Two-Person Repeated Games Where Players Know Their Own Payoffs*

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Abstract

We consider repeated games where each player knows his own payoff matrix but has uncertainty about his opponent's payoff matrix. In the *One-Sided Case*, where one of the players has complete information, we give a characterization for *Correlated Equilibria (CE)* and a new proof for the characterization of *Nash Equilibria (NE)*. Also, in this case *NE* = *CE* . In the *Two-Sided Case*, where each player has doubts about his opponent's payoffs, we give a characterization for *NE*. Like in the One-Sided Case, every *NE* point is payoff equivalent to a completely revealing equilibrium. We then show that some games have no Nash equilibrium points.

1. Introduction

Repeated games with incomplete information have received considerable attention during the last 20 years. One reason for this interest is that this model is a game-theoretical idealization of many social and economic relationships. These are often developed over time and they are made difficult by the asymmetry of the information of the individuals involved. The main research interest is the characterization of equilibrium points (Nash, correlated and others).

A repeated game is a multi-stage game which is composed of a one-shot game (sub-game) played repeatedly. A game with incomplete information is a game in which at least one of the players lacks part of the relevant data. In the *standard-information* case each player can observe all actions taken during the play but he might not be certain about the payoffs, neither his nor those of the other players.

A standard two-players game with lack of information on one side consists of: two players - player *1* and player *2* - a finite set K of states of nature, each state k in K being described by a (one shot) two-person game (i.e. two payoff matrices, one for each player). The state of nature is chosen according to a probability vector[1] $p \in \Delta^K$. Only player *1*, the "informed player", is told which game k was chosen. This game is then played repeatedly.

The first research on repeated games of incomplete information was done in the Mathematica reports (1966-1968) by Aumann, Maschler and Stearns. These articles dealt mainly with the case of two-person zero-sum games.

Since then, the two-person *zero-sum* case has been extensively studied, as detailed in Sorin (1980) and the forthcoming book of Mertens, Sorin and Zamir.

The two-person *non-zero-sum* case was first studied by Aumann, Maschler and Stearns (1968). A complete characterization of the equilibria in these games, with standard one-sided information, was given by Hart (1985). Sorin (1983) proved the existence of an equilibrium point in any such game with only two possible states of nature (i.e., $|K| = 2$).

A drawback of the general case in this model is that player *2* - the "uninformed player"

[1] Δ^K is the $|K| - 1$ dimensional simplex, see the notations in the beginning of next section.

- does not know what his own payoffs are. He receives his payoffs in "sealed envelopes" that go directly into his safe-deposit box, and he may never find out what they contain.

To overcome this drawback, we would like to inform the uninformed player of his payoffs at each stage. We wish to keep the framework of standard information, that is - to inform the player of the choices (actions) made at each stage. But known actions, together with known payoffs, could help player *2* to deduce the chosen state of nature. To prevent this, while still assuming standard information, we will formulate a game with a single payoff matrix for the uninformed player for all states of nature. Thus, he can determine his own payoffs from the actions. But still, this does not help him to deduce the state of nature. This kind of game will be called a *Game where Players Know Their Own Payoffs*.

It is not to be confused with the model of games with *observable payoffs* in which the payoffs are known, but the actions are not, i.e., without standard information. Lehrer (1986) studied two-player repeated games with observable (known) payoffs and non-observable actions in the *complete* information case, and characterized the equilibrium-payoff sets.

Shalev (1988) has shown that in the model with lack of information on one side and known payoffs[2] (the standard information case) every game has a Nash-equilibrium point. He gave a characterization of all the Nash-equilibrium points as (payoff-) equivalent to a completely revealing equilibrium (i.e. an enforceable joint plan – as first defined in Aumann, Maschler and Sterns (1968), see also Sorin (1983, Remark 1, p.199) – where the "informed player" reveals k immediately).

Shalev based his result on Hart's characterization (1985) which is quite complex and involves bi-martingales. Our *first result* is a direct (and simpler) proof of Shalev's characterization.

A natural extension of the one sided information case is the model with *Lack of Information on Both Sides* . In this model a state of nature is a pair (k, l) in $K \times L$ (where K and L are finite sets). To each state of nature there corresponds a pair of payoff matrices (a game). In the *independent case*, a state of nature is chosen according to probability vectors $p \in \Delta^K$ and $q \in \Delta^L$ (i.e. the probability of (k, l) is $p^k q^l$). Player 1 is told only k while player *2* is told only l.

[2] Shalev actually used the term *Observable Payoffs* in his work.

The *zero-sum* case of this model was completely analyzed by Mertens and Zamir (1971-72). A *non-zero-sum* game with *lack of information on both sides* where *the players know their own payoffs* is a game where the payoff matrices of player 1 depend only on k (i.e., not on l) whereas the matrices of player *2* depend only on l. (The reader may find an example in section 6.6).

Our *second result* is a characterization of all Nash-equilibrium points in this model. As in the one-sided information case (Shalev 1988), all the Nash-equilibrium points are (payoff-) equivalent to a completely revealing equilibrium. As a consequence of this characterization we will be able to show that there are games with *no* Nash-equilibrium points.

It should be pointed out that these are the first results in a model of non-zero-sum games with lack of information on *both sides* .

Up to this point we have dealt with Nash-equilibria; we will now discuss a different concept - correlated equilibria.

Correlated equilibria were introduced by Aumann (1974); a correlated equilibrium for the game Γ consists of a Nash equilibrium of an extension of Γ where, before the beginning of the game, a pair of signals - one for each player - is transmitted. This pair is the output of some correlation device (thus, the signals of the two players may be statistically correlated).

Forges (1988) studied the model of two players with lack of information on one side (in the general case - not necessarily with known payoffs). Forges introduced the following concept: a *communication device* for the game Γ acts at every stage of Γ by receiving an *input* from each player and then it selects a pair of *outputs*, one for each player, as a function of its past memory (i.e. all the past inputs and outputs). A communication device thus enables the players to exchange information and also to coordinate their strategic choices at every stage. *A communication equilibrium* for Γ can then be defined as a Nash equilibrium in an extension of Γ obtained by adding a communication device to the game.

Let NE be the set of Nash-equilibrium payoffs. Let C (respectively D) be the set of correlated (respectively communication) equilibrium payoffs. Our *third result*: In the known payoffs model, the sets of Nash, correlated and communication equilibrium payoffs all coincide (i.e., $NE = C = D$).

It is known (Forges 1985, 1988) that $C = D$ for the games of information transmission. In these games the only role of the informed player consists of sending signals to his opponent. But still, for the information transmission games, $NE \neq C$. Hence, this is the first general case where all the sets coincide, i.e., communication, where payoffs are known, does not increase the set of equilibria.

The paper is organized as follows: In section 2, we will give a formal description of the model. Section 3 states the main results. Sections 4 and 5 are devoted to the case of lack of information on one side: a direct proof of Shalev's result is the subject of section 4; section 5 deals with the correlated and communication equilibria. Section 6 is devoted to the case of lack of information on both sides; it consists of the proof of the characterization of the equilibria in this model. It also contains an example of the complete analysis of a specific game; in some instances, the game has no Nash equilibrium.

2. The Model

Notation.

$I\!R$ is the real line, and $I\!R^n$ the n-dimensional Euclidean space. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two vectors; $x \geq y$ means $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$. The scalar product $\sum_{i=1}^{n} x_i y_i$ is denoted by $x \cdot y$. For a finite set L, $|L|$ is the number of elements of L and $I\!R^L$ is the $|L|$-dimensional Euclidean space with coordinates indexed by the members of L (thus, we write $x = (x_l)_{l \in L}$ for x in $I\!R^L$). The unit simplex in $I\!R^L$ will be denoted[3] by Δ^L,

$\Delta^L := \{x \in I\!R^L : x_l \geq 0 \text{ for all } l \text{ in } L, \sum_{l \in L} x_l = 1\}$. Finally, $I\!N$ is the set of positive integers $\{1, 2, \ldots\}$.

2.1 Standard Information with Lack of Information on One Side and Known Own Payoffs.

The model given here is based on the model in Hart (1985), with appropriate changes made for known payoffs; it has been introduced in Shalev (1988).

The class of games we study is given by the following:

(i) Two players, player *1* and player *2*.

(ii) A finite set I of choices for player *1* and a finite set J of choices for player *2*; I and J each contain at least two elements.

(iii) A finite set K of games; to each k in K (state of nature) there corresponds an $I \times J$ matrix A^k, and there is an $I \times J$ matrix B common to all the games: $A^k = (A^k(i,j))_{i \in I; j \in J}$ and $B = (B(i,j))_{i \in I; j \in J}$.

(iv) A probability vector $p = (p^k)_{k \in K}$ on the set K (i.e., $p \in \Delta^K$); without loss of generality, we assume $p^k > 0$ for all k in K; otherwise, we may discard those k that have zero probability.

Based on (i)-(iv), a *game with lack of information on one side and known payoffs* $\mathbf{\Gamma_\infty(p)}$ is given as follows:

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[3] The symbol := means that the expression on the left is being defined.

(v) An element κ of K is chosen according to the probability vector p; κ is told to player *1*, but not to player *2*.

(vi) At each stage $t = 1, 2, \ldots,$ player *1* chooses an element i_t in I and player *2* chooses an element j_t in J; the choices are made simultaneously (or, without either player knowing what the other did).

(vii) Both players are then told the pair (i_t, j_t), and they get the payoffs $A^{\kappa}(i_t, j_t)$ and $B(i_t, j_t)$, respectively (player *1* can calculate both payoffs, player *2* can calculate only his own payoff).

(viii) Both players have perfect recall (i.e., they do not forget what they were told at all previous stages).

(ix) All of (i)-(viii) is common knowledge to both players (see Aumann (1976) for a precise definition of common knowledge).

Next we describe the sets of strategies of the players in $\Gamma_{\infty}(p)$.

For each $t = 1, 2, \ldots,$ let H_t be the set of histories up to (but not including) stage t, namely, $H_t = (I \times J)^{t-1}$.

A *pure strategy* σ *of player 1* is a collection $\sigma = (\sigma_t)_{t=1}^{\infty}$, where

$$\sigma_t : H_t \times K \to I \tag{2.1}$$

for all $t = 1, 2, \ldots,$. Thus, for every history h_t in H_t and every k in K (the "true" game chosen), $\sigma_t(h_t; k)$ is the choice i_t made by player *1* at stage t. In a similar way a *pure strategy* τ *of player 2* is $\tau = (\tau_t)_{t=1}^{\infty}$, where

$$\tau_t : H_t \to J \tag{2.2}$$

for all $t = 1, 2, \ldots,$.

A *mixed strategy* is, as usual, a probability distribution over the set of pure strategies. Since $\Gamma_{\infty}(p)$ is a game with perfect recall, one can restrict the study to behavior strategies (cf. Kuhn (1953) and Aumann (1964)) where players make independent randomizations at each move.

A *behavior strategy* is thus defined in the same way as a pure strategy, with (2.1) replaced by

$$\sigma_t : H_t \times K \to \Delta^I \tag{2.3}$$

and (2.2) replaced by

$$\tau_t : H_t \to \Delta^J \,. \tag{2.4}$$

Since we never use pure strategies specifically, the term "strategy" will henceforth mean behavior or mixed strategy.

We have not yet defined payoffs in $\Gamma_\infty(p)$, only sequences of payoffs. Given a pair of strategies (σ, τ) of the two players we denote

$$a_T^k = \frac{1}{T} \sum_{t=1}^{T} A^k(i_t, j_t) \tag{2.5}$$

$$\beta_T = \frac{1}{T} \sum_{t=1}^{T} B(i_t, j_t) \tag{2.6}$$

for all $T = 1, 2, \cdots$ and all k in K. Thus a_T^k is the average payoff up to (and including) stage T to player 1, if the true game is $\kappa = k$; this depends on the choices of i_t's and j_t's, made according to σ and τ (actually, only $\sigma(\cdot; k)$ and τ matter); let $E_{\sigma,\tau}^k(a_T^k)$ denote its expectation. For player *2*, β_T is his average payoff up to stage T; it depends on σ, τ and also on the choice of κ (because the i_t's may depend on κ) according to p; let $E_{\sigma,\tau,p}(\beta_T)$ be its expectation.

A pair (σ, τ) of strategies is a *(Nash) equilibrium point* in $\Gamma_\infty(p)$ if

$$\liminf_{T\to\infty} E_{\sigma,\tau}^k(a_T^k) \geq \limsup_{T\to\infty} E_{\sigma',\tau}^k(a_T^k) \tag{2.7}$$

for all strategies σ' of player *1* and all k in K, and

$$\liminf_{T\to\infty} E_{\sigma,\tau,p}(\beta_T) \geq \limsup_{T\to\infty} E_{\sigma,\tau',p}(\beta_T) \tag{2.8}$$

for all strategies τ' of player *2*.

