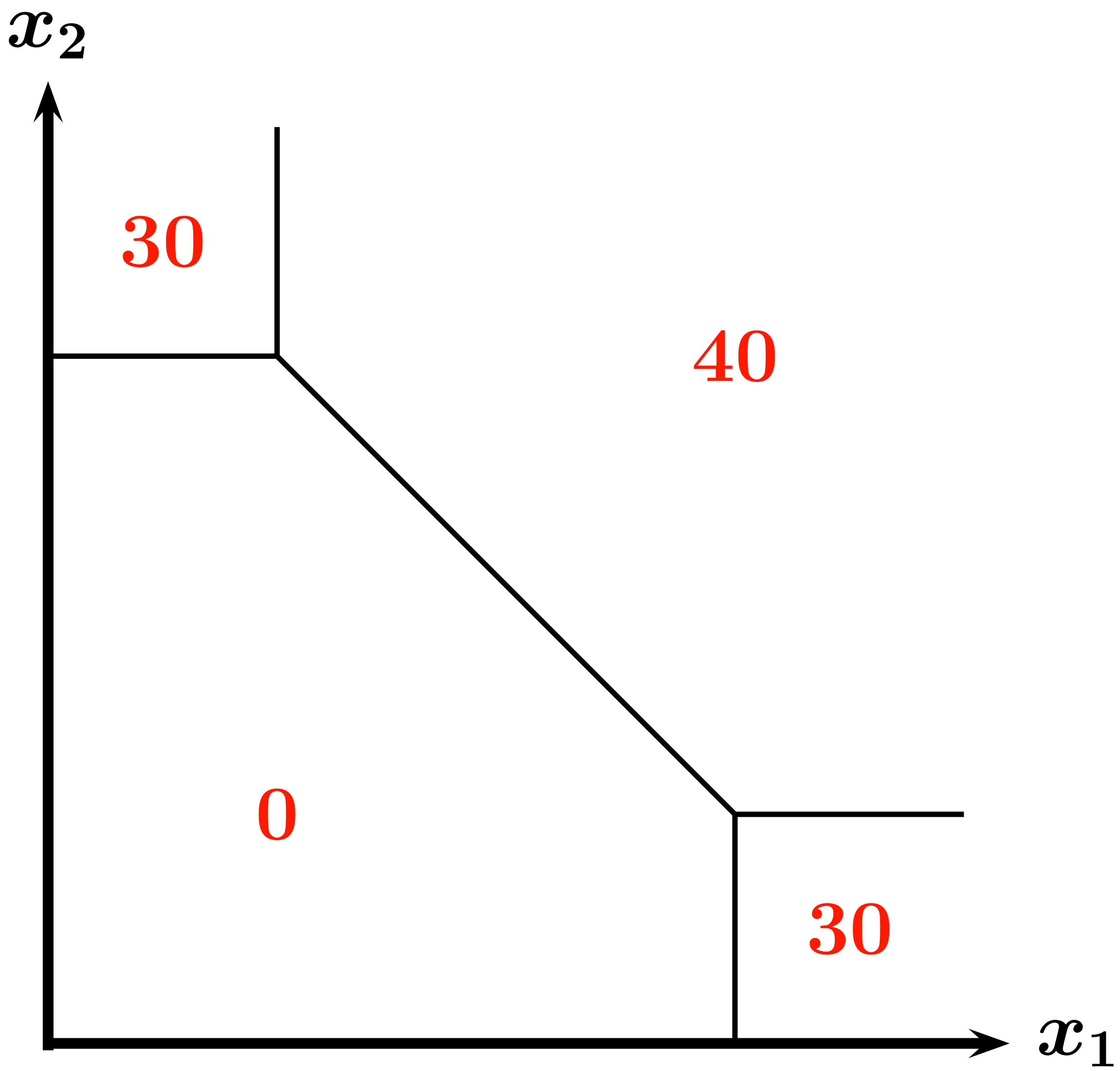
MONOTONIC

Monotonic Mechanism



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MONOTONIC

Subadditive (Submodular)

