CHAPTER I: THE NASH COOPERATIVE SOLUTION OF TWO PERSON GAMES

1. Bargaining Problems - Fixed Threats

1.1 Definition: A (two person) bargaining problem is a pair (a,C) where $a \in \mathbb{R}^2$ is the nonagreement point and C, a convex compact subset of $\mathbb{R}^2$ containing a, is the set of all agreement points. The collection of all such pairs is denoted $\mathcal{B}$.

1.2 Notation: If x and y are elements of $\mathbb{R}^n$ we write $x \gg y$ iff $x_i > y_i$ for $i = 1,\ldots,n$, $x \geqq y$ iff $x_i \geqq y_i$ for $i = 1,\ldots,n$, and $x > y$ iff $x \geqq y$ and $x \neq y$. Also, $\mathbb{R}^n_+ \equiv \{x \in \mathbb{R}^n\colon\ x \geqq 0\}$, and if C is a convex compact subset of $\mathbb{R}^2$, $C_p = \{x \in C\colon$ there is no $y \in C$ with $y > x\}$.

1.3 Definition: If $f\colon \mathcal{B} \to \mathbb{R}^2$ satisfies the following axioms and $f(a,C) \in C$ for all $(a,C) \in \mathcal{B}$ then f is a Nash solution function and $f(a,C)$ is a Nash solution of the bargaining problem (a,C).

Axiom 1: $f(a,C) \in C_p$ (Pareto efficiency).

Axiom 2: If $a_1 = a_2$ and $(x_1,x_2) \in C \Rightarrow (x_2,x_1) \in C$ then $f_1(a,C) = f_2(a,C)$ (symmetry).

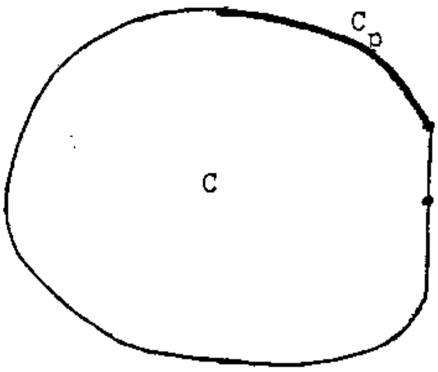
Axiom 3: Let $\alpha_1, \alpha_2 \in \mathbb{R}_+$ and $\beta_1, \beta_2 \in \mathbb{R}$. Then if $C' = \{(\alpha_1 x_1 + \beta_1, \alpha_2 x_2 + \beta_2)\colon x \in C\}$ and $a' = (\alpha_1 a_1 + \beta_1, \alpha_2 a_2 + \beta_2)$, $f(a',C') = (\alpha_1 f_1(a,C) + \beta_1, \alpha_2 f_2(a,C) + \beta_2)$ (invariance under linear transformations).

Axiom 4: If $C \subset D$ then $f(a,D) \in C \Rightarrow f(a,C) = f(a,D)$ (independence of irrelevant alternatives).

1.4 <u>Theorem</u> (Nash [1950]): <u>There exists a Nash solution function</u> f. <u>Moreover it is unique and</u> $f(a,C) = \arg\max_{x \in C_p} (x_1 - a_1)(x_2 - a_2)$.

<u>Proof</u>: See Nash [1950], or Luce and Raiffa [1957], p. 127.

1.5 <u>Remark</u>: If there is an $x \in C$ such that $x >> a$, then clearly $f(a,C) = \arg\max_{\substack{x \in C \\ x \geqq a}} (x_1 - a_1)(x_2 - a_2)$. In Diagram 1 there is no $x \in C$ such that $x >> a$; $f(a,C) = \arg\max_{x \in C_p} (x_1 - a_1)(x_2 - a_2)$ is as shown, whilst $\arg\max_{\substack{x \in C \\ x \geqq a}} (x_1 - a_1)(x_2 - a_2)$ is the line segment joining $f(a,C)$ and a. ~~If there is an $x \in C$ such that $x >> a$ then clearly $\arg\max_{\substack{x \in C_p \\ x \geqq a}} (x_1 - a_1)(x_2 - a_2)$ $= \arg\max_{\substack{x \in C \\ x \geqq a}} (x_1 - a_1)(x_2 - a_2)$. It is also clear that one replacing~~



<u>Diagram 1</u>

Moreover, one may replace "$x \in C_p$" by "$x \in C_p$ and $x \geqq a$".