If we take $\sigma' = \sigma$ in (2.7), we get a vector $a = (a^k)_{k \in K}$ such that

$$\lim_{T\to\infty} E_{\sigma,\tau}^k(a_T^k) = a^k \tag{2.9}$$

for all k in K. Similarly, $\tau' = \tau$ in (2.8) gives β with

$$\lim_{T \to \infty} E_{\sigma,\tau,p}(\beta_T) = \beta \ . \tag{2.10}$$

We will call a and β the ***payoffs*** of the equilibrium point (σ, τ).

2.2 Standard Information with Lack of Information on Both Sides and Known Own Payoffs

This model is an extension of the model in section 2.1. Therefore we will only point out the differences between the two models.

Definition of the model:

(i) and (ii) as in section 2.1.

(iii) Finite sets of K and L. $K \times L$ represents the set of possible states of nature (games). To each k in K there corresponds an $I \times J$ matrix A^k, and to each l in L an $I \times J$ matrix B^l.

If the game chosen is $(k, l) \in K \times L$, then $(A^k(i,j))_{i \in I, j \in J}$ and $(B^l(i,j))_{i \in I, j \in J}$ are the payoff matrices to player *1* and player *2* (respectively).

(iv) Probability vectors $p = (p^k)_{k \in K}$ on the set K and $q = (q^l)_{l \in L}$ on the set L (i.e. $p \in \Delta^K$ and $q \in \Delta^L$). Without loss of generality we assume that $p^k > 0$ and $q^l > 0$ for all k in K and $l \in L$, otherwise we may discard those k and l that have zero probability.

(v) Elements κ in K and ℓ in L are chosen *independently*. κ is chosen according to the probability vector p and ℓ is chosen according to the probability vector q. κ is told to player *1*, but not to player *2*. ℓ is told to player *2*, but not to player *1*.

(vi) as in section 2.1.

(vii) as in section 2.1 except that the payoff to player *2* is $B^{\ell}(i_t, j_t)$, and that each player knows his own payoff only.

(viii) to (ix) as in section 2.1.

The game based on (i)-(ix) will be denoted by $\mathbf{\Gamma}_{\infty}(\mathbf{p}, \mathbf{q})$.

There are, of course, some differences in the definitions of strategies, payoffs, and equilibria. However, these differences are purely technical. For example: the average payoff up to (and including) stage T to player *2* is

$$b_T^l = \frac{1}{T} \sum_{t=1}^{T} B^l(i_t, j_t) \tag{2.11}$$

for all $T = 1, 2, \cdots$, when the true game is $\ell = l$, b^l_T depends on σ, τ and the choice of κ according to p (because the i_t's may depend on κ). Let $E^l_{\sigma,\tau,p}(b^l_T)$ be its expectation. Similarly, the expectation of a^k_T when the true game is $\kappa = k$ will be denoted by $E^k_{\sigma,\tau,q}(a^k_T)$.

3. Statement of The Main Results

3.1 Nash equilibrium points in a game $\Gamma_\infty(p)$ of information on one-side (Shalev 1988).

First, let us define the notion of individually rational payoffs in this model (presented in Shalev (1988)). The Folk Theorem in the *complete* information case states that the set of equilibrium payoffs coincides with the set of feasible and individually rational payoffs. Following Hart (1985), we will now characterize individual rationality in $\Gamma_\infty(p)$, using the methods from the investigation of the zero-sum case by Aumann and Maschler (1966).

Notation.

Let p be a probability vector in Δ^K; let $p \cdot A$ be the matrix[4] $\sum_{k\in K} p^k A^k$. Consider the two-person zero-sum game with payoffs to player *1* given by $p \cdot A$ and let $(\text{Val}_1 A)(p)$ denote its value (when played once). Thus,

$$(\text{Val}_1 A)(p) = \max_{x\in\Delta^I} \min_{y\in\Delta^J} (p\cdot A)(x,y) = \min_{y\in\Delta^J} \max_{x\in\Delta^I} (p\cdot A)(x,y) \tag{3.1}$$

for $x = (x_i)_{i\in I}$ and $y = (y_j)_{j\in J}$ where $(p\cdot A)(x,y) = \sum_{i\in I}\sum_{j\in J} x_i y_j \sum_{k\in K} p^k A^k(i,j)$.

Let $\text{Val}_2 B$ be the value to player *2* of the two-person zero-sum game with payoff matrix B to player*2*. i.e.,

$$\text{Val}_2 B = \max_{y\in\Delta^J} \min_{x\in\Delta^I} (B)(x,y) = \min_{x\in\Delta^I} \max_{y\in\Delta^J} (B)(x,y) \tag{3.2}$$

Where x and y are defined as above, and $(B)(x,y) = \sum_{i\in I}\sum_{j\in J} x_i y_j B(i,j)$.

Definitions.

1) A vector $a = (a^k)_{k\in K}$ in $\mathbb{R}^K$ is an *individually rational payoff vector (IR) to player 1* in $\Gamma_\infty(p)$ if

$$\overline{p}\cdot a \geq (\text{val}_1 A)(\overline{p}) \text{ for all } \overline{p} \text{ in } \Delta^K \tag{3.3}$$

Note that (3.3) is a necessary and sufficient condition for the set $Q_a = \{x \in \mathbb{R}^K : x \leq a\}$ to be approachable by player *2*. This was proved by Blackwell (1956).

[4] I.e., whose $(i,j)'$th. element is $\sum_{k\in K} p^k A^k(i,j)$.

2) A scalar β in $I\!R$ is an *individually rational payoff to player 2* in $\Gamma_\infty(p)$ if

$$\beta \geq \mathrm{Val}_2 B \ . \tag{3.4}$$

This follows from the fact that player *2*'s payoffs are independent of k. Therefore, his level of individual rationality (in a two-person game) is exactly the value of his payoff matrix.

Note that neither of the above definitions depends on p.

3) For any $\delta \in \Delta^{I\times J}$, define

$$A(\delta) := \left(A^k(\delta)\right)_{k\in K} = \Big(\sum_{i\in I}\sum_{j\in J}\delta^{ij}A^k(i,j)\Big)_{k\in K} \tag{3.5}$$

and

$$B(\delta) := \sum_{i\in I}\sum_{j\in J}\delta^{ij}B(i,j) \ . \tag{3.6}$$

Thus $A^k(\delta)$ is the payoff to player *1* in the k'th game, where the frequencies used for payoff accumulation are given by δ.

Theorem A (Shalev 88).

Let $\Gamma_\infty(p)$ be a repeated game with lack of information on one side and known payoffs. Let $a = (a^k)_{k\in K} \in I\!R^K$ and $\beta \in I\!R$. Then (a,β) is a Nash-equilibrium payoff vector in $\Gamma_\infty(p)$ if and only if

For every $k \in K$ there exists a probability vector δ^k on $I \times J$ (i.e. $\delta^k \in \Delta^{I\times J}$), satisfying the following 5 conditions:

(i)	$A^k(\delta^k) = a^k$	for all $k \in K$.	
(ii)	$\sum_{k\in K} p^k B(\delta^k) = \beta$		(a and β are the payoffs).
(iii)	$a\cdot q \geq (\mathrm{Val}_1 A)(q)$	for all $q \in \Delta^K$	(a is IR to player *1*).
(iv)	$B(\delta^k) \geq \mathrm{Val}_2 B$	for all $k \in K$	(player *2*'s payoffs are IR).
(v)	$A^k(\delta^k) \geq A^k(\delta^{\widehat{k}})$	for all $k, \widehat{k} \in K$	(cheating does not pay).

Remarks.

In view of the proof, we can interpret the conditions (i)-(v) as follows:

* conditions (i) and (ii) ensure that (a, β) is, in fact, an equilibrium payoff.
* condition (iii) forces a to be an individually rational payoff vector to player *1*.
* condition (iv) states that independently of the game chosen, player *2* gets at least his value in his one matrix payoff.
* condition (v) states that "cheating" is not worthwhile for player *1*. That is, it is not worthwhile for him to act as thought the state of nature is $\widehat{k}$ when, in fact, it is k, $k \neq \widehat{k}$.

3.2 Correlated and communication equilibrium in the model with lack of information on one-side.

Let NE be the set of Nash-equilibrium payoffs. Let C be the set of correlated equilibrium payoffs and and let D be the set of communication equilibrium payoffs (see Forges (1988)).

Theorem B. *Let Γ be a game with lack of information on one-side and known payoffs. Then*

$$NE = C = D \tag{3.7}$$

Hence, communication and correlation, where payoffs are known, does not increase the set of equilibria.

3.3 Nash equilibrium points in the model of lack of information on both sides

Theorem C. *Let $\Gamma_\infty(p,q)$ be a repeated game with lack of information on both sides and known payoffs. Let $a = (a^k)_{k \in K} \in \mathbb{R}^K$ and $b = (b^l)_{l \in L} \in \mathbb{R}^L$.*

Assume $|I| \geq |K|$ and $|J| \geq |L|$.

Then (a, b) is a Nash-equilibrium payoff vector in $\Gamma_\infty(p,q)$ if and only if for each k in K and l in L there is a probability vector δ^{kl} on $I \times J$ (i.e., $\delta^{kl} \in \Delta^{I \times J}$, $\delta^{kl} = \left(\delta^{kl}(i,j)\right)_{i \in I, j \in J}$) satisfying:

(i) $\sum_{l \in L} q^l A^k(\delta^{kl}) = a^k$ for all k in K

(ii) $\sum_{k\in K} p^k B^l(\delta^{kl}) = b^l$ for all l in L

(iii) For each $\widehat{k}$ in K and l in L there is an individually rational payoff vector to player *1* $c_{\widehat{k}l} = (c^k_{\widehat{k}l})_{k\in K}$, satisfying:

$$a^k = \sum_{l\in L} q^l A^k(\delta^{kl}) \geq \sum_{l\in L} q^l \operatorname{Max}\left\{A^k(\delta^{\widehat{k}l}), c^k_{\widehat{k}l}\right\}$$

for all k in K

(iv) For each k in K and $\widehat{l}$ in L there is an individually rational payoff vector to player *2* $d_{k\widehat{l}} = (d^l_{k\widehat{l}})_{l\in L}$ satisfying

$$b^l = \sum_{k\in K} p^k B^l(\delta^{kl}) \geq \sum_{k\in K} p^k \operatorname{Max}\left\{B^l(\delta^{k\widehat{l}}), d^l_{k\widehat{l}}\right\}$$

for all l in L.

Remarks.

1. The assumptions $|I| \geq |K|$ and $|J| \geq |L|$ are used only to show that if conditions (i) to (iv) are satisfied than a Nash-Equilibria- payoff-vector can be obtained. The opposite direction (i.e., a N.E payoff vector implies conditions (i) ~~to~~ (iv)) holds even without this assumption. The equilibrium constructed in the proof is completely revealing in one stage. The revealing of the information in only one stage is only possible when $|I| \geq |K|$ and $|J| \geq |L|$.

   This restriction may be overcome in the following manner:

   * Allow the players to use a "rich enough" alphabet at the first stage.

   or

   * Instead of game $\Gamma_\infty(p, q)$, play an equivalent game (with respect to strategies and payoffs), in which $|I| \geq |K|$ and $|J| \geq |L|$ (e.g., duplicate rows and columns in the game's matrices).

2. Conditions (iii) and (iv) may be replaced by the following condition:

   (v) For each k in K and l in L define

$$G_{kl} := \left\{ (e,f) \in \mathbb{R}^K \times \mathbb{R}^L : \begin{array}{c} \text{there exists } \delta \in \Delta^{I\times J} \text{ such that} \\ e^{\widehat{k}} \geq A^{\widehat{k}}(\delta) \text{ for all } \widehat{k}, \quad e^k = A^k(\delta) \\ f^{\widehat{l}} \geq B^{\widehat{l}}(\delta) \text{ for all } \widehat{l}, \quad f^l = B^l(\delta) \\ e \text{ is IR vector for player 1.} \\ f \text{ is IR vector for player 2.} \end{array} \right\} \tag{3.8}$$

For all k and $\widehat{k}$ in K and for all l and $\widehat{l}$ in L there exists $(e_{\widehat{k}l}, f_{\widehat{k}l})$ in $G_{\widehat{k}l}$ and $(e_{k\widehat{l}}, f_{k\widehat{l}})$ in $G_{k\widehat{l}}$ satisfying

$$\begin{aligned} a^k &\geq \sum_{l\in L} q^l e^k_{\widehat{k}l} \\ b^l &\geq \sum_{k\in K} p^k f^l_{k\widehat{l}} \end{aligned} \tag{3.9}$$

Conditions (iii) and (iv) are equivalent to condition (v). Suppose there are d_{kl}s and c_{kl}s which satisfy conditions (iii) and (iv) we can assume that $d^l_{kl} = B^l(\delta^{kl})$ and $c^l_{kl} = A^k(\delta^{kl})$ for each k in K and l in L.