Classes of Monotonic Mechanisms

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Hart and Reny 2014

Classes of Monotonic Mechanisms

- Symmetric deterministic mechanisms

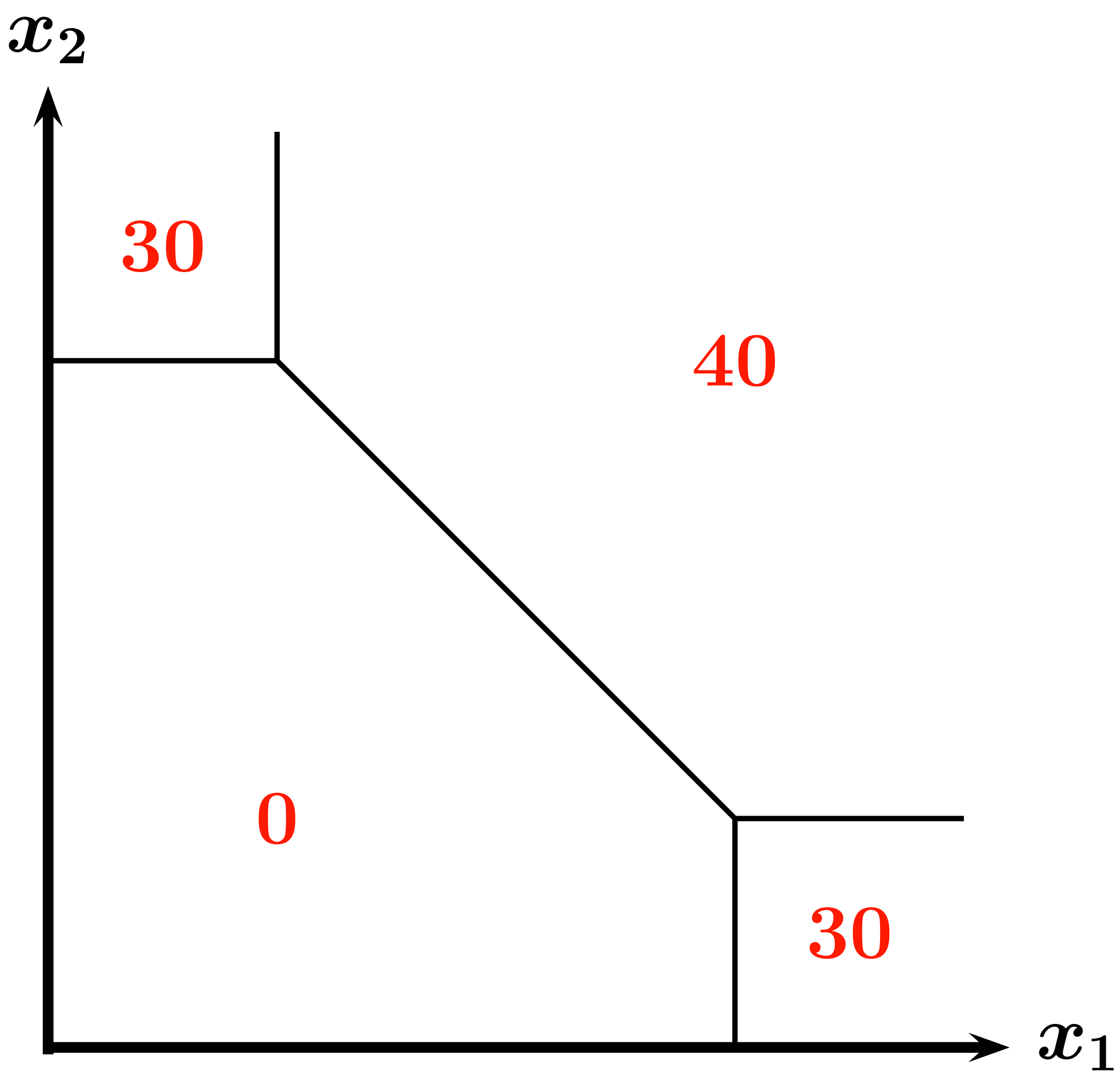
Hart and Reny 2014