2. Variable Threats

Let C be a convex compact subset of $\mathbb{R}^2$. We will first examine the nature of the set $\{a \in C: f(a,C) = x\}$ for any given $x \in C_p$. Define $g: C \to C_p$ by

$$g(a) = f(a,C) \quad .$$

Then $g^{-1}(x) = \{a \in C: f(a,C) = x\}$ is the set in which we are interested.

2.1 <u>Lemma</u>: g <u>is a continuous function of</u> a.

<u>Proof</u>: Let $\{a^n\}$ be a sequence of members of C, with $a^n \to a$. Let $d^n = g(a^n)$, and assume ~~(without loss of generality)~~ that $d^n \not\to g(a)$. ~~Then~~ Then there is a subsequence, without loss of generality the whole sequence, converging to some $d \neq g(a)$. Now $(x_1 - a_1^n)(x_2 - a_2^n) \leqq (d_1^n - a_1^n)(d_2^n - a_2^n)$ for all $x \in C_p$, so that, taking limits, $(x_1 - a_1)(x_2 - a_2) \leqq (d_1 - a_1)(d_2 - a_2)$ for all $x \in C_p$, or $d = g(a)$. Contradiction, thus ~~Thus~~ g is continuous.

2.2 <u>Lemma</u>: (a) <u>If</u> $a \in C \backslash C_p$ <u>and</u> d <u>is in the intersection of</u> C <u>and the line through</u> a <u>and</u> $g(a)$ <u>then</u> $g(d) = g(a)$;

(b) <u>if</u> $a, b \in C$ <u>and</u> $0 \leq \alpha \leq 1$ <u>then</u> $g_i(\alpha a + (1 - \alpha)b) \in [g_i(a), g_i(b)]$ <u>for</u> $i = 1,2$.

<u>Proof</u>: (i) Let C be the isosceles triangle in Diagram 2, with a as shown. Moving the origin to a, and using Axiom 3, and then Axioms 1 and 2 we see that $g(a)$ is the midpoint of PQ.

(ii) Let C be the triangle in Diagram 3, with a as shown. We can rescale the payoffs and shift the origin to create a new problem for which the set of agreement points is an isosceles triangle with corner at