Define:

$$\begin{aligned} e_{kl} &:= \text{Max}\left\{A(\delta^{kl}), c_{kl}\right\} \\ f_{kl} &:= \text{Max}\left\{B(\delta^{kl}), d_{kl}\right\} \end{aligned} \tag{3.10}$$

(a maximum for two vectors is defined coordinate by coordinate).

The pair (e_{kl}, f_{kl}) thus defined satisfies condition (v) where, the δ for (e_{kl}, f_{kl}) in the definition of G_{kl} is taken as δ^{kl}.

If, on the other hand, there are pairs (e_{kl}, f_{kl}) which satisfy condition (v), then conditions (iii) and (iv) will be satisfied for $d_{kl} = f_{kl}$ and $c_{kl} = e_{kl}$.

Condition (v) seems to have a simpler form than conditions (iii) and (iv), and, in fact, it is derived from the proof of Proposition 6.5. However, the form of conditions (iii) and (iv) reflects better the structure of an equilibrium point constructed in the proof of Theorem C.

3. Here, as in §3.1, a vector $a = (a_k)_{k\in K}$ is IR to player *1* in $\Gamma_\infty(p, q)$ if:

$$\overline{p} \cdot a \geq (\text{Val}_1\, A)(\overline{p}) \quad \text{for all} \quad \overline{p} \quad \text{in } \Delta^K$$

and a vector $b = (b_l)_{l\in L}$ is IR to player *2* if:

$$\overline{q} \cdot b \geq (\text{Val}_2\, B)(\overline{q}) \quad \text{for all} \quad \overline{q} \quad \text{in } \Delta^L$$

Conditions (iii) and (iv) imply that: a and b are *individually rational* (IR) payoff vectors to players 1 and 2, respectively[5]. In particular conditions (iii) and (iv) imply

$$A^k(\delta^{kl}) \geq \text{Val}_1\, A^k \quad \text{and} \quad B^l(\delta^{kl}) \geq \text{Val}_2\, B^l$$

[5] Indeed, take $\widehat{k} = k$ in condition (iii) and $\widehat{l} = l$ in condition (iv).

for all k in K and l in L.

4. If $\mathrm{Val}_1\, A(\cdot)$ is a convex function then if condition (iii) is true for some c_{kl}, it will also be true for $c_{kl} = (\mathrm{Val}_1\, A^k)_{k \in K}$; a similar statement holds for player *2*.
5. Condition (iii) requires $|K \times L|$ vectors c_{kl}. Remark 4 states that if $(\mathrm{Val}_1\, A)(\cdot)$ is a convex function, only one vector is sufficient. Is it possible, in general, that less than $|K \times L|$ vectors will suffice? (We do not know the answer.)
6. There are games $\Gamma_\infty(p, q)$ without a Nash equilibrium point (see §6.6).

4. Nash equilibrium points in a game $\Gamma_\infty(p)$ of information on one-side

The original proof was given by Shalev (1988). It is based on the general characterization for equilibrium payoffs in the general case of incomplete information on one side (not necessarily with known payoffs). The general characterization given by Hart (1985) is quite complicated and includes use of bimartingales.

The following proof is direct.

Proof (Theorem A).

4.1 The existence of δ^ks $\Longrightarrow$ existence of Nash equilibrium point (σ, τ) with payoff vector (a, β).

We will construct a completely revealing equilibrium in the following manner (presented by Shalev (1988)):

For each $k \in K$ we will set a plan to achieve the frequencies of δ^k on $I \times J$. The plan will consist of a sequence of choices for player *1* and a sequence of choices for player *2*. These sequences will be deterministic, i.e., without randomizations.

Suppose that the chosen state of nature is κ. Player *1*'s strategy σ is as follows: signal κ in the first $\lceil \log_2 |K| \rceil$ stages (by converting k to a binary number and playing $i = i_1$ to signal zero digits, and $i = i_2$ to signal ones, where $i_1 \neq i_2$), then play to achieve frequencies δ^κ, according to the plan for δ^κ, as long as player *2* does not deviate from τ. If player *2* deviates from τ, play a mixed strategy holding player *2* to $\text{Val}_2\, B$ at every stage t following the deviation.

Player *2*'s strategy τ is as follows: play arbitrarily in the first $\lceil \log_2 |K| \rceil$ stages, receiving player *1*'s signal of k, then play according to the plan to achieve frequencies δ^k, as long as player *1* does not deviate from σ. If player *1* deviates from σ, play a ***Blackwell strategy*** ensuring that player *1* will get no more than a (simultaneously in all the games). The existence of such a Blackwell strategy is ensured by condition (iii) (Blackwell 1956, see Aumann and Maschler, 1966).

Proposition 4.2. *(σ, τ) is a Nash equilibrium point with a payoff vector (a, β).*

Proof:

1. If nature chooses $\kappa \in K$, the payoff to player *1*, according to (σ, τ), will be $a^\kappa = A^\kappa(\delta^\kappa)$, the payoff to player *2* will be $B(\delta^\kappa)$. Therefore, the payoff vector to player *1* in $\Gamma_\infty(p)$ will be $a = (a^k)_{k\in K} = \left(A^k(\delta^k)\right)_{k\in K}$ and the payoff vector to player *2* in $\Gamma_\infty(p)$ will be $\beta = \sum_{k\in K} p^k B(\delta^k)$. Hence, (a, β) is the payoff vector for (σ, τ).
2. Suppose that player *2* deviates from τ and uses $\widehat{\tau}$. τ is deterministic (as $t \geq \log_2 |K|$), i.e., has no randomizations. Therefore, player *1* can detect the first stage (move) in which player *2* is not playing according to τ. Upon detecting a deviation player *1* holds player *2* at $\mathrm{Val}_2 B$. Condition (iv), therefore, guarantees that player *2* cannot benefit from any deviation.[6]

   For player *1*, only the following two kinds of deviation are possible:
   a. A detectable deviation.
   b. Player *1* "lies" when revealing his information and continues to play according to the false information. For example, suppose that the true game is k. In the first stages player *1* signals $\widehat{k} \in K$, $\widehat{k} \neq k$, then continues to play according to $\delta^{\widehat{k}}$ (this is the only deviation which cannot be detected).

   Condition (v) ensures that player *1* cannot benefit from this undetectable deviation. When the deviation is detectable, player *2* will use the Blackwell strategy which keeps player *1*'s payoffs at a for all the payoff matrices simultaneously (possible according to condition (iii)). Thus player *1* cannot benefit from any deviation.

Q.E.D.

---

[6] Player 1 holds player *2*'s payoff at $\mathrm{Val}_2\, B$ in the following manner: After the deviation from τ, player *1* uses an optimal strategy in game B, viewed as a zero-sum game, at each stage independently.
Consequently, from the time of the deviation, the payoff expectation for player *2* is at each stage at most $\mathrm{Val}_2\, B$. The stages prior to deviation are negligible and therefore the expectation of the average payoff will not be, in the limit, greater than $\mathrm{Val}_2\, B$.

4.3 The existence of the Nash equilibrium point (σ, τ) with payoff vector $(a, b) \Longrightarrow$ the existence of δ^ks.

Certain preliminary work is required before we proceed to this proof. First, we shall define the probability space with which we are dealing. Then, we shall briefly refer to the concept of Banach limits, a concept which this proof (and those following) makes much use of (Hart, 1985).

4.4 The probability space. For each $t \in I\!N$ (the set of positive integers), we have defined $H_t = (I \times J)^{t-1}$, the set of histories before stage t. We also define the set of infinite histories $H_\infty = \prod_{t=1}^{\infty}(I \times J)$, an element of H_∞ being a sequence $\{(i_t, j_t)\}_{t=1}^{\infty}$ of moves made by the two players at all stages.

On H_∞ we define for each $t \in I\!N$ the finite field generated by H_t, and call it $\mathcal{H}_t$; thus, two infinite histories belong to the same *atom* in $\mathcal{H}_t$ if and only if they coincide up to (but not including) t. Let $\mathcal{H}_\infty$ be the σ-field generated by all the $\mathcal{H}_t$'s (usually called the *cylindrical* or the *product σ-field* on the space H_∞).

The basic probability space will also include the choice of κ in K by chance. Thus, let $\Omega = H_\infty \times K$ be endowed with the σ-field $\mathcal{H}_\infty \otimes 2^K$. Each pair of strategies (σ, τ) and each probability vector $p \in \Delta^K$ for the initial chance move determine a probability distribution on this space. We denote it by $P_{\sigma,\tau,p}$; note that $E_{\sigma,\tau,p}$ used in §2.1 is precisely the expectation with respect to $P_{\sigma,\tau,p}$, and $E^k_{\sigma,\tau}$ is the conditional expectation given $\kappa = k$.

Note that $\{\mathcal{H}_t\}_{t \in I\!N}$ is an increasing sequence of finite subfields of $\mathcal{H}_\infty$, converging to $\mathcal{H}_\infty$ as $t \to \infty$.

We will denote the field generated by H_t on Ω also by $\mathcal{H}_t$; this will generate no confusion.

4.5 Banach limit.

Banach limit is an extension of the notion of limit to all bounded sequences.

Let l_∞ be the space of all real sequences. A *Banach limit* $L[\cdot]$ is a linear operator on l_∞ such that

$$\liminf_{n \to \infty} x_n \leq L[x_n] \leq \limsup_{n \to \infty} x_n \quad \text{for every} \quad \{x_n\}_n \text{ in } l_\infty \tag{4.1}$$

(see Dunford and Schwartz (1958), p. 73 or Hart (1985) for more details). The Banach limit operator is not unique, so let L be *one fixed* Banach limit throughout this paper.

Now we can continue with the proof.

4.6 Conditions (i) and (ii).

Let (σ, τ) be a Nash equilibrium point with payoff vector (a, β) in $\Gamma_\infty(p)$.

We will denote $P_{\sigma,\tau,p}$ by P and $E_{\sigma,\tau,p}$ by E. We will also use E^k for the conditional expectation $E(\cdot|\kappa = k)$. We want to find suitable δ^ks

Define random variable $m_T(i,j)$ on $I \times J$; for all $T \in I\!N$ and $(i,j) \in I \times J$.

$$m_T(i,j) := \left\{ \begin{array}{c} \text{the number of times in which} \\ \text{the pair of choices } (i,j) \text{ appears in the first } T \text{ stages of the game} \end{array} \right\}$$

m_T depends on the players' strategies and the chosen k.

Hence, $E^k(m_T(i,j)) = E_{\sigma,\tau,p}(m_T(i,j)|\kappa = k)$ is the expectation of the number of times that (i,j) appears in the first T stages, where the chosen state of nature is $\kappa = k$ and the strategy of players 1 and 2 is σ (in fact σ^k) and τ respectively.

The next stage is to define δ^k by:

$$\delta^k(i,j) := \lim_{T\to\infty} \frac{1}{T}(E^k(m_T(i,j))) \tag{4.2}$$

Unfortunately, the limit does not necessarily exist. Therefore, it is necessary to use the Banach limit $L[\cdot]$ to define the δ^ks.

$$\delta^k(i,j) := L\left[\frac{1}{T}(E^k(m_T(i,j)))\right] \tag{4.3}$$

The δ^k's satisfy the conditions of Theorem A:

1. $\delta^k \in \Delta^{I\times J}$ for all k in K, that is $0 \leq \delta^k(i,j) \leq 1$ for all $(i,j) \in I \times J$ and $\sum_{\substack{i\in I \\ j\in J}} \delta^k(i,j) = 1$.
2. conditions (i) and (ii) of Theorem A, that is

$$\beta = \sum_{k\in K} p^k B(\delta^k)$$

and $$a^k = A^k(\delta^k) \quad \text{for all } k \text{ in } K\ .$$

4.7 Condition (iii).

Definition .

A *Component strategy (C_strategy)* σ *of player 1* is a collection $\sigma = (\sigma_t)_{t=1}^{\infty}$, where

$$\sigma_t : H_t \to \Delta^I$$

for all $t = 1, 2, \ldots,$. Thus, C_strategy is a mixed strategy that disregards the state of nature. A mixed strategy σ can be seen as $|K| - tuple$ of C_strategies. We will denote the k'th component of σ (i.e., $\sigma(\cdot, k)$) by σ^k. A C_strategy σ, on the other hand, correspond to a mixed strategy in which all the components are identical.

Suppose that condition (iii) does not hold. The set $Q_a = \{x \in \mathbb{R}^K, x \leq a\}$, therefore, is not an approachable set by player *2*. Hence, given strategy τ of player *2*, player *1* has a strategy $\widehat{\sigma}$, which ensures him, at least for one k in K, a payoff greater than a^k. The existence of strategy $\widehat{\sigma}$ justified by the Blackwell Theorem (Blackwell 1956). The model discussed by Blackwell is different from our model in that the information is complete, but the payoffs are vectors (vector payoffs). In our model this means that $\widehat{\sigma}$ completely disregards the state of nature, i.e. $\widehat{\sigma}$ is a C_strategy.