Classes of Monotonic Mechanisms

- Symmetric deterministic mechanisms
- Submodular mechanisms

Hart and Reny 2014

Monotonic Mechanism



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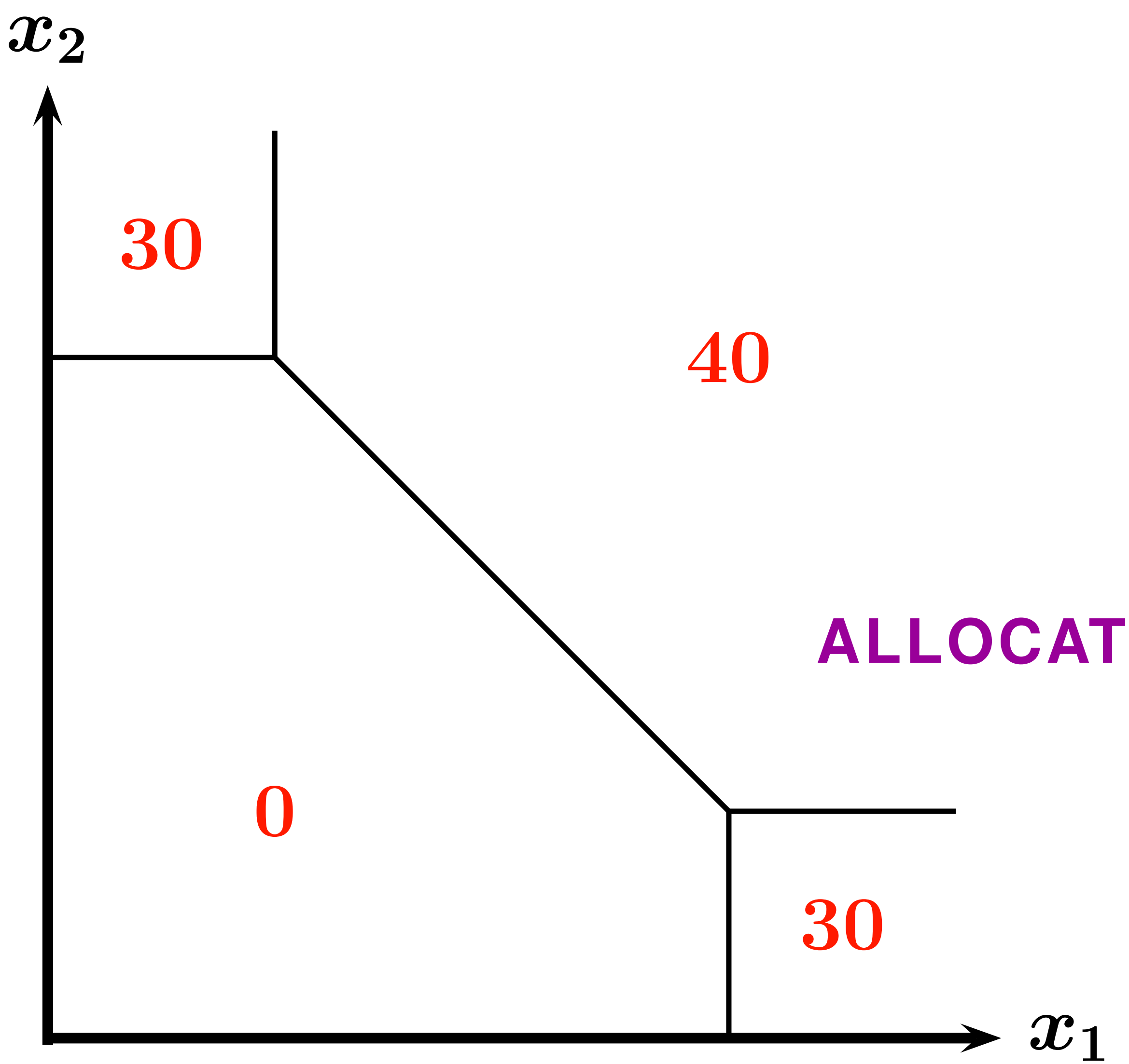
MONOTONIC

Subadditive (Submodular)