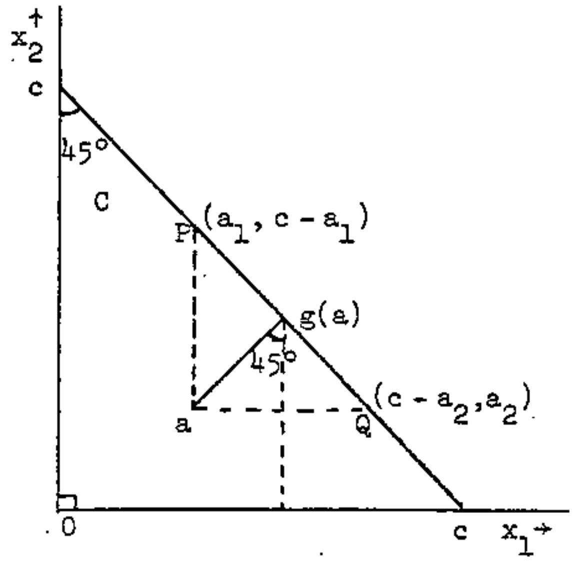


Diagram 2

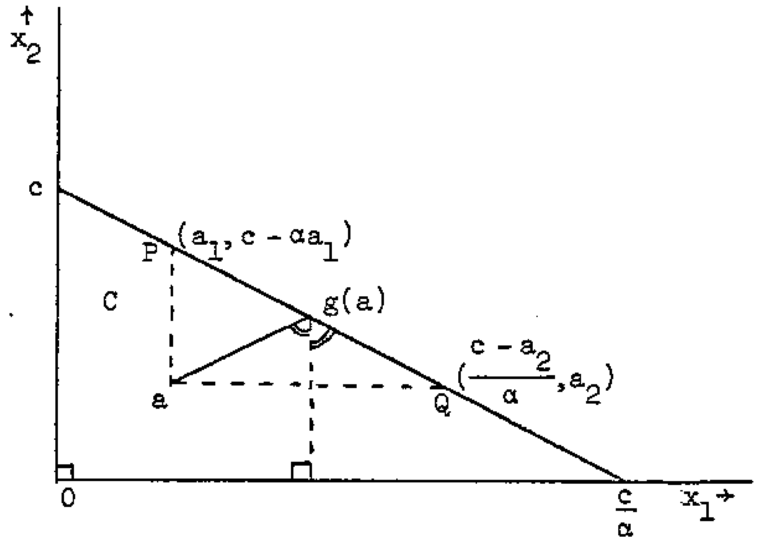


Diagram 3

the origin. Then we can apply the result of (i), together with Axiom 3, to deduce that $g(a)$ is precisely the midpoint of PQ.

(iii) Consider an arbitrary bargaining problem (a,C). We want to show that if d is in the intersection of C and the line through a and $g(a)$ then $g(d) = g(a)$. Note that if there is no $x \in C$ such that $x >> a$, then the result is immediate. So we need only consider the case where there is an $x \in C$ with $x >> a$. In this case consider the tangent at $g(a)$ to the rectangular hyperbola through $g(a)$ with center a and asymptotes parallel to the axes of $\mathbb{R}^2$. The tangent is unique since the rectangular hyperbola is differentiable. Furthermore, it has a negative finite slope, and separates $S \equiv \{x \in \mathbb{R}^2: (x_1 - a_1)(x_2 - a_2) \geq (g_1(a) - a_1) \times (g_2(a) - a_2)\}$ and C: if there were a point in C above the line, the convexity of C would ensure that there would be a point C in the interior of S, which would contradict the fact that $g(a)$ is the Nash solution of

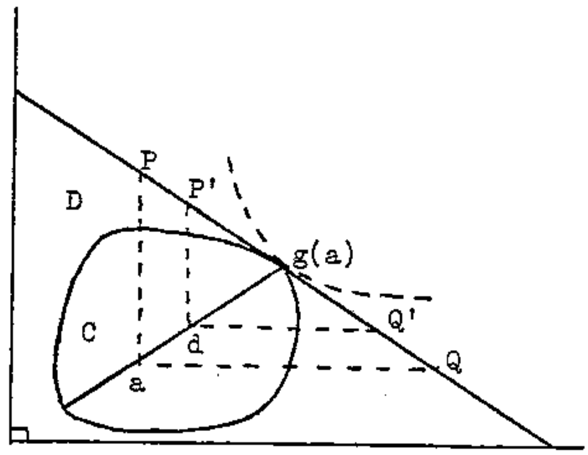


Diagram 4

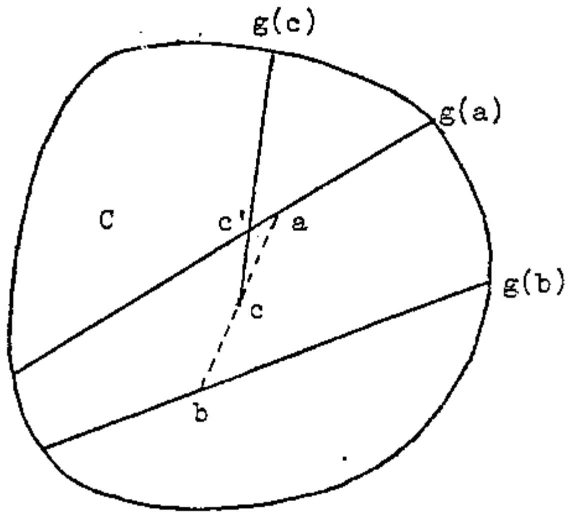


Diagram 5a

(a,C). Hence it is possible to construct a right-angled triangle containing C, as in Diagram 4. Let the points in the triangle comprise the set D. Give the point $a \in \mathbb{R}^2$ the label A, and let the points in the triangle APQ comprise the set E. Then by construction we have $f(a,C) = g(a)$ and $f(a,E) = g(a)$; so by (ii), $g(a)$ is the midpoint of PQ. Also by (ii), $f(d,D)$ is the midpoint of P'Q', which by similar triangles is the midpoint of PQ. But $g(a) \in C$, so by Axiom 4, $g(a) = f(d,D) = f(d,C) = g(d)$. Hence $g(d) = g(a)$, proving part (a) of the lemma.