Suppose that the C_strategy $\widehat{\sigma}$ enables player *1* (in game k_0) to get (against τ) more than a^{k_0}. Player *1* will swap the k_0 component of σ for $\widehat{\sigma}$ and in this manner improve his payoffs, despite the fact that (σ, τ) is an equilibrium point; a contradiction.

4.8 Condition (iv).

Lemma 4.9. *Let $\Gamma_\infty(p)$ be a game with lack of information on one-side and known payoffs. Let (σ, τ) be a Nash equilibrium point in $\Gamma_\infty(p)$. Then, for every $s \in \mathbb{N}$,*

$$L\big[E_{\sigma,\tau}(\beta_T|\mathcal{H}_s)\big] \geq \mathrm{Val}_2\, B \quad \text{a.s. } (P_{\sigma,\tau,p}) \ .$$

Proof: Suppose there is $s_0 \in \mathbb{N}$ and $h_{s_0} \in H_{s_0}$ such that $P(h_{s_0}) \neq 0$ and

$$L\big[E(\beta_T|H_s)\big](h_{s_0}) < \mathrm{Val}_2\, B \tag{4.5}$$

define strategy $\widehat{\tau}$ for player *2*:

$\widehat{\tau}$: play according to τ until stage s_0.

If the history to this point is exactly h_{s_0}, change to a strategy which ensures $\text{Val}_2\, B$ (i.e., play each stage the optimal strategy in the one-shot game).

Otherwise, continue according to τ.

Denote the probability distribution and the expectation induced by $(\sigma, \widehat{\tau})$, by $\widehat{P_{\sigma,\tau}}$ and $\widehat{E_{\sigma,\tau}}$. The definition of $\widehat{\tau}$ together with (4.5) yields:

$$L[\widehat{E_{\sigma,\tau}}(\beta_T|\mathcal{H}_{s_0})](x) = L[E_{\sigma,\tau}(\beta_T|\mathcal{H}_{s_0})](x) \tag{4.6}$$

for all $x \in H_{s_0}$ such that $x \neq h_{so}$, but for $x = h_{so}$ we get

$$L[\widehat{E_{\sigma,\tau}}(\beta_T|\mathcal{H}_{s_0})](h_{s_0}) \geq \text{Val}_2 B > L[E_{\sigma,\tau}(\beta_T|\mathcal{H}_{s_0})](h_{s_0}) \tag{4.7}$$

The properties of the Banach limit and of the conditional expectation, therefore, give

$$\begin{aligned}\limsup_{T\to\infty} \widehat{E_{\sigma,\tau}}(\beta_T) \geq L[\widehat{E_{\sigma,\tau}}(\beta_T)] &= L[\widehat{E_{\sigma,\tau}}(\widehat{E_{\sigma,\tau}}(\beta_T|\mathcal{H}_{s_0}))] \\ &= \widehat{E_{\sigma,\tau}}\big(L[\widehat{E_{\sigma,\tau}}(\beta_T|\mathcal{H}_{s_0})]\big) = E_{\sigma,\tau}\big(L[\widehat{E_{\sigma,\tau}}(\beta_T|\mathcal{H}_{s_0})]\big)\end{aligned} \tag{4.8}$$

The last equality in (4.8) results from the fact that $P_{\sigma,\tau}$ and $\widehat{P_{\sigma,\tau}}$ coincide until stage s_0, since the difference between τ and $\hat{\tau}$ is from stage s_0 onwards.

The fact that $P(h_{s_0}) > 0$ yields

$$\begin{aligned}\limsup_{T\to\infty} \widehat{E_{\sigma,\tau}}(\beta_T) &\geq E_{\sigma,\tau}(L[\widehat{E_{\sigma,\tau}}(\beta_T|\mathcal{H}_{s_0})]) && \text{by (4.8)} \\ &> E_{\sigma,\tau}(L[E_{\sigma,\tau}(\beta_T|\mathcal{H}_{s_0})]) && \text{(4.6) and (4.7)} \\ &= L[E_{\sigma,\tau}(E_{\sigma,\tau}(\beta_T|\mathcal{H}_{s_0}))] && \text{L is linear} \\ &= L[E_{\sigma,\tau}(\beta_T)] \quad \geq \liminf_{t\to\infty} E_{\sigma,\tau}(\beta_T)\end{aligned} \tag{4.9}$$

We obtained a contradiction to the fact (σ, τ) is a Nash-equilibrium point.

Q.E.D.(Lemma 4.9)

Before going any further we must introduce the concept of martingales.

4.10 The martingale $\{p_s\}$. (based on Hart 1985)

For each $k \in K$, $s \in \mathbb{N}$ and an history $h_s \in H_s$, let $p_s^k = p_s^k(h_s)$ be the conditional probability of the "true" game κ of being k, given σ, τ, p and h_s. We can thus write $p_s^k = P_{\sigma,\tau,p}(\kappa = k|\mathcal{H}_s) = P(k|\mathcal{H}_s)$ (on each atom $h_s \in H_s$ of $\mathcal{H}_s, p_s^k$ is a.s. constant, thus a.s. equal to $p_s^k(h_s)$). We put $p_s = (p_s^k)_{k\in K}$.

Proposition . *The sequence $\{p_s\}_{s\in\mathbb{N}}$ is a Δ^K-valued martingale with respect to $\{\mathcal{H}_s\}_s$ satisfying:*

1. $p_1 = p$

2. There exists a Δ^k-valued random variable p_∞ such that $p_s \to p_\infty$ a.s. $(P_{\sigma,\tau,p})$ as $s \to \infty$.

Proof: (see Hart(1985)) The convergence is due to the Martingale Convergence Theorem and the fact that all the $\{p_s\}$ are uniformly bounded. (For further details on martingales see, for instance Breiman (1968), chap 5)

The following lemma states that *given a game* k and an history then after a "sufficient" number of stages the pair of strategies (σ^k, τ) induces almost the same payoff as the pair (σ, τ). Corollary 4.12 states that the expectation of the payoff to player *2*, given that the game is any k in K, is at least $\mathrm{Val}_2 B$, for all k's.

Lemma 4.11. *Let $\Gamma_\infty(p)$ be a game with lack of information on one side and known payoffs.*

Let (σ, τ) be a Nash-equilibrium point in $\Gamma_\infty(p)$

Then for every $\varepsilon > 0$ there is $s_0 = s_0(\varepsilon) \in \mathbb{N}$ such that:

$$s > s_0(\varepsilon) \Rightarrow \left|E^k(L[E(\beta_T|\mathcal{H}_s)]) - L[E^k(\beta_T)]\right| < \varepsilon \tag{4.10}$$

for every $k \in K$.

Corollary 4.12.

$$L[E^k(\beta_T)] \geq \mathrm{Val}_2 B \tag{4.11}$$

for all $k \in K$.

The corollary will be proved below, following the proof of Lemma 4.11.

The idea of the proof of Lemma 4.11 is that the matrix payoff B for player 2 is not dependent on κ. At the beginning of the game, the choices of player *1* may be dependent on κ. But, after a "sufficient" number of stages (i.e., when the martingale $\{p_s\}$ is already "close" to p_∞), player *1* plays an (almost) non-revealing strategy. Hence, the strategies of both players do not depend on the state of nature.

Consequently, after a "sufficient" number of stages, the expected payoff for player *2*, given a *certain history* and the state of nature κ, is (almost) equal to the expected payoff when κ is not given. Lemma 4.9 implies that the last expectation is at least $\mathrm{Val}_2\, B$. Given $\kappa = k$, we want to show that the expected payoff (taking all the possible histories) is at least $\mathrm{Val}_2\, B$. This is true because the above expectation is merely an average (according to P^k) of the payoffs for each history. The average of values, all of which are not less than $\mathrm{Val}_2\, B$, must be not less than $\mathrm{Val}_2\, B$. Hence, proving the corollary.

Proof. (Lemma 4.11)

1. Fix $s \in \mathbb{N}$ and $k \in K$.

   For all $h_s \in H_s$ such that $P_s^k(h_s) > 0$

   $$P[h_{T+1}|h_s, \kappa = k] = \frac{P[\kappa = k|h_{T+1}, h_s] \cdot P[h_{T+1}|h_s]}{P[\kappa = k|h_s]} \tag{4.12}$$

   holds for all $T \geq s$.

   For $T \geq s$, if h_{T+1} is a possible continuation of h_s (i.e. $P_{\sigma,\tau}[h_{T+1}|h_s] \neq 0$), then conditioning on h_s and h_{T+1} is actually conditioning only on h_{T+1}, so (4.12) becomes

   $$\frac{P[\kappa = k|h_{T+1}] \cdot P[h_{T+1}|h_s]}{P[\kappa = k|h_s]} = \frac{p_{T+1}^k(h_{T+1})}{p_s^k(h_s)} \cdot P[h_{T+1}|h_s] \tag{4.13}$$

   which holds for every $h_s \in H_s$ such that $p_s^k(h_s) \neq 0$.

2. $E^k(B(i_T, j_T)|h_s)$ is by definition equal to $\sum_{h_{T+1} \in H_{T+1}} B(i_T, j_T) \cdot P[h_{T+1}|h_s, \kappa = k]$ which, using (4.13), can be rewritten as

   $$\sum_{h_{T+1} \in H_{T+1}} B(i_T, j_T) \cdot \frac{p_{T+1}^k(h_{T+1})}{p_s^k(h_s)} \cdot P(h_{T+1}|h_s) \tag{4.14}$$

3. Let M be the maximum absolute value of any possible payoff to player *2*

   $$M := \mathrm{Max}\{|B(i,j)| : i \in I,\ j \in J\}$$

The equality $E(B(i_T, j_T)|h_s) = \sum_{h_{T+1}\in H_{T+1}} B(i_T, j_T)\cdot P(h_{T+1}|h_s)$ together with (4.14) above and the fact that p_s^k is $\mathcal{H}_s$-measurable yields:

$$
\begin{aligned}
&|E^k(B(i_T, j_T)|h_s) - E(B(i_T, j_T)|h_s)| \leq \\
&\sum_{h_{T+1}\in H_{T+1}} M\cdot P(h_{T+1}|h_s)\left|\frac{p_{T+1}^k(h_{T+1})}{p_s^k(h_s)} - 1\right| \\
&= M\cdot\frac{1}{p_s^k(h_s)}\cdot E[|p_{T+1}^k - p_s^k||h_s]
\end{aligned}
\tag{4.15}
$$

which holds for every $h_s \in H_s$ such that $p_s^k(h_s) \neq 0$.

4. Recalling the definition of $\beta_T = \frac{1}{T}\sum_{t=1}^{T} B(i_t, j_t)$ we get :

$$
\begin{aligned}
&E^k(\beta_T|h_s) - \left|E(\beta_T|h_s)\right| \\
&\leq \frac{1}{T}\left|\sum_{t=1}^{s-1}\{E^k(B(i_t, j_t)|h_s) - E(B(i_t, j_t)|h_s)\}\right| \\
&+ \frac{1}{T}\left|\sum_{t=s}^{T} M\frac{1}{p_s^k(h_s)}\cdot E(|p_{t+1}^k - p_s^k||h_s)\right|
\end{aligned}
\tag{4.16}
$$

when $t < s$ then (i_t, j_t) conditioned on h_s is the t-th element of h_s so the first summand of (4.16) equals zero. The second summand of (4.16) equals to

$$
\begin{aligned}
&\frac{1}{T}\cdot M\frac{1}{p_s^k(h_s)}\cdot\sum_{t=s}^{T} E(|p_{t+1}^k - p_s^k||h_s) \\
&\leq \frac{1}{T}\cdot M\frac{1}{p_s^k(h_s)}\cdot T\cdot E(\sup_{t\geq s}|p_{t+1}^k - p_s^k||h_s)
\end{aligned}
\tag{4.17}
$$

The right hand side of (4.17) does not depend on T so it yields (see lemma 4.5, Hart 1985),

$$
\begin{aligned}
&\left|L[E(\beta_T|h_s)] - L[E^k(\beta_T|h_s)]\right| \\
&\leq \frac{1}{T}\cdot M\frac{1}{p_s^k(h_s)}\cdot T\cdot E(\sup_{t\geq s}|p_{t+1}^k - p_s^k||h_s)
\end{aligned}
\tag{4.19}
$$

5.

$$
\left|E^k\{L[E(\beta_T|h_s)]\} - L[E^k(\beta_T)]\right|
$$

$$
\begin{aligned}
&= \left|E^k\{L[E(\beta_T|h_s)]\} - E^k\{L[E^k(\beta_T|h_s)]\}\right| \\
&= \left|E^k\{L[E(\beta_T|h_s)] - L[E^k(\beta_T|h_s)]\}\right| \\
&\leq E^k\left\{M \cdot \frac{1}{p_s^k(h_s)} \cdot E\left(\sup_{t\geq s} |p_{t+1}^k - p_s^k||h_s\right)\right\} \\
&= \sum_{P^k(h_s)\neq 0} P^k(h_s) \cdot \left\{M \cdot \frac{1}{p_s^k(h_s)} \cdot E\left(\sup_{t\geq s} |p_{t+1}^k - p_s^k||h_s\right)\right\}
\end{aligned} \tag{4.20}
$$

in (4.20) we used (4.19) and the finite additivity of L.