Allocation-Monotonic Mechanism



Menu	
good 1	$ 30
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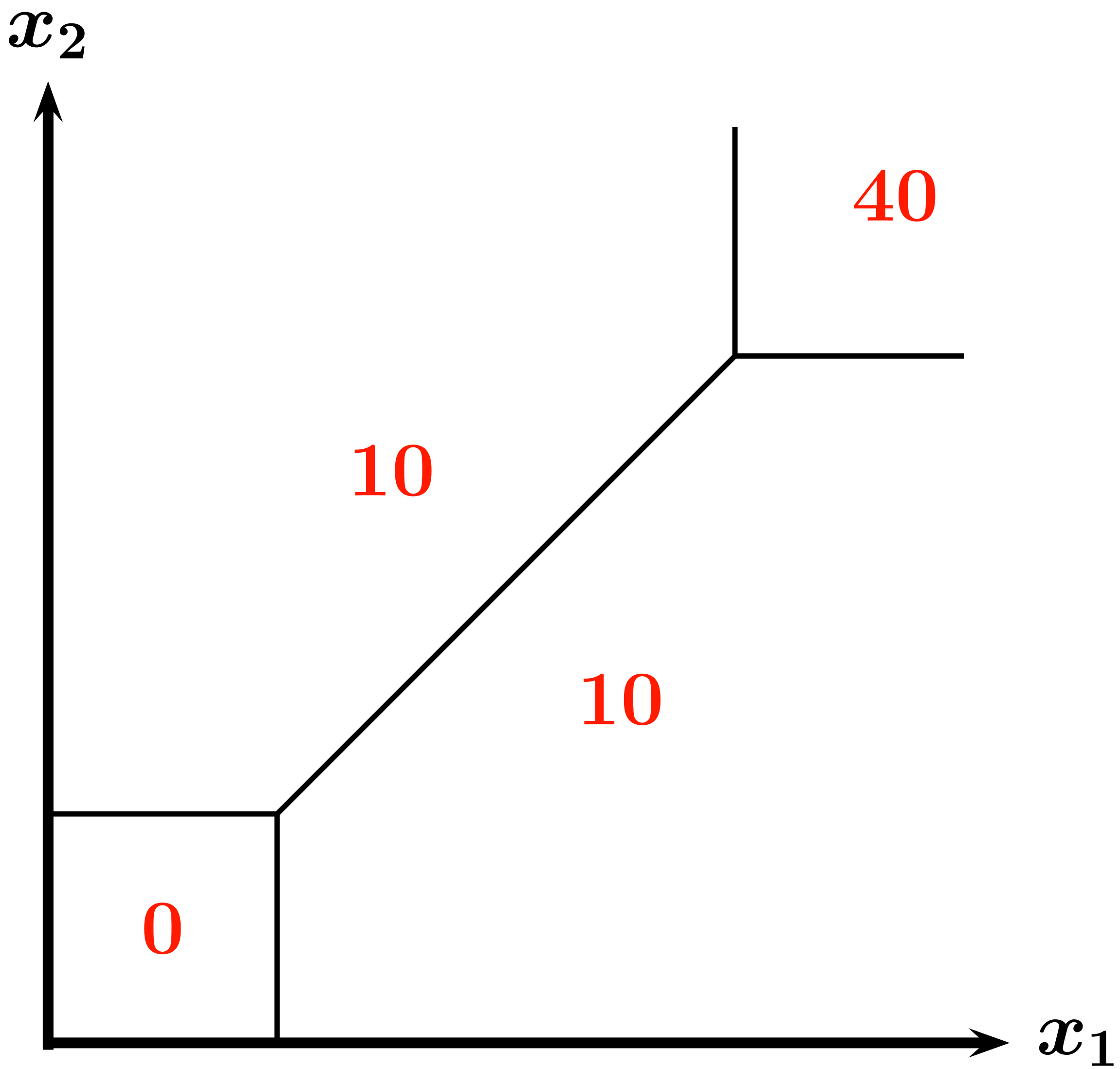
both	$ 40

ALLOCATION-MONOTONIC

Subadditive (Submodular)