(iv) Let $a, b \in C$ and $c = \alpha a + (1 - \alpha)b$, $\alpha \in [0,1]$. First let $a, b \in C \setminus C_p$ and suppose $g_i(c) \notin [g_i(a), g_i(b)]$ for some i. Then the segment $[c, g(c)]$ has a point in common with the line through a and $g(a)$, or that through b and $g(b)$. Let this point be c' (see Diagram 5a). Then by (iii), $g(c') = g(a)$ and $g(c') = g(c)$, which is not possible since $g(a) \neq g(c)$ by assumption. Hence $g_i(c) \in [g_i(a), g_i(b)]$

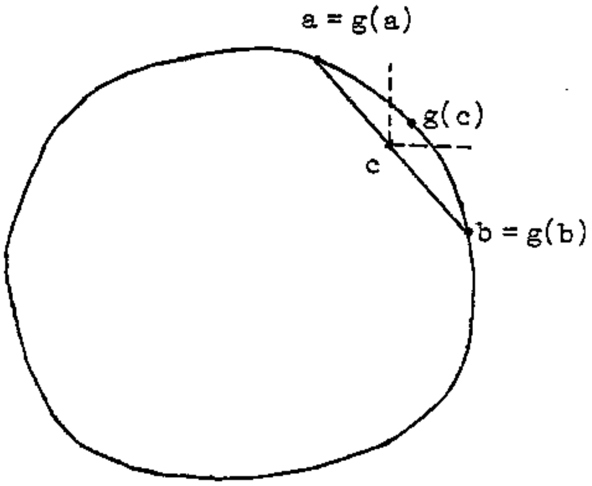


Diagram 5b

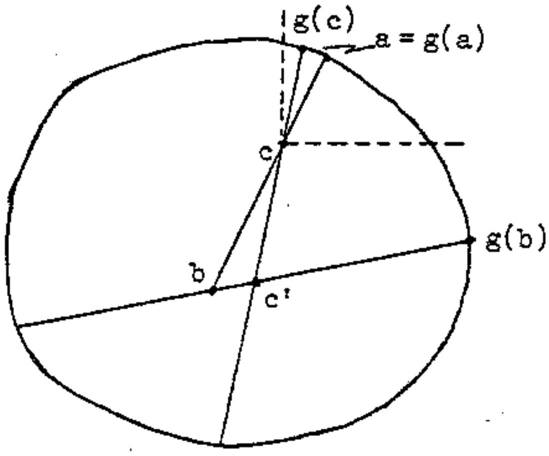


Diagram 5c

for $i = 1,2$. Second, if a or b, or both, are elements of C_p, the result follows from the first case since g is continuous (see Lemma 2.1).

This completes the proof of the lemma.

We are now in a position to establish the nature of $g^{-1}(x)$ for $x \in C_p$. Henceforth if a lower case letter denotes a point in $\mathbb{R}^2$ the corresponding upper case letter is the label of that point. We have:

2.3 Lemma: If $x \in C_p$, let $\ell(x)$ be the set of lines which support C at x. Then $g^{-1}(x) = \bigcup_{\ell \in \ell(x)} \{a \in C:$ the line through A and X is the reflection of ℓ with respect to a perpendicular from X onto an axis$\}$.

Proof: Let C_p' be the set of points in C_p all of whose supporting lines have negative finite slope. We will first show that if $x \in C_p'$

and a is such that the line through A and X is the reflection of some $\ell \in \ell(x)$ with respect to a perpendicular from X onto an axis, then $a \in g^{-1}(x)$. Construct the right-angled triangle PQR of Diagram 6, the hypotenuse of which is a segment of ℓ. Since RX is, by construction, the reflection of XQ with respect to XS, RS = SQ. Similarly, PT = TR, so X is the midpoint of PQ. Let D be the set of points in PQR. Then by (ii) of the proof of Lemma 2.2, $f(r,D) = x$, so by Lemma 2.2, $f(a,D) = x$. Hence by Axiom 4, $f(a,C) = g(a) = x$, as was to be shown. If $x \in C_p \setminus C_p'$, then for any supporting line with negative finite slope the construction can be carried out. For a supporting line with zero or infinite slope the result follows immediately from Lemma 2.2 (since in this case AX coincides with the supporting line).