Recall that,

$$
P^k(h_s) = P_{\sigma^k,\tau}(h_s) = P(h_s|\kappa = k) = \frac{P(k|h_s)\cdot P(h_s)}{P(\kappa = k)} = P(h_s)\cdot\frac{p_s^k(h_s)}{p^k} \tag{4.21}
$$

which yields

$$
\begin{aligned}
&\left|E^k\{L[E(\beta_T|h_s)]\} - L[E^k(\beta_T)]\right| \\
&\leq \frac{M}{p^k} \cdot \sum_{P^k(h_s)>0} P(h_s) \cdot E\left(\sup_{t\geq s} |p_{t+1}^k - p_s^k||h_s\right) \\
&= \frac{M}{p^k} \cdot E\left(\sup_{t\geq s} |p_{t+1}^k - p_s^k|\right)
\end{aligned} \tag{4.22}
$$

The last equality is due to the fact that, whenever $P^k(h_s) = 0$, then $|p_{t+1}^k - p_s^k| = 0$.

The Bounded Convergence Theorem and the convergence of $\{p_t^k\}_t$ (a.s) states that

$$
E\left(\sup_{t\geq s} |p_{t+1}^k - p_s^k|\right)_{s\to\infty} \to 0 \tag{4.23}
$$

thus we get

$$
|E^k\{L[E(\beta_T|\mathcal{H}_s)]\} - L[E^k(\beta_T)]| \underset{s\to\infty}{\to} 0 \tag{4.24}
$$

This is true for all k in K. So for each $\varepsilon > 0$ one can find an $s(\varepsilon, k)$ such that

$$
s > s(\varepsilon,k) \Longrightarrow \left|E^k\{L[E(\beta_T|\mathcal{H}_s)]\} - L[E^k(\beta_T)]\right| < \varepsilon \tag{4.25}
$$

taking $s(\varepsilon) = \max_k \{s(\varepsilon, k)\}$ will give the desired result. Q.E.D. (Lemma 4.11)

Proof of Corollary 4.12.

Lemma 4.9 states that

$$E^k(L[E(\beta_T|\mathcal{H}_S)]) \geq \mathrm{Val}_2 B \tag{4.26}$$

hence, using Lemma 4.11:

$$L[E^k(\beta_T)] \geq \mathrm{Val}_2 B \ . \tag{4.27}$$

and corollary 4.12 is proven which immediately gives condition (iv) since:

$$B(\delta^k) = B(L[E^k(m_T)]) = L[B(E^k(m_T))] = L[E^k(\beta_T)] \geq \mathrm{Val}_2 B \tag{4.28}$$

Remarks.

1) A similar proof implies that

$$L\left[E^k(\beta_T|h_s)\right] \geq \mathrm{Val}_2\, B \tag{4.29}$$

for all $s \in \mathbb{N}$ and $h_s \in H_s$ such that $P_s^k(h_s) \neq 0$

2) Condition (iv) holds because the payoffs are *known* and does *not* hold in the general case.

4.13 Condition (V).

Suppose there are $\widehat{k}$ and k in K, $\widehat{k} \neq k$, such that $A^k(\delta^k) < A^k(\delta^{\widehat{k}})$.
Player *1* prefers the following strategy $\widehat{\sigma}$ to σ.
$\widehat{\sigma}$ coincides with σ except that when $\kappa = k$, the C_strategy $\sigma^{\widehat{k}}$ is used instead of σ^k.
When $\kappa \neq k$, the payoffs according to (σ, τ) are the same as the payoffs for $(\widehat{\sigma}, \tau)$, but when $\kappa = k$, the payoff to player *1*, according to $(\widehat{\sigma}, \tau)$, is greater than the payoff according to (σ, τ), which contradicts the fact that (σ, τ) is a Nash equilibrium.

Q.E.D. (Theorem A)

5. Correlated and Communication Equilibrium

The main result of this section is that in the model with lack of information on one side and known payoffs, the possibility of using correlation or communication does not increase the set of equilibrium payoffs.

Forges (1988) gives a characterization of the correlated equilibrium payoffs in the general model of games with lack of information on one side (i.e., where payoffs are not necessarily known : the payoff matrix B^k of player *2* may depend on the state of nature k).

Our first proof (Koren ,1988) was based on Forges' theorem (Theorem 5.3, Forges 1988). Forges, later, suggested a direct proof using Theorem A. We will present Forges' proof, the idea of the original proof is given below as a remark.

First, let us present the set up as given in Forges (1988).

5.1 Communication Equilibria.

Let $\Gamma = \Gamma_\infty(p)$ be a two-person repeated game with lack of information on one side. A communication device d for Γ consists of sets I_t^1 and I_t^2 of *inputs* for player *1* and player *2* respectively at stage t ($t = 1, 2, \ldots$), sets O_t^1 and O_t^2 of *outputs* for player *1* and player *2* respectively at stage t and transition probabilities P_t to choose the pairs of outputs as a function of all the preceding inputs and outputs.

Adding d to Γ, one can form the extended game Γ_d. Stage t ($t = 1, 2, \ldots$) of Γ_d can be described as follows:

- player *1* and player *2* transmit simultaneously an input (in I_t^1 and I_t^2 respectively) to the device
- recalling all past outputs and inputs, the device selects a pair of outputs in $O_t^1 \times O_t^2$ and tells player *1* only the first component and player *2* the only second component.
- the players make their moves in I and respectively J, as in the original game.

The payoffs in Γ_d are the same as in Γ, i.e., depend only on the moves in $I \times J$. Observe that the only connection between Γ and d is through the players (e.g., the memory of the device does not contain the moves made by the players, unless the players report these moves as inputs) and that a communication device has two simultaneous roles: to enable the players to coordinate their strategies and to exchange information.

A *communication equilibrium* for Γ is a Nash equilibrium in an extension Γ_d induced by a communication device d.

Let D_∞ (denoted by D in the introduction) be the set of all communication equilibrium payoffs in Γ; i.e., $D_\infty = \bigcup_d D_\infty(d)$, where $D_\infty(d)$ is the subset of $\mathbb{R}^K \times \mathbb{R}$ of all Nash equilibrium payoffs (a, β) in Γ_d, and d ranges over all communication devices d.

We now turn to subclasses of communication equilibria.

r-Communication Equilibria.

An *r-device* ($r = 1, 2, \dots$) for Γ is a communication device for Γ where no input is possible after stage r (formally, the sets I_t^1 and I_t^2 of inputs are singletons for $t = r+1, r+2, \dots$). One can proceed exactly as above to define *r-communication equilibria* in Γ. The set of all r-communication equilibrium payoffs in Γ is denoted as D_r ($r = 1, 2, \dots$).

Correlated Equilibria.

Taking $r = 0$ in the previous definition yields a device which acts only by sending a stream of signals to the players who cannot send *any* input. Such a device will be called *autonomous*. The associated set D_0 of equilibrium payoff is referred to as the set of *extensive form correlated equilibrium* payoffs.

Finally, a *correlation device* for Γ is an autonomous device which acts only once, by selecting a pair of signals, one for each player, before the beginning of the game or at the first stage, just before the players make their first move (it makes no difference for player *1* to get a signal before or after learning the state of nature if this signal is independent of it). A correlation device can thus be defined as a particular communication device where all the sets of inputs and outputs except O_1^1 and O_1^2 are singletons.

A (normal form) *correlated equilibrium* for Γ is a Nash equilibrium in an extension Γ_d induced by a correlation device d. This is equivalent to Aumann's original definition (see Aumann, 1974).

Let NE (respectively C) be the set of all Nash (respectively normal form correlated) equilibrium payoffs in Γ.

Theorem B. *Let Γ be a game with lack of information on one-side and observable payoffs.*

Then

$$NE = C = D_0 = D_1 = \ldots = D_r = \ldots = D_\infty \tag{5.1}$$

for every $0 \leq r \leq \infty$.

Proof: (Forges)

It is clear that

$$NE \subseteq C \subseteq D_0 \subseteq D_1 \subseteq \ldots \subseteq D_\infty \tag{5.2}$$

Therefore, it is sufficient to prove that $D_\infty \subseteq NE$. Given a game Γ, a communication device d and an equilibrium point (σ, τ) with payoff vector (a, β), it is possible to define, in the usual way, probability vectors δ^k – one for each k in K. The δ^ks thus obtained satisfy condition (i)-(v) of Theorem A, hence (a, β) is a Nash equilibrium payoff in Γ.

The probability space in this case is somewhat different from the probability space defined in §4.4.

For each $t \in I\!N$ we define (as in §4.4) $H_t = (I \times J)^{t-1}$ the set of histories before stage t. However, in this case, where there is also a communication device, H_t describes only part of the history. This is due to the fact that each of the players transmitted t messages (inputs) to the device and received t responses (outputs) during the t stages. Denote the list of player *1*'s messages (player *2*, respectively) by M_t^1 (M_t^2 respectively). Denote the list of the device responses to player *1* (player *2* respectively) by R_t^1 (R_t^2, respectively). That is:

$$\begin{array}{ll} R_t^2 \in \Pi_{s=1}^t O_s^2 & \quad R_t^1 \in \Pi_{s=1}^t O_s^1 \\ M_t^2 \in \Pi_{s=1}^t I_s^2 & \quad M_t^1 \in \Pi_{s=1}^t I_s^1 \end{array} \tag{5.3}$$

Note that the list up to stage t also includes the input and output of stage t. This is because the players make their choice at stage t, after the "dialogue" with the communication device.

Let us define the set of player *2*'s *comprehensive histories* (CH) (i.e., including the dialogue between player *2* and the device):

$$(CH)_t := \Pi_{s=1}^{t-1}(I \times J) \times \Pi_{s=1}^t (R_s^2 \times M_s^2) \tag{5.4}$$

On $(CH)_\infty$ we define for each $t \in I\!N$ the finite field generated by $(CH)_t$, and call it $(\mathcal{CH})_t$. Let $(\mathcal{CH})_\infty$ be the σ-field generated by all the $(\mathcal{CH})_t$s (i.e., the cylindrical σ-field on the space $(CH)_\infty$).

As in §4.4 the basic probability space will also include the choice of κ in K by chance. Thus, let $\Omega = (CH)_\infty \times K$ be endowed with the σ-field $(\mathcal{CH})_\infty \otimes 2^K$. Each communication device d and each pair of strategies (σ, τ) and each probability vector $p \in \Delta^K$ for the initial chance move determine a probability distribution on this space. We denote it by $P_{\sigma,\tau,p,d}$ and by $E_{\sigma,\tau,p,d}$ the expectation with respect to $P_{\sigma,\tau,p,d}$. Let $E^k_{\sigma,\tau,p,d}$ be the conditional expectation give $\kappa = k$.

Let (σ, τ) be an equilibrium point in Γ_d with payoff vector (a, β). Denote $P_{\sigma,\tau,p,d}$ (respectively $E_{\sigma,\tau,p,d}$) by P (respectively E) and denote $E^k_{\sigma,\tau,p,d}$ by E^k.
Define the δ^ks as in section 4.6.
Conditions (i),(ii),(iii) and (iv) of Theorem A are proven for the above δ^ks in exactly the same way as they are proved in Theorem A.

The proof of condition (iii) must be adapted to the new probability space, but basically it remains the same. The principal change is: Each time the finite field $\mathcal{H}_s$ is used, it should be replaced by $(\mathcal{CH})_s$. In particular, as in §4.8, the sequence of random variables $\{p^k_s\}_s$ defined by

$$p^k_s = P(\kappa = k | (\mathcal{CH})_s)$$

is a martingale with respect to $(\mathcal{CH})_\infty$ and therefore converges a.s. P.

Remark. The original proof (Koren, 1988) used theorem 5.3 of Forges (1988) which is a characterization of the communication equilibria payoffs in the general model.

Given a payoff vector (a, β) in D_∞ we can define the δ^ks in the same way it is done in the proof of Theorem A. The only change is that now the probability space must take into account the communication device d. Then using Forges' characterization the δ^ks are shown to satisfy the conditions of Theorem A, hence (a, β) is in NE.

6. Nash equilibrium points in the model of lack of information on both sides

Proof (Theorem C).

6.1. The existence of δ^{kl}s (and c_{kl}s, d_{kl}s) $\Longrightarrow$ existence of Nash equilibrium point (σ,τ) in $\Gamma_\infty(p,q)$ with payoff vector (a,b).

As for the proof of Theorem A (§4.1), we will construct a pair of strategies (σ,τ) which are completely revealing. But here we require that the revealing will be at a single stage – the first stage. The players reveal their information *simultaneously*, then play according to a pre-set plan.