Monotonic



Menu

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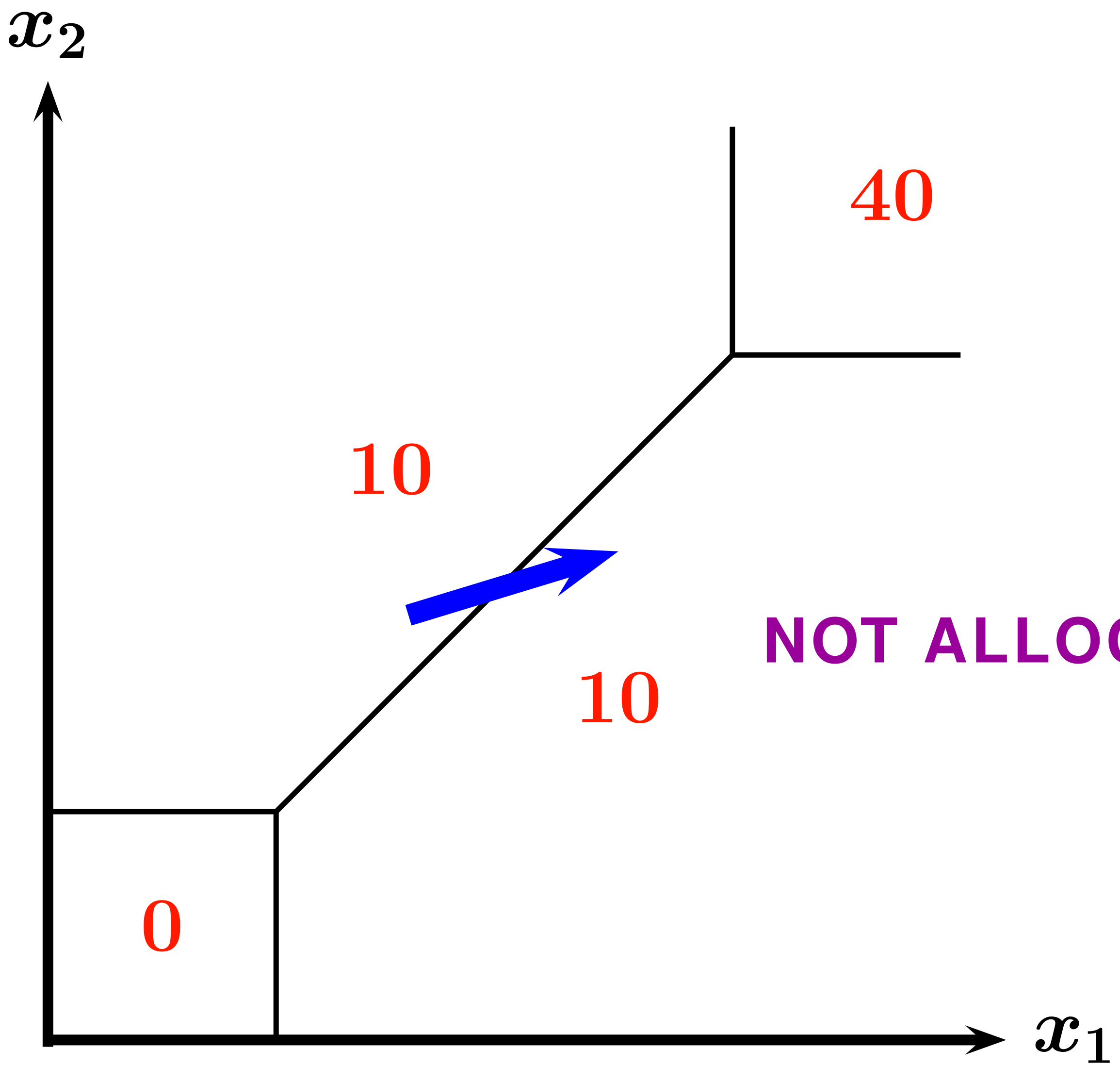
MONOTONIC

Symmetric

Deterministic

NOT Allocation-Monotonic



Menu	
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NOT ALLOCATION-MONOTONIC

Symmetric

Deterministic

Allocation-Monotonic Mechanisms

Monotonic Mechanisms

Monotonic Mechanisms

- A mechanism $\mu = (q, s)$ is **MONOTONIC** if its payment function s is nondecreasing:

$$x \geq y \text{ implies } s(x) \geq s(y)$$

Allocation-Monotonic Mechanisms

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- **allocation monotonicity** $\Rightarrow$ **monotonicity**

Allocation-Monotonic Mechanisms

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2-Good Deterministic Mechanisms