Now we will show that if $a \in g^{-1}(x)$, AX is a segment of the reflection of a supporting line at X with respect to a perpendicular from X onto an axis. If $x_i = a_i$ for $i = 1$ or $i = 2$, the result is immediate. If not, then the rectangular hyperbola through x with center at a and asymptotes parallel to the axes of $\mathbb{R}^2$ has a tangent at x which supports C at that point (see part (iii) of the proof of Lemma 2.2). Construct the right-angled triangle AFG, as in Diagram 6, the hypotenuse of which is a segment of the tangent. Let E be the set of points in AFG. Then $f(a,E) = x$ by construction, so that by (ii) of the proof of Lemma 2.2 X is the midpoint of PQ. But this means that AX is a segment of the reflection of XQ with respect to XS.

This completes the proof of the lemma.

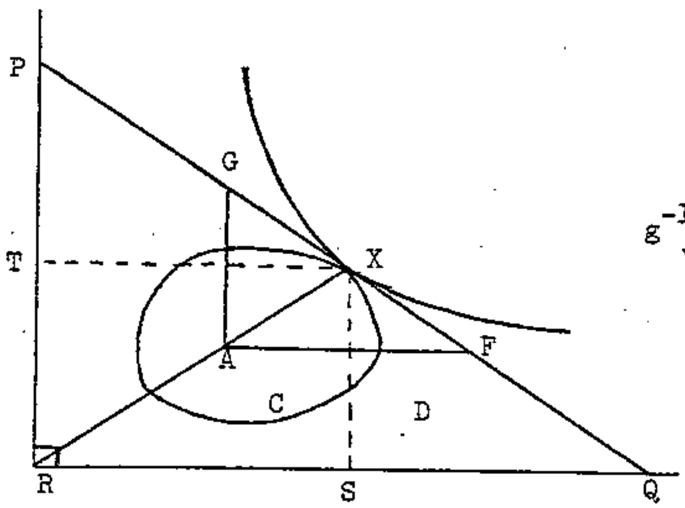


Diagram 6

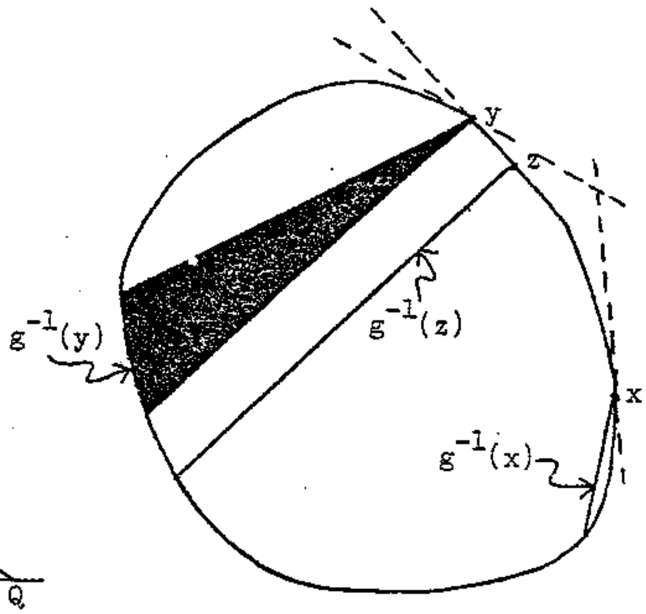


Diagram 7

2.4 Remark: If there is a unique supporting line at x, $g^{-1}(x)$ is the intersection of a line and C; if there are multiple supporting lines at x, $g^{-1}(x)$ is the union of an uncountable collection of such intersections--in fact, a cone (see Diagram 7). Since each such cone contains a point both of whose coordinates are rational, there are countably many such (nondegenerate) cones for any given C.

We can now discuss the ("variable threat") Nash solution of a two person game. Let G be a two person game in strategic form in which the set of pure strategies of player 1 (resp. 2) is $M = \{1,\dots,m\}$ (resp. $N = \{1,\dots,n\}$) and the payoff function of player i is $h_i: M \times N \to \mathbb{R}$ for $i = 1,2$. Let X (resp. Y) be the set of mixed strategies of player 1 (resp. 2) in G, and let $H_i: X \times Y \to \mathbb{R}$ be the payoff function of player i in the mixed extension of G. Finally, let C be the convex hull of $H(X,Y)$, and let $g: C \to C_p$ be the function defined at the beginning of this chapter.