For each k in K and l in L, we choose a plan to achieve the frequencies of δ^{kl} on $I\times J$. The plan will consist of a sequence of choices for player *1* and a sequence of choices for player *2*. These sequences will be deterministic, i.e., the plan for δ^{kl} has no randomizations. Suppose that the chosen state of nature is (κ,ℓ); player *1*'s strategy σ is then as follows: Signal κ in the first stage[7]. Signaling in a single stage is possible because[8] $|I|\geq|K|$. Then if player *2*'s signal at the first stage was $l\in L$, play to achieve frequencies $\delta^{\kappa l}$ according to the plan, as long as player *2* does not deviate from the plan. If player 2 deviates from τ, play a Blackwell strategy, ensuring that player *2* will get no more than $d_{\kappa l}$. This is possible because $d_{\kappa l}$ is individually rational to player *2*, that is approachable by player *1*.

Player *2*'s strategy is defined in the same manner. In particular, if player *1* deviates from σ, player *2* "punishes" him with the Blackwell strategy which keeps his payments at $c_{k\ell}$ (where k is player *1*'s signal at the first stage).

Proposition 6.2. *(σ,τ) is a Nash Equilibrium Point with payoff vector (a,b).*

Proof:

1. The fact that the payoff vector of (σ,τ) is (a,b) is easily established by conditions (i) and (ii) (as for Theorem A).

---

[7] Fix any mapping f of I *onto* K. Signaling $\kappa=k$ will be carried out by playing any $i\in I$ such that $f(i)=k$.

[8] See remark 1. of Theorem C (§3.3).

2. The roles played by players 1 and 2 are completely symmetrical, therefore it is enough to prove that player *1* will not deviate from σ.

For player *1*, the best deviations from σ are of the following type: When the true game is $k \in K$, signal $\widehat{k} \in K$ ($\widehat{k}$ is not necessarily different from k). Then watch which $l \in L$ player *2* signals in the first stage. From now, player *1* may, on the one hand, stick to his original signal $\widehat{k}$, i.e., play according to $\delta^{\widehat{k}l}$. On the other hand, he may deviate from the plan to achieve $\delta^{\widehat{k}l}$ (in which case his deviation will be detected).

In the first instance, his payoff is $A^k(\delta^{\widehat{k}l})$. In the second instance, his payoff is, at most $c^k_{\widehat{k}l}$ (The vector $c_{\widehat{k}l}$ is used since player *2* punishes according to player *1*'s signal i.e., $\widehat{k}$). Player *1* will, of course, prefer the greater of the two (depending on $k, \widehat{k}, l$).

Both players produce their signals simultaneously, therefore, the expectation of player *1*'s payoff, when the true game is $\kappa = k$, but he has signaled $\widehat{k}$, is at most

$$\sum_{l \in L} q^l \operatorname{Max}\left\{A^k(\delta^{\widehat{k}l}), c^k_{\widehat{k}l}\right\}$$

Condition (iii) guarantees that this expression is less than a^k, so that σ is optimal against τ.

Q.E.D. (Proposition 6.2)

6.3. (σ,τ) **is a Nash equilibrium point in** $\Gamma_\infty(p,q) \Longrightarrow$ **the existence of** δ^{kl}**s,** c_{kl}**s and** d_{kl}**s for each** k **in** K **and** l **in** L.

Before we start the proof, let us define the probability space with which we are dealing. The probability space here is the necessary extension for the space defined in §4.4. Here, $\Omega = H_\infty \times K \times L$ is endowed with the σ-field $\mathcal{H}_\infty \otimes 2^K \otimes 2^L$. Each pair of strategies (σ,τ) and probability vectors $p \in \Delta^K$ and $q \in \Delta^L$, for the initial chance move, determine a probability distribution on this space. We denote it by $P_{\sigma,\tau}$. Denote by $E_{\sigma,\tau}$ the expectation with respect to $P_{\sigma,\tau}$. We denote by $E^{k,\cdot}_{\sigma,\tau}$ (respectively $E^{\cdot,l}_{\sigma,\tau}$) the conditional expectation given $\kappa = k$ (respectively $\ell = l$), and by $E^{k,l}_{\sigma,\tau}$ the conditional expectation given $\kappa = k$ and $\ell = l$.

For the given (σ,τ) we denote $P^{k,l}_{\sigma,\tau}$ and $E^{k,l}_{\sigma,\tau}$ by $P^{k,l}$ and $E^{k,l}$. Let us define the δ^{kl}s and show that conditions (i)-(iv) are satisfied.

Conditions (i) and (ii).

We will define δ^{kl} in the same manner as we defined δ^k, in the case of lack of information on one side (see §4.6).

Define random variable $m_T(i,j)$ on $I \times J$, such that for all $T \in I\!N$ and (i,j) in $I \times J$.

$$m_T(i,j) := \left\{ \begin{array}{c} \text{The number of times in which the} \\ \text{pair of choices } (i,j) \text{ appears} \\ \text{in the first } T \text{ stages of the game} \end{array} \right\} \tag{6.1}$$

define δ^{kl} by:

$$\delta^{kl}(i,j) := L\left[\frac{1}{T}E^{kl}(m_T(i,j))\right] . \tag{6.2}$$

The δ^{kl} thus obtained are probability vectors on $I \times J$ and satisfy conditions (i) and (ii).

Conditions (iii) and (iv).

Conditions (iii) and (iv) are symmetrical with respect to the roles of players 1 and 2. Therefore, we will prove here condition (iv) only.

Define, as in §4.8, the martingale $\{p_s\}_s$.

$$p_s^k := P_{\sigma,\tau}(\kappa = k|\mathcal{H}_s) = P(k|\mathcal{H}_s) \tag{6.3}$$

First, before the full proof, we will draw a sketch of it, applied to a simple case.

Suppose that:

1) $K = L = \{1,2\}$
2) $\{p_s\}_s$ converges after a finite number of stages, say s_0 (for all histories). This means that after s_0 stages, player *1* completely disregards his information and plays a strategy which depends only on the history to date.

In §6.1, player *1* "punishes" player *2* with d_{kl}. Here also, we will try to find d_{kl}, which will be the optimal achievement of player *2*, following deviation from τ^l, when player *1* plays according to σ^k.

Suppose that $\ell = 1$ and instead of starting with τ^1, player *2* plays according to τ^2 for the first s_0 stages. After s_0 stages, a history has been obtained. From this stage on, player *1* plays a completely non-revealing strategy, so that this is a good time for player *2* to assess his future moves.

On one hand, player *2* can continue according to τ^2, thus his deviation remains undetectable. We denote his expected payoff vector[9] according to τ^2 and σ, by: $\left(e^1(h_{s_0}), e^2(h_{s_0})\right)$ where $e^1(h_{s_0})$ is the payoff according to payoff matrix $l = 1$ and $e^2(h_{s_0})$ is the payoff according to payoff matrix $l = 2$.

On the other hand, it is possible that player *2* may improve his payoff (in matrix $l = 1$) to more than $e^1(h_{s_0})$ by deviating from τ^2. Denote the maximal possible payoff against σ, given h_{s_0}, as $d^1(h_{s_0})$ (note, always $d^1(h_{s_0}) \geq e^1(h_{s_0})$). It is possible that $d^1(h_{s_0})$ cannot be achieved by any strategy (i.e., $d^1(h_{s_0})$ is "sup", not "max"). Assume here, for simplicity, that the "sup" may be achieved by some deviation.

Therefore, given history h_{s_0}, player *2*'s expected payoff is at least $d^1(h_{s_0})$.

Note:.

1. $\left(d^1(h_{s_0}), e^2(h_{s_0})\right)$ is an IR vector for player *2* since, for $\ell = 2$, τ^2 is an optimal strategy against σ, therefore $e^2(h_{s_0})$ cannot be improved and for $\ell = 1$, $d^1(h_{s_0})$ cannot be improved because it is maximal against σ. Therefore, also:

$$(d_{12}^1, d_{12}^2) := E^{1,2}(d^1(h_{s_0}), e^2(h_{s_0})) \tag{6.4}$$

$$(d_{22}^1, d_{22}^2) := E^{2,2}(d^1(h_{s_0}), e^2(h_{s_0})) \tag{6.5}$$

are IR vectors for player *2*, since the set of IR vectors to player *2* is convex.

2. $e^1(h_{s_0})$ is the expectation of the payoff for player *2* in game $l = 1$ according to τ^2 and σ. However, after s_0 stages, there is no difference between σ and σ^1 (and between σ and σ^2). Thus, $e^1(h_{s_0})$ is also the expectation according to σ^1 or σ^2, (i.e., $e^1(h_{s_0})$ is the expectation even if $\kappa = 1$ or $\kappa = 2$ is given). Hence

$$E^{1,2}(e^1(h_{s_0})) = E^{1,2}(L[E^{\cdot,2}(b_T^1|h_{s_0})]) = E^{1,2}(L[(E^{1,2}(b_T^1|h_{s_0})]) = B^1(\delta^{12}) \tag{6.6}$$

$$E^{2,2}(e^1(h_{s_0})) = E^{2,2}(L[E^{\cdot,2}(b_T^1|h_{s_0})]) = E^{2,2}(L[(E^{2,2}(b_T^1]|h_{s_0})]) = B^1(\delta^{22}) \tag{6.7}$$

Now, we have reached the final stage of the sketched proof. Using the defined deviation from τ (i.e., to begin according to τ^2 and to reassess the situation after s_0 stages), we will

[9] The expected payoff means the Banach limit of the expected payoffs of the finite histories, i.e., $e^1(h_{s_0}) = L[E^{\cdot,2}(b_T^1|h_{s_0})]$ and $e^2(h_{s_0}) = L[E^{\cdot,2}(b_T^2|h_{s_0})]$.

attempt to estimate the payoff to player *2* when $\ell = 1$ (but player *2* starts according to τ^2).

The payoff to player *2* is:

$$E^{\cdot,2}(d^1(h_{s_0})) = \sum_{k=1,2} p^k E^{k,2}(d^1(h_{s_0})) = \sum_{k=1,2} p^k d^1_{k2} \tag{6.8}$$

We know that $d^1(h_s) \geq e^1(h_s)$ for all histories h_s; this together with (6.6) and (6.7) yields $d^1_{k2} = E^{k,2}(d^1(h_s)) \geq E^{k,2}(e^1(h_s)) = B^1(\delta^{k2})$ for all k. (σ, τ) is an equilibrium point, that is τ^1 is optimal against σ, which yields

$$b^1 \geq \sum_{k=1,2} p^k d^1_{k2} = \sum_{k=1,2} p^k \operatorname{Max}\{B^1(\delta^{k2}), d^1_{k2}\} \tag{6.9}$$

which is one of four conditions of (iv) $[l = 1, \bar{l} = 2]$. The fact that, $d^2_{k2} = B^2(\delta^{k2})$ for all k in K, and $b^2 = \sum_{k=1,2} p^k B^2(\delta^{k2})$ yields condition (iv) for $l = \bar{l} = 2$ (actually, as an equality). The additional 2 conditions (with d_{k1}) are obtained using the symmetrical deviation from τ (i.e., play τ^1 when $\ell = 2$).

Now we turn to the actual proof of the general case.

Lemma 6.4. *Let $\Gamma_\infty(p,q)$ be a game and (σ, τ) Nash equilibrium point in $\Gamma_\infty(p,q)$. Then, for any $\bar{\varepsilon} > 0$ there is $s_0 = s_0(\bar{\varepsilon}) \in I\!N$ such that*

$$s > s_0 \Longrightarrow \left| E^{k,\bar{l}}(L[E^{\cdot,\bar{l}}(b^l_T|\mathcal{H}_s)]) - L[E^{k,\bar{l}}(b^l_T)] \right| < \bar{\varepsilon} \tag{6.10}$$

for all $l, \bar{l} \in L$ and $k \in K$.

Proof: Same as for Lemma 4.11, where each τ is replaced by $\tau^{\bar{l}}$, β_T by b^l_T and M is defined by $M := \operatorname{Max}_{\substack{l \in L \\ i \in I \\ j \in J}} |B^l(i,j)|$

Q.E.D. (Lemma 6.4)

Proposition 6.5. *For each k in K and $\bar{l}$ in L there is an IR vector $d_{k\bar{l}} = (d^l_{k\bar{l}})_{l \in L}$ for player 2 satisfying*

$$b^l = \sum_{k \in K} p^k B^l(\delta^{kl}) \geq \sum_{k \in K} p^k \operatorname{Max}\{B^l(\delta^{k\bar{l}}), d^l_{k\bar{l}}\} \tag{6.11}$$

for all l in L.

Proof:

(1) We will begin by constructing a sequence of IR vectors for player *2*.