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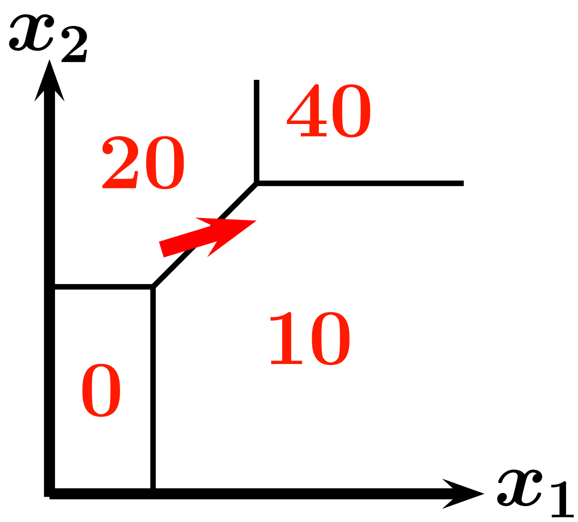
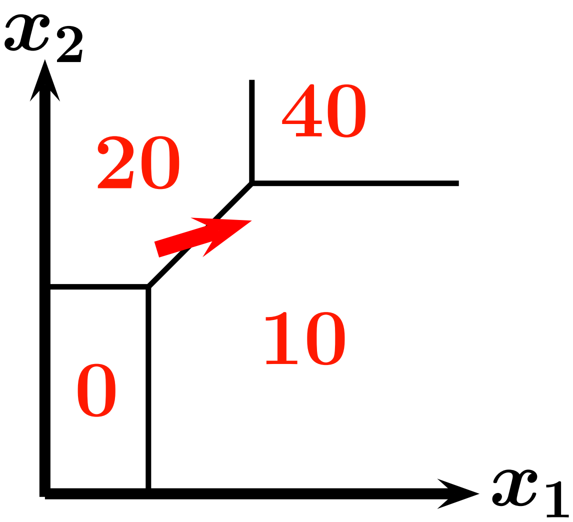
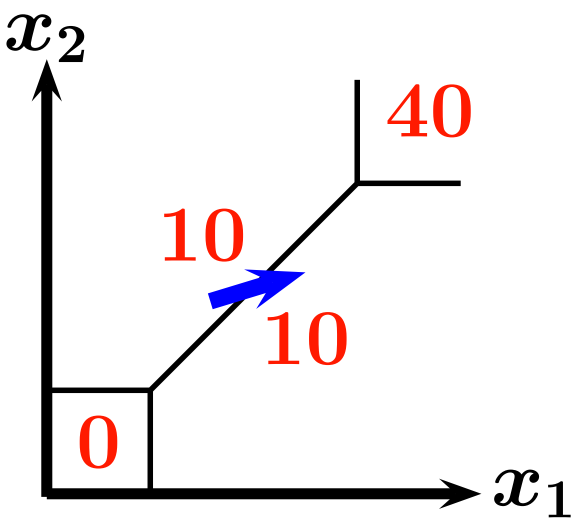


not mon

not alloc-mon

2-Good Deterministic Mechanisms

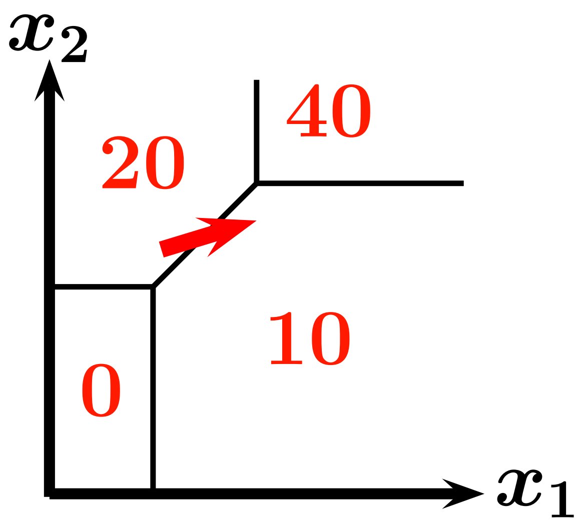




not **mon**
not **alloc-mon**

mon
not **alloc-mon**

2-Good Deterministic Mechanisms



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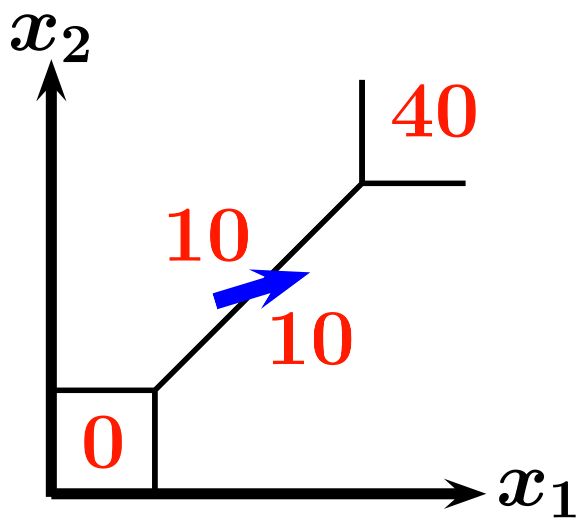