2.5 Remark: Since the strategy sets of both players in G are finite, C is a polyhedron; it is the set of points which the players can reach via correlated strategies--i.e. the set of agreement points.

2.6 <u>Definition</u>: The (two person) <u>bargaining game</u> derived from G is the (two person) game G^* in which the set of pure strategies of player 1 (resp. 2) is X (resp. Y) and the payoff function of player i is $g_i \circ H: X \times Y \to \mathbb{R}$ for $i = 1,2$.

2.7 <u>Remark</u>: The following passage in Luce and Raiffa [1957], p. 140, explains what is going on.

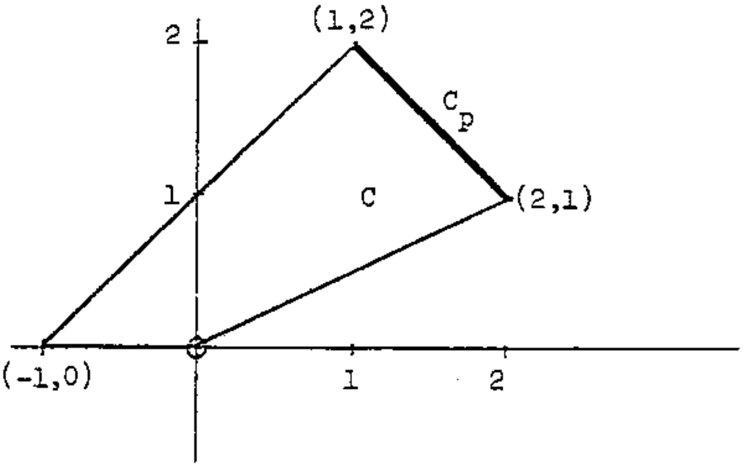
> [In G^*] each player adopts a mixed strategy [of G] as a "threat"; the pair of threats establishes a payoff, which, in turn, acts as the [nonagreement point] for future bargaining; and the bargaining problem is resolved in the manner discussed in [Chapter 1 above]. Therefore, the problem is reduced to selecting the threat strategies so as to influence the [nonagreement point]--which controls the ultimate payoff--in the most favorable manner.

2.8 <u>Example</u>: Let G be as in Table 1. Then C is as in Diagram 8. $H(X,Y)$ is a subset of C which contains all the vertices of C. A pair of strategies (x,y) in G^* determines a point $a = H(x,y)$ in $H(X,Y)$, and consquently a pair of payoffs $g(a) = g(H(x,y))$ in C_p.

		N: 1	N: 2
M	1	2,1	0,0
M	2	-1,0	1,2

Payoffs in G

<u>Table 1</u>



<u>Diagram 8</u>

2.9 Theorem (Nash [1953]): There exists a point $d \in C$ (the agreement point) and points $x_o \in X$ and $y_o \in Y$ (the optimal threats) such that

(1) $d \in C_p$,

(2) for all $y \in Y$, $g_1(H(x_o,y)) \geqq d_1$, and

(3) for all $x \in X$, $g_2(H(x,y_o)) \geqq d_2$.

2.10 Remark: Since $d \in C_p$, (2) implies that for all $y \in Y$, $g_2(H(x_o,y)) \leqq d_2$ and (3) implies that for all $x \in X$, $g_1(H(x,y_o)) \leqq d_1$. So we have a point (x_o,y_o) in $X \times Y$ such that $g_1(H(x_o,y_o)) = d_1$ and

$$\forall (x,y) \in X \times Y \quad , \quad g_1(H(x,y_o)) \leqq g_1(H(x_o,y_o)) \leqq g_1(H(x_o,y)) \quad ;$$

similarly for player 2.

2.11 Remark: We cannot directly apply Nash's theorem on the existence of an equilibrium point in the mixed extension of a noncooperative game with finitely many pure strategies (Nash [1951]; see also Theorem 2.23 of Aumann [1976]) because $g_i \circ H$ for $i = 1,2$ is not linear in x and y.

Proof of Theorem: Recall the following:

(i) for all a in C, $g(a)$ is Pareto efficient;

(ii) g is continuous (Lemma 2.1); and

(iii) for all $a, b \in C$ and $0 \leqq \alpha \leqq 1$, $g_i(\alpha a + (1 - \alpha)b) \in [g_i(a),g_i(b)]$ for $i = 1,2$ (Lemma 2.2).