Let $s \in \mathbb{N}$ and $l \in L$; for each $h_s \in H_s$ (history of $s-1$ stages) such that $P(h_s) \neq 0$, define:

$$d_s^l(h_s) := \sup_{\overline{\tau}} L\big[E_{\sigma,\overline{\tau}}(b_T^l | h_s)\big] \tag{6.12}$$

where the sup is over all strategy $\overline{\tau}$ of player *2*. That is, $d_s^l(h_s)$ is the best payoff that player *2* can achieve against σ, where his payoff matrix is B^l, given that the history to date is h_s. Note that the vector

$$d_s(h_s) := \big(d_s^l(h_s)\big)_{l\in L} \tag{6.13}$$

is IR to player *2*, for every h_s such that $P(h_s) \neq 0$, because otherwise player *2* could improve his payoffs in at least one game, which is contrary to the definition of d_s^l in (6.12). Define

$$d_{kl}(s) = \big(d_{kl}^{\bar{l}}(s)\big)_{\bar{l}\in L} := E^{k,l}\big(d_s(h_s)\big) \tag{6.14}$$

$d_{kl}(s)$ is IR to player *2*, because the set of IR vectors to player *2* is convex.

(**Note:** $d_s(h_s)$ is defined only for h_s, such that $P(h_s) \neq 0$, but if $P(h_s) = 0$, then also $P^{k,l}(h_s) = 0$. Hence, (6.14) is well defined.)

(2) The definition of $\overline{\tau}$.

For each $l \in L$ and for each $h_s \in H_s$ such that $P(h_s) \neq 0$ we have defined $d_s^l(h_s)$. In the definition (6.12) "sup" is used and it may be that the "sup" cannot be achieved by any strategy $\overline{\tau}$. Therefore, for $\varepsilon > 0$ and $h_s \in H_s$ let $\overline{\tau} \equiv \overline{\tau}(l, h_s, \varepsilon)$ be a strategy for player *2*, which satisfies

$$\big|d_s^l(h_s) - L\big[E_{\sigma,\overline{\tau}}(b_T^l | h_s)\big]\big| < \varepsilon \tag{6.15}$$

(hence $\overline{\tau}$ is "ε-optimal").

(3) The definition of $\hat{\tau}$.

For every l and $\bar{l}$ in L, and for every $\varepsilon > 0$ and $s \in \mathbb{N}$ we define a deviation

$\widehat{\tau} = \widehat{\tau}(s, \varepsilon, l, \bar{l})$ of player *2* from τ as follows:

$\widehat{\tau}$: In all states of nature except l, $\widehat{\tau}$ is identical to τ (i.e for all states of nature except l, τ and $\overline{\tau}$ has the same C_strategies). If the state of nature is $\ell = l$, $\widehat{\tau}$ coincides with $\tau^{\bar{l}}$ until stage s. At stage s an history h_s has been obtained, from this stage on $\widehat{\tau}$ continues according to $\overline{\tau}(l, h_s, \varepsilon)$. (Hence, if the state of nature is l, playing according to $\widehat{\tau}$ means : play according to $\tau^{\bar{l}}$ up to stage s. At stage s, we have obtained a history, h_s. $\widehat{\tau}$ selects a strategy $\overline{\tau}$ which is "ε-optimal" against σ. The next moves will be played according to $\overline{\tau}$).

Denote $E_{\sigma,\widehat{\tau}(s,\varepsilon,l,\bar{l})^{l_0}}$ by $\widehat{E}^{\cdot,l_0}$. The pair of strategies (σ, τ) is an equilibrium point, therefore:

$$b^l = L\big[E_{\sigma,\tau^l}(b^l_T)\big] \geq L\big[E_{\sigma,\widehat{\tau}(s,\varepsilon,l,\bar{l})^l}(b^l_T)\big] = L\big[\widehat{E}^{\cdot,l}(b^l_T)\big] \tag{6.16}$$

for all $s \in \mathbb{N}$ and $\varepsilon > 0$ and for all $l, \bar{l} \in L$.

We will estimate the right side of (6.16) and this will, eventually, give us (6.9).

(4) Estimation of $\widehat{E}^{\cdot,l}(b^l_T)$.

For every l and $\bar{l}$ in L, fix $\varepsilon > 0$. For every $s \in \mathbb{N}$ we will estimate $L\big[\widehat{E}^{\cdot,l}(b^l_T|\mathcal{H}_s)\big]$. The properties of the conditional expectation and the finite additivity of the Banach limit yield

$$\begin{aligned} L\big[\widehat{E}^{\cdot,l}(b^l_T)\big] =& L\big[\widehat{E}^{\cdot,l}(\widehat{E}^{\cdot,l}(b^l_T|\mathcal{H}_s))\big] = \widehat{E}^{\cdot,l}\big(L\big[\widehat{E}^{\cdot,l}(b^l_T|\mathcal{H}_s)\big]\big) = \\ &= E^{\cdot,\bar{l}}\big(L\big[E_{\sigma,\overline{\tau}}(b^l_T|\mathcal{H}_s)\big]\big) \end{aligned} \tag{6.17}$$

The last equality in (6.17) is due to the fact that up to stage s $\widehat{\tau}^l$ and $\tau^{\bar{l}}$ coincide. $\overline{\tau}$ is the strategy $\overline{\tau}(l, h_s, \varepsilon)$ of $\widehat{\tau}$'s definition. ($\overline{\tau}$ may depend on the history h_s.) The definition of $E^{\cdot,\bar{l}}$ implies that (6.17) is equal to:

$$\sum_{k\in K} p^k E^{k,\bar{l}}\big(L\big[E_{\sigma,\overline{\tau}}(b^l_T|\mathcal{H}_s)\big]\big) \tag{6.19}$$

Now, for each h_s such that $P^{\cdot,l}(h_s) > 0$ the definition of $\overline{\tau} = \overline{\tau}(l, h_s, \varepsilon)$ gives

$$\big|L\big[E_{\sigma,\overline{\tau}}(b^l_T|h_s)\big] - d^l_s(h_s)\big| < \varepsilon \tag{6.20}$$

Using the equality $d^l_{k,\bar{l}}(s) = E^{k,\bar{l}}(d^l_s(h_s))$ we get that

$$E^{k,\bar{l}}(L[E_{\sigma,\widehat{\tau}}(b^l_T|h_s)]) \geq E^{k,\bar{l}}(d^l_s(h_s) - \varepsilon) = d^l_{k,\bar{l}}(s) - \varepsilon \; . \tag{6.21}$$

In conclusion,

$$b^l \geq L[\widehat{E}^{\cdot,l}(b^l_T)] = \sum_{k\in K} p^k E^{k,\bar{l}}(L[E_{\sigma,\widehat{\tau}}(b^l_T|\mathcal{H}_s)]) \geq \sum_{k\in K} p^k(d^l_{k\bar{l}}(s) - \varepsilon) \tag{6.22}$$

For every k in K, every $l, \bar{l}$ in L and every $\varepsilon > 0$.

(5) The definition of d_{kl}.

For every $n \in I\!N$, let s be the $s(1/n)$ obtained from Lemma 6.4 (for $\bar{\varepsilon} = 1/n$). I.e.,

$$t > s(1/n) \Longrightarrow \left| E^{k,\bar{l}}(L[E^{\cdot,\bar{l}}(b^l_T|\mathcal{H}_t)]) - L[E^{k,\bar{l}}(b^l_T)] \right| < \frac{1}{n}$$

for all l and $\bar{l}$ in L and k in K).

Consider the sequence $\{d_{k\bar{l}}(s(1/n))\}_{n=1}^{\infty}$ .It is bounded (for every $k, \bar{l}$), therefore there is a sub-sequence $(n_m)_{m=1}^{\infty}$ such that $\{d_{k\bar{l}}(s(1/n_m))\}_{m=1}^{\infty}$ converges for every $k, \bar{l}$ (for all k and $\bar{l}$ *simultaneously*). W.l.g. assume that the original sequence is converging. (otherwise consider $(n_m)_{n=1}^{\infty}$ instead of $\{n\}_{n=1}^{\infty}$ in the first place.) Let $d_{k\bar{l}}$ denote the limit of this sequence

$$d_{k\bar{l}} := \lim_{n\to\infty} d_{k\bar{l}}(s(1/n)) \tag{6.23}$$

Conclusions

1. $d_{k\bar{l}}$ is an IR vector for player *2* for all k and $\bar{l}$. This is because the set of the IR vectors is a closed set.
2. For each $\varepsilon_1 > 0$ there is $N = N(\varepsilon_1)$ such that

$$n > N \Longrightarrow |d^l_{k\bar{l}} s(1/n) - d^l_{k\bar{l}}| < \varepsilon_1 \tag{6.24}$$

for every k in K and for every l and $\bar{l}$ in L.

(6) Fix $\varepsilon > 0$ and $\varepsilon_1 > 0$. Let $N = N(\varepsilon_1)$ of (6.24) then (6.22) for $\widehat{\tau} = \widehat{\tau}(s(1/n), \varepsilon, l, \bar{l})$ yields:

$$n > N \Longrightarrow b^l \geq \sum_{k\in K} p^k(d^l_{k\bar{l}}(s) - \varepsilon) \geq \sum_{k\in K} p^k(d^l_{k\bar{l}} - \varepsilon - \varepsilon_1) \tag{6.25}$$

This is true for all $\varepsilon > 0$ and $\varepsilon_1 > 0$, hence

$$b^l \geq \sum_{k \in K} p^k d^l_{k\bar{l}} \tag{6.26}$$

(7) Fix $\varepsilon_1 > 0$. Let N be $N(\varepsilon_1)$ of (6.24) (denote $s(1/n)$ by s), then (6.24) and (6.14) yield

$$n > N \Longrightarrow d^l_{k\bar{l}} \geq d^l_{k\bar{l}}(s) - \varepsilon_1 = E^{k\bar{l}}(d^l_s(h_s)) - \varepsilon_1 \tag{6.27}$$

Since $d^l_s(h_s)$ is the optimal payoff of player *2*, where $\ell = l$, against σ given the history h_s (i.e., particularly better than $\tau^{\bar{l}}$) we get $d^l_s(h_s) \geq L[E^{\cdot,\bar{l}}(b^l_T|h_s)]$ hence

$$n > N \Longrightarrow d^l_{k\bar{l}} \geq E^{k,\bar{l}}(L[E^{\cdot,\bar{l}}(b^l_T|h_s)]) - \varepsilon_1 \tag{6.28}$$

the way in which $s = s(1/n)$ is chosen in Lemma 6.4 yields

$$n > N \Longrightarrow d^l_{k\bar{l}} \geq L[E^{k,\bar{l}}(b^l_T)] - \frac{1}{n} - \varepsilon_1 = B^l(\delta^{k\bar{l}}) - \frac{1}{n} - \varepsilon_1 \tag{6.29}$$

This is true for all $n > N(\varepsilon_1)$ hence $d^l_{k\bar{l}} \geq B^l(\delta^{k\bar{l}}) - \varepsilon_1$ which is true for all $\varepsilon_1 > 0$, hence $d^l_{k\bar{l}} = \text{Max}\{B^l(\delta^{k\bar{l}}), d^l_{k\bar{l}}\}$ for all k in K and $l, \bar{l}$ in L. Hence, (6.26) yields the desired conditions

$$b^l \geq \sum_{k \in K} p^k \,\text{Max}\{B^l(\delta^{k\bar{l}}), d^l_{k\bar{l}}\} \tag{6.30}$$

for all k in K and $l, \bar{l}$ in L.

Note. $d^l_s(h_s) = L[E^{\cdot,l}(b^l_T|h_s)]$ hence $d^l_{kl} = B^l(\delta^{kl})$ (we use it in Remark 3 of Theorem C).

Q.E.D. (Proposition 6.5)

Q.E.D. (Theorem C)

6.6 Example. We analyze the game $\Gamma_\infty(p,q)$, where

$$K = L = \{1,2\}\ , \quad p = (p^1, p^2) >> 0\ , \quad q = (q^1, q^2) >> 0$$

$$A^1 = \begin{array}{|cc|} \hline 3 & 0 \\ 0 & 1 \\ \hline \end{array}$$

$$A^2 = \begin{array}{|cc|} \hline 3 & 3 \\ 1 & 1 \\ \hline \end{array}$$

$$B^1 = \begin{array}{|cc|} \hline 1 & 0 \\ 0 & 3 \\ \hline \end{array} \qquad B^2 = \begin{array}{|cc|} \hline 1 & 3 \\ 1 & 3 \\ \hline \end{array}$$

	q^1		q^2	
p^1	3,1	0,0	3,1	0,3
	0,0	1,3	0,1	1,3
p^2	3,1	3,0	3,1	3,3
	1,0	1,3	1,1	1,3

Figure 1. $\Gamma_\infty(p,q)$

We will determine for which (p,q) there is an equilibrium in the game $\Gamma_\infty(p,q)$.