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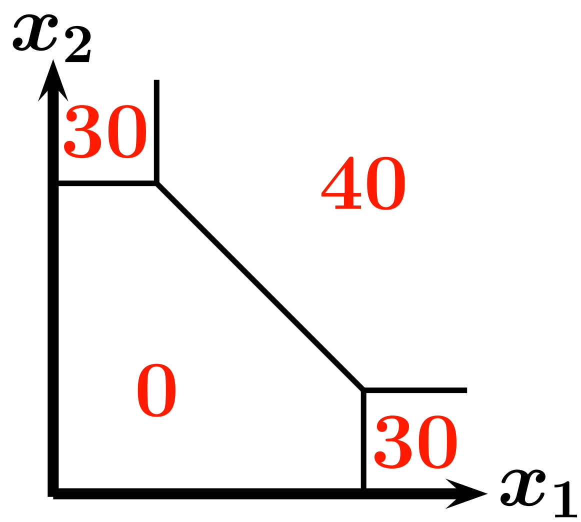
not alloc-mon



mon

alloc-mon

Deterministic: Pricing

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Let $\mu = (q, s)$ be a **deterministic** mechanism for k goods. Put $K := \{1, ..., k\}$.

Deterministic: Pricing

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- $p : 2^K \to [0, \infty]$ is the (**canonical**) **PRICING FUNCTION** of μ (nondecreasing function)

Deterministic: Submodular

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- The function p is **SUBMODULAR** if for all A, B :

$$p(A) + p(B) \geq p(A \cup B) + p(A \cap B)$$

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$\Leftrightarrow$ for all $i, j \notin A$:

$$p(A \cup \{i\}) - p(A) \geq p(A \cup \{i, j\}) - p(A \cup \{j\})$$

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Decreasing marginal price

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Decreasing marginal price

- The mechanism μ is **SUBMODULAR** if its (canonical) pricing function is submodular

Deterministic: Allocation Monotonic

Deterministic: Allocation Monotonic

Let μ be a tie-favorable **deterministic** mechanism

Theorem.

μ is **allocation monotonic**

$\Leftrightarrow$ μ is **submodular**

$$\{\textbf{AMon}\} = \{\textbf{Submod}\}$$

General: Allocation Monotonicity

General: Allocation Monotonicity

Let μ be a tie-favorable general (probabilistic) mechanism

Theorem.

μ is **submodular**

$\Rightarrow$ μ is **allocation monotonic**

$\Rightarrow$ μ is **separably subadditive**

$\{\text{SUBMOD}\} \subset \{\text{AMON}\} \subset \{\text{SEP SUBADD}\}$

General: Pricing

General: Pricing

Let $\mu = (q, s)$ be a mechanism for k goods.

General: Pricing

Let $\mu = (q, s)$ be a mechanism for k goods.

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- $p : [0,1]^k \to [0,\infty]$ is the (**canonical**) **PRICING FUNCTION** of μ (nondecreasing, convex, closed function)
  - The convex functions b and p are Fenchel conjugates

Submodular Pricing

Submodular Pricing

The function p is **SUBMODULAR** if

- for all g, h in $[0,1]^k$:

$$p(g) + p(h) \geq p(g \vee h) + p(g \wedge h)$$

Submodular Pricing

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$\Leftrightarrow$ for all g and orthogonal $d_1, d_2 \geq 0$:

$$p(g+d_2) - p(g) \geq p(g+d_1+d_2) - p(g+d_1)$$

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Marginal price of good i decreases as allocation of good $j \neq i$ increases

Submodular Pricing

The function p is **SUBMODULAR** if

- for all g, h in $[0,1]^k$:

$$p(g) + p(h) \geq p(g \vee h) + p(g \wedge h)$$

$\Leftrightarrow$ for all g and orthogonal $d_1, d_2 \geq 0$:

$$p(g+d_2) - p(g) \geq p(g+d_1+d_2) - p(g+d_1)$$

Marginal price of good i decreases as allocation of good $j \neq i$ increases

- If p is differentiable: $\dfrac{\partial^2 p}{\partial g_i \partial g_j} \leq 0$ for all $i \neq j$

Separably Subadditive Pricing

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The function p is **SEPARABLY SUBADDITIVE** if

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$$p(g + h) \leq p(g) + p(h)$$

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- Weaker than submodularity (by $p(0) = 0$)

Sub... Mechanisms

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- A mechanism μ is **SUBMODULAR** if its (canonical) pricing function is submodular

- A mechanism μ is **SEPARABLY SUBADDITIVE** if its (canonical) pricing function is separably subadditive

Allocation Monotonicity

Allocation Monotonicity

Let μ be a tie-favorable **deterministic** mechanism

Theorem.