Using induction on (iii) we obtain

(iii') for all $a_1,\ldots,a_r$ in C and $\alpha_1,\ldots,\alpha_r \geqq 0$ with $\sum_{s=1}^{r} \alpha_s = 1$
$\min_{1\leqq s\leqq r} g_i(a_s) \leqq g_i(\sum_{s=1}^{r} \alpha_s a_s) \leqq \max_{1\leqq s\leqq r} g_i(a_s)$ for $i = 1,2$.

Define, for all $x \in X$, $y \in Y$, $i \in M$, and $j \in N$,

$$c_i^1(x,y) = \max \{g_2(H(x,y)) - g_2(H(e_i^1,y)), 0\} \quad \text{and}$$

$$c_j^2(x,y) = \max \{g_1(H(x,y)) - g_1(H(x,e_j^2)), 0\} \quad ,$$

where e_k^1 (resp. e_k^2) is the k-th unit vector in X (resp. Y)

<u>Assertion 1</u>: For all $x \in X$ and $y \in Y$ there is an $i \in M$ such that

(a) $x_i > 0$, and
(b) $c_i^1(x,y) = 0$.

<u>Proof</u>: By definition, $c_i^1(x,y) \geqq 0$. If there is no $i \in M$ for which (a) and (b) hold, then

$$x_i > 0 \text{ implies } c_i^1(x,y) > 0 \quad , \text{ or}$$

$$x_i > 0 \text{ implies } g_2(H(x,y)) > g_2(H(e_i^1,y)) \quad .$$

Since $x = \sum_{i \in M} x_i e_i^1 = \sum_{\substack{i \in M \\ x_i > 0}} x_i e_i^1$, this contradicts (iii').

<u>Assertion 2</u>: For all $x \in X$ and $y \in Y$, there is a $j \in N$ such that

(a) $y_j > 0$, and
(b) $c_j^2(x,y) = 0$.

<u>Proof</u>: As for Assertion 1.

Now for each $x \in X$ and $y \in Y$ let

$$x_i^* = (x_i + c_i^1(x,y))/(1 + \sum_{i \in M} c_i^1(x,y)) \quad \text{for } i \in M \text{ , and}$$

$$y_j^* = (y_j + c_j^2(x,y))/(1 + \sum_{j \in N} c_j^2(x,y)) \quad \text{for } j \in N \text{ .}$$

It is clear that $x_i^*, y_j^* \geqq 0$ for all $i \in M$, $j \in N$, and $\sum_{i \in M} x_i^* = \sum_{j \in N} y_j^* = 1$. Hence the transformation $(x,y) \to (x^*,y^*)$ is from $X \times Y$ into itself, and is continuous (since c_i^1 and c_j^2 are continuous); $X \times Y$ is convex and compact, so we can apply Brouwer's fixed point theorem to obtain a point (x_o,y_o) in $X \times Y$ such that $(x_o^*,y_o^*) = (x_o,y_o)$. Let $d = g(H(x_o,y_o))$. Then we have:

<u>Assertion 3</u>: For all $x \in X$, $g_2(H(x,y_o)) \geqq g_2(H(x_o,y_o)) = d_2$.

<u>Proof</u>: Applying Assertion 1 to x_o and y_o, we get an $i \in M$ such that $x_{oi} > 0$ and $c_i^1(x_o,y_o) = 0$. Hence

$$0 < x_{oi} = x_{oi}^* = \frac{x_{oi} + 0}{1 + \sum_{i \in M} c_i^1(x_o,y_o)} \quad .$$

So $1 + \sum_{i \in M} c_i^1(x_o,y_o) = 1$, and $c_i^1(x_o,y_o) = 0$ for all $i \in M$. By the definition of c_i^1, this implies $g_2(H(x_o,y_o)) \leqq g_2(H(e_i^1,y_o))$ for all $i \in M$. Hence for all $x \in X$, using (iii'),

$$g_2(H(x,y_o)) \geqq \min_{i \in M} g_2(H(e_i^1,y_o)) \geqq g_2(H(x_o,y_o)) \quad ,$$

as was to be shown.

Assertion 4: (x_o, y_o) are optimal threats, and d is the agreement point.

Proof: Apply Assertion 2 to obtain an inequality similar to that in Assertion 3 for player 1.