The motivation for choosing this example is due to Shalev (1988). He noticed an interesting phenomenon. Consider the following game $\Gamma_\infty(p)$ with lack of information on *one* side and known payoffs:

$$A^1 = \begin{array}{|cc|} \hline 3 & 0 \\ 0 & 1 \\ \hline \end{array} \qquad B^1 = \begin{array}{|cc|} \hline 1 & 0 \\ 0 & 3 \\ \hline \end{array}$$

$$A^2 = \begin{array}{|cc|} \hline 3 & 3 \\ 1 & 1 \\ \hline \end{array}$$

p^1	3,1	0,0
	0,0	1,3
p^2	3,1	3,0
	1,0	1,3

Figure 7. $\Gamma_\infty(p)$

The set of equilibria payoffs for this game is determined using Theorem A. One can see that the minimum payoff to player *1*, when $\kappa = 1$, is $9/4$. Since a payoff of less than $9/4$ is an incentive to player *1* to cheat when $\kappa = 1$ (see Shalev for details).

A game known as "The Battle of the Sexes" is defined by the following pair of matrices:

$$A^1 = \boxed{\begin{matrix} 3 & 0 \\ 0 & 1 \end{matrix}} \qquad\qquad B^1 = \boxed{\begin{matrix} 1 & 0 \\ 0 & 3 \end{matrix}}$$

Figure 8. "The Battle of The Sexes"

$\Gamma_\infty(p)$ may be seen as a game "derived" (as defined by Shalev) from the game Battle of the Sexes, by adding the payoff matrix A^2 and the vector p.

The "Folk Theorem" implies that the minimum payoff to player 1 in the "Battle of the Sexes" is $3/4$. Hence by adding a doubt of player *2* as to the state of nature, the set of equilibrium payoffs may be changed to favor player 1 (even when p^1 is close to 1).

A conjecture was raised (by R.J. Aumann) that if this process of adding doubt is duplicated for player *2*, giving the game $\Gamma_\infty(p, q)$ of our example, there will be no equilibrium point. This is due to the fact that in the state of nature $\kappa = 1$ and $\ell = 1$, both players will "have" to receive at least $9/4$ as their payoff. However, $9/4 + 9/4 \geq 4$, therefore there will be no equilibrium point. Indeed this is true for p^1 and q^1 close to 1, but when one of them is "small" enough, an equilibrium point does in fact exist and at least one player is satisfied with less than $9/4$.

We start the analyzing of the game. Let (σ, τ) be an equilibrium in $\Gamma_\infty(p, q)$ with payoff vector (a, b). Then, there are four probability vectors $(\delta^{kl})_{\substack{k=1,2\\ l=1,2}}$ in $\Delta^{I \times J}$, satisfying

$$a^k = q^1 A^k(\delta^{k1}) + q^2 A^k(\delta^{k2}) \qquad k = 1, 2 \tag{6.31}$$

and

$$b^l = p^1 B^l(\delta^{1l}) + p^2 B^l(\delta^{2l}) \qquad l = 1, 2 \tag{6.32}$$

Remark 3 in Theorem C ensures that: $a^{kl} := A^k(\delta^{kl}) \geq \mathrm{Val}_1\, A^k$ and $b^{kl} := B^l(\delta^{kl}) \geq \mathrm{Val}_2\, B^l$ for $k = 1, 2$ and $l = 1, 2$. Note that $Val_1 A^1 = Val_2 B^1 = 3/4$ and $Val_1 A^2 = Val_2 B^2 = 3$, hence the δ^{kl}s must be of the following form[10] :

δ^{11}	δ^{12}
δ^{21}	δ^{22}

=

δ^{11}	$\begin{matrix} 0 & 1-e \\ 0 & e \end{matrix}$
$\begin{matrix} t & 1-t \\ 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$

Figure 2.

for some e and t, where $3/4 \le e, t \le 1$.

Since $(\mathrm{Val}_1\, A)(\cdot)$ and $(\mathrm{Val}_2\, B)(\cdot)$ are convex functions in this case, conditions (iii) and (iv), in this game are (see Remark 4 of Theorem C)

$$a^1 = q^1 a^{11} + q^2 a^{12} \ge q^1 \cdot \mathrm{Max}\left\{A^1(\delta^{21}), 3/4\right\} + q^2\, \mathrm{Max}\left\{A^1(\delta^{22}), 3/4\right\} \tag{6.35}$$

$$b^1 = p^1 b^{11} + p^2 b^{21} \ge p^1 \cdot \mathrm{Max}\left\{B^1(\delta^{12}), 3/4\right\} + p^2\, \mathrm{Max}\left\{B^1(\delta^{22}), 3/4\right\} \tag{6.36}$$

The remaining conditions (for a^2 and b^2) are clearly satisfied because $a^2 = b^2 = 3 = \mathrm{Max}_{\substack{i\in I\\ j\in J\\ k\in K\\ l\in L}} \left\{A^k(i,j), B^l(i,j)\right\}$.

From figure 2 we get :

$$A^1(\delta^{21}) = 3t \quad A^1(\delta^{22}) = 0 \quad B^1(\delta^{12}) = 3e \quad B^1(\delta^{22}) = 0 \quad a^{12} = e \quad b^{21} = t \tag{6.37}$$

thus (6.35) and (6.36) get the following form:

$$(6.35) \iff q^1 a^{11} + q^2 e \ge 3tq^1 + \frac{3}{4} q^2 \tag{6.38}$$

$$(6.36) \iff p^1 b^{11} + p^2 t \ge 3ep^1 + \frac{3}{4} p^2 \tag{6.39}$$

Dividing (6.38) by q^1 and (6.39) by p^1 yields [11] :

$$(6.35) \iff a^{11} + \frac{e}{q^1} - e \ge 3t + \frac{3}{4q^1} - \frac{3}{4}$$

$$\iff a^{11} + \frac{3}{4} \ge 3t + e + \frac{1}{q^1}\left(\frac{3}{4} - e\right) \tag{6.40}$$

$$(6.36) \iff b^{11} + \frac{3}{4} \ge 3e + t + \frac{1}{p^1}\left(\frac{3}{4} - t\right) \tag{6.41}$$

[10] The numbers in the matrices' entries represent probabilities, for example, if $\kappa = \ell = 2$ only the top right corner will be played.

[11] Recall that $p^2 = 1 - p^1$ and $q^2 = 1 - q^1$.

We will find a necessary condition for (p,q) in order that (6.40) and (6.41) will be satisfied. Later, we will show that the condition thus obtained is sufficient for the existence of an equilibrium in $\Gamma_\infty(p,q)$.

a^{11} and b^{11} always satisfy $a^{11}+b^{11} \le 4 = \text{Max}_{\substack{i\in I\\ j\in J}} \{A^1(j,i)+B^1(i,j)\}$ so

$$(6.31)+(6.32) \Rightarrow 4+6/4 \ge a^{11}+b^{11}+6/4 \ge 4(e+t)+\frac{1}{q^1}\left(\frac{3}{4}-e\right)+\frac{1}{p^1}\left(\frac{3}{4}-t\right)$$
$$= e\left(4-\frac{1}{q^1}\right)+t\left(4-\frac{1}{p^1}\right)+\frac{3}{4}\left(\frac{1}{p^1}+\frac{1}{q^1}\right) \tag{6.42}$$

1. If $\left(4-\frac{1}{q^1}\right) \ge 0$, i.e., $q^1 \ge 1/4$ then if condition (6.42) is true for some $e \ge 3/4$, then it is also true for $e = 3/4$ so set $e = 3/4$
2. If $\left(4-\frac{1}{q^1}\right) < 0$, i.e., $q^1 < 1/4$, set $e = 1$.
3. If $\left(4-\frac{1}{p^1}\right) \ge 0$, i.e., $p^1 \ge 1/4$ set $t = 3/4$.
4. If $\left(4-\frac{1}{p^1}\right) < 0$, i.e., $q^1 < 1/4$ set $t = 1$.

Let us divide the (p,q)s plane with respect to the existence or non-existence of an equilibrium point.

1. $q^1 < 1/4$ and $p^1 < 1/4$ $\quad [e = t = 1]$

$$(6.42) \Longrightarrow 5.5 \ge \left(8-\frac{1}{p^1}-\frac{1}{q^1}\right)+\frac{3}{4}\left(\frac{1}{p^1}+\frac{1}{q^1}\right) = 8-\frac{1}{4}\left(\frac{1}{p^1}+\frac{1}{q^1}\right)$$
$$\Longrightarrow \frac{1}{p^1}+\frac{1}{q^1} \ge 10 \tag{6.43}$$

2. $q^1 < 1/4$ and $p^1 \ge 1/4$ $\quad [e = 1, t = 3/4]$

$$(6.42) \Longrightarrow 5.5 \ge 4-\frac{1}{q^1}+3-\frac{3}{4}\cdot\frac{1}{p^1}+\frac{3}{4}\left(\frac{1}{p^1}+\frac{1}{q^1}\right) = 7-\frac{1}{4q^1}$$
$$\Longrightarrow q^1 \le 1/6 \tag{6.44}$$

3. $q^1 \ge 1/4$ and $p^1 < 1/4$ $\quad [e = 3/4, t = 1]$

$$(6.42) \Longrightarrow p^1 \le 1/6 \tag{6.45}$$

4. $q^1 \ge 1/4$ and $p^1 \ge 1/4$ $\quad [e = t = 3/4]$

$$(6.42) \Longrightarrow 5.5 \ge \frac{3}{4}\left(8-\frac{1}{p^1}-\frac{1}{q^1}\right)+\frac{3}{4}\left(\frac{1}{p^1}+\frac{1}{q^1}\right)$$
$$\Longrightarrow 5.5 \ge 6 \tag{6.46}$$

Conclusions.

We have found that a necessary condition for an equilibrium point in $\Gamma_\infty(p,q)$ is:

$$p^1 \leq 1/6 \quad \text{or} \quad q^1 \leq 1/6 \quad \text{or} \quad \frac{1}{p^1}+\frac{1}{q^1} \geq 10 \tag{6.47}$$

Now, we will show that if this condition is satisfied then $\Gamma_\infty(p,q)$ has an equilibrium point.

1.

$$(\delta^{kl})_{k,l} = \begin{array}{|c|c|} \hline \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 1/4 \\ 0 & 3/4 \end{matrix} \\ \hline \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \\ \hline \end{array}$$

Figure 3.

Figure 3 is an example for an equilibrium point where $p^1 \leq 1/6$. (can be checked using (6.40) and (6.41))

2.

$$(\delta^{kl})_{k,l} = \begin{array}{|c|c|} \hline \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \\ \hline \begin{matrix} 3/4 & 1/4 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \\ \hline \end{array}$$

Figure 4.

Figure 4 is an example for an equilibrium point where $q^1 \leq 1/6$.

3.

$$(\delta^{kl})_{k,l} = \begin{array}{|c|c|} \hline \begin{matrix} \alpha & 0 \\ 0 & 1-\alpha \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \\ \hline \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \\ \hline \end{array}$$

Figure 5.

Figure 5 is an example for an equilibrium point where $\frac{1}{p^1}+\frac{1}{q^1} \geq 10$, here $\frac{9q^1-1}{8q^1} \leq \alpha \leq \frac{1-p^1}{8p^1}$
Thus $\Gamma_\infty(p,q)$ has an equilibrium point if and only if condition (6.47) is satisfied.

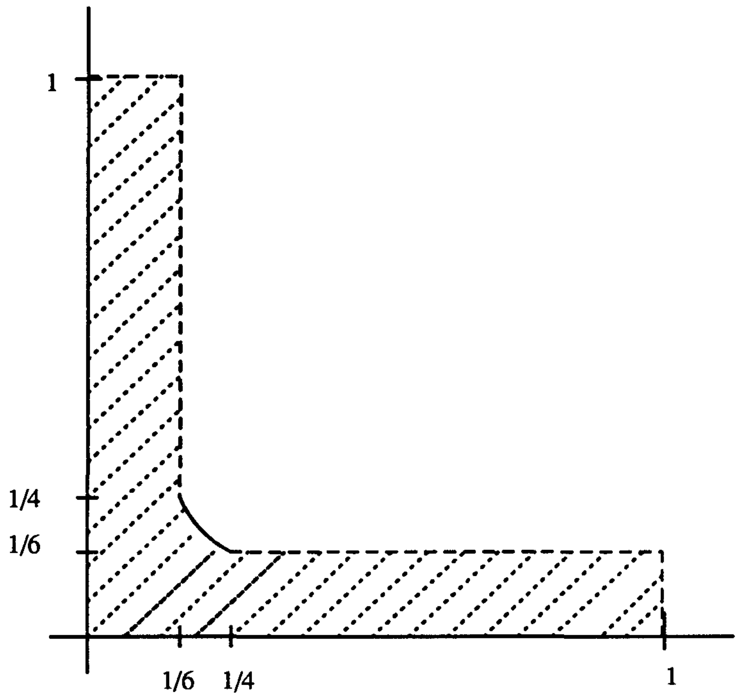

Figure 6. Nash equilibrium points in $\Gamma_\infty(p, q)$.
The x-axis is p^1 and the y-axis is q^1. The equilibrium points of $\Gamma_\infty(p, q)$ are all the points in the shaded area (including the boundaries) and the lines $p^1 = 1$ and $q^1 = 1$.

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