μ is **allocation monotonic**

$\Leftrightarrow$ μ is **submodular**

$$\{\textbf{AMON}\} = \{\textbf{SUBMOD}\}$$

Proof

Proof

Deterministic mechanisms

μ allocation monotonic

$\Updownarrow$ [*]

b supermodular

$\Updownarrow$ [FC]

p submodular

(μ submodular)

Proof

Deterministic mechanisms
μ allocation monotonic $\Updownarrow$ [*] b supermodular $\Updownarrow$ [FC] p submodular (μ submodular)

[FC] = p and b are Fenchel Conjugates

Proof of [*]

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- **Proof.** Assume that b is differentiable, then $q = \nabla b$ (because $b(x) = \max_y (q(y) \cdot x - s(y))$
  $= q(x) \cdot x - s(x)$).

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  Without differentiability: “tie-favorable”

Proof

μ allocation monotonic

$\Updownarrow$ [*]

b supermodular

$\Updownarrow$ [FC]

p submodular

(μ submodular)

[FC] = p and b are Fenchel Conjugates

General: Allocation Monotonicity

General: Allocation Monotonicity

Let μ be a tie-favorable general (probabilistic) mechanism

Theorem.

μ is **submodular**

$\Rightarrow$ μ is **allocation monotonic**

$\Rightarrow$ μ is **separably subadditive**

$\{\textsc{Submod}\} \subset \{\textsc{AMon}\} \subset \{\textsc{Sep Subadd}\}$

Proof

Proof

<table>
<tr><th colspan="3">General mechanisms</th></tr>
<tr><td>μ allocation monotonic</td><td></td><td></td></tr>
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- $\not\Leftarrow$: already for **QUADRATIC** mechanisms

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- $\not\Leftarrow$: already for **QUADRATIC** mechanisms

$$q(x)=Ax,\ s(x)=b(x)=\tfrac{1}{2}x^\top Ax,\ p(g)=\tfrac{1}{2}g^\top A^{-1}g$$

Proof

General mechanisms		
μ allocation monotonic		
$\Updownarrow$ [*]		
b supermodular	$\Rightarrow$	b separably superadditive
$\Uparrow$ [FC]		$\Updownarrow$ [FC]
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Allocation-Monotonic Revenue

Allocation-Monotonic Revenue

Theorem. For every k-good valuation X

$$\textbf{AMonRev}(X) \leq 2\ln(2k) \cdot \textbf{SRev}(X)$$

Proof

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Proof. Let p be the canonical pricing function of an allocation monotonic mechanism.

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- Then: p' is separable, and

$$\frac{1}{k} p' \le p \le p'$$

(p nondecreasing and separably subadditive)

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(p nondecreasing and separably subadditive)
- Apply a result of Chawla, Teng, and Tzamos

Proof

Proof

Theorem (*Chawla, Teng, and Tzamos 2019*)
Let $\mathcal{P}'$ be a cone of nondecreasing and closed k-good pricing functions. Assume that there are constants $0 < c_1 < c_2 < \infty$ such that for every $p \in \mathcal{P}$ there is $p' \in \mathcal{P}'$ satisfying

$$c_1 p'(g) \leq p(g) \leq c_2 p'(g)$$

for every g; then

$$\mathcal{P}\text{-}\textsc{Rev}(X) \leq 2 \ln \left(2 \, \frac{c_2}{c_1} \right) \cdot \mathcal{P}'\text{-}\textsc{Rev}(X)$$

for every k-good valuation X.

Proof

Proof. Let p be the canonical pricing function of an allocation monotonic mechanism.

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Symmetric Deterministic

Symmetric Deterministic Revenue

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Theorem. For every k-good valuation X

$$\text{SymDRev}(X) \leq O(\log^2 k) \cdot \text{SRev}(X)$$

Proof

Proof

- **Theorem.** For every k-good valuation X
$\text{SUPERMODSYMDREV}(X) \leq H(k) \cdot \text{SREV}(X)$
where $H(k) := 1 + \frac{1}{2} + ... + \frac{1}{k} \sim \ln k$

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- Let p be the canonical pricing function of a symmetric deterministic mechanism

Proof

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  where $H(k) := 1 + \frac{1}{2} + ... + \frac{1}{k} \sim \ln k$
- Let p be the canonical pricing function of a symmetric deterministic mechanism
  - Put: $d(m) := p(m) - p(m-1)$
    $d'(m) := \max\{d(n) : 1 \leq n \leq m\}$
    $p'(m) := d'(1) + ... + d'(m)$

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  - Then: p' supermodular and $\frac{1}{k}p' \leq p \leq p'$

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The End

Next time, get

a NON-MONOTONIC speaker ...

Thank You!

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