This completes the proof of the theorem.

2.12 Remark: It is easy to see that the theorem holds if instead of letting C = convex hull of $H(X,Y)$, it is merely assumed that C is convex and compact and contains $H(X,Y)$.

2.13 Remark: What the theorem says is that player 1 can guarantee himself a final payoff of at least d_1 by playing the strategy x_o, and this strategy is a best response when player 2 uses the strategy y_o; and player 2 can guarantee himself a final payoff of at least d_2 by playing the strategy y_o, and this strategy is a best response when player 1 uses the strategy x_o.

2.14 Remark: The agreement point d is unique. To see this, suppose that there were two such points, d and d'. Then player 1 could guarantee himself $\max(d_1, d_1')$, which would mean that player 2 could not guarantee himself $\max(d_2, d_2')$, in which case one of the points would not be an agreement point. The optimal threats, however, need not be unique (see Example 2.18 below).

2.15 Remark: The use of mixed strategies in G^* does not expand the set of feasible payoffs. Indeed, given the pure strategy of the other player, randomization between pure strategies (in G^*) by a player will yield a payoff which is a linear combination of the payoffs he gets to the pure strategies, while if C is strictly convex in the appropriate region, he could obtain a higher payoff by using some other pure strategy.

In the examples below, we will use (a,b) to denote a point in $\mathbb{R}^2$, reserving the notation (x,y) for a pair of mixed strategies in G. In the tables each row corresponds to a strategy of player 1, and each column to a strategy of player 2, as usual.

2.16 Example: Let G be as in Example 2.3. C is then as in Diagram 9. C_p is a segment of the line $a + b = 3$, so for $(a^*,b^*) \in C_p$, $g^{-1}(a^*,b^*)$ is a segment of $a - b = a^* - b^* = K$, say. The choice of threats by the players determines a value for K, and hence the outcome. Player 1 prefers higher values of K, and player 2 prefers lower values, so the "K-game"

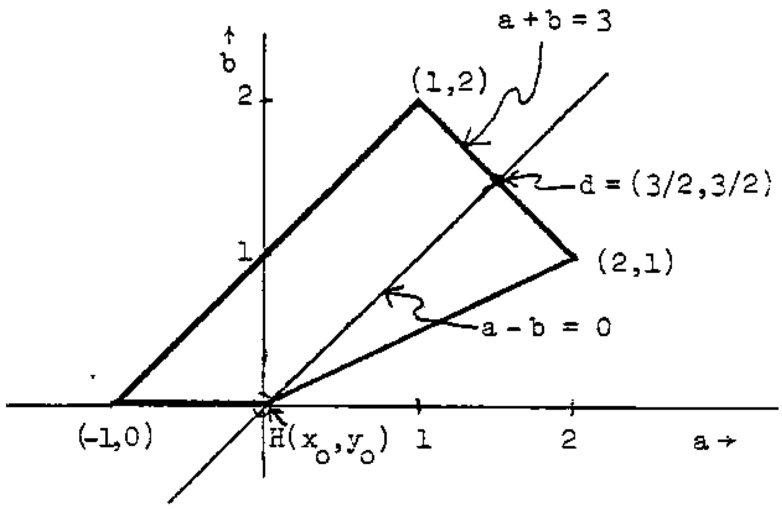


Diagram 9

a - b

1	0
-1	-1

Table 2

involved is strictly competitive. To find the optimal threats in G^* we calculate $a - b$ for the payoff (a,b) to each pure strategy combination in G; these are given in Table 2. The equilibrium point of this K-game is seen to be $(x_o,y_o) = ((1,0),(0,1))$ which is thus the pair of optimal threats in G^*; the nonagreement point is $H(x_o,y_o) = (0,0)$. The minmax value of the K-game is $K^* = 0$, so the agreement point in G^* is (d_1,d_2) such that $d_1 - d_2 = 0$ and $d_1 + d_2 = 3$, or $(d_1,d_2) = (3/2,3/2)$.

2.17 Example: Let the payoff matrix of G be as in Table 3. C is then as in Diagram 10. C_p consists of a segment of the line $a + 2b = 6$, and a segment of the line $2a + b = 6$. We divide C into three regions, as shown in the diagram, and examine the possibility that an optimal threat lies in each one in turn. Proceeding as in the previous example for region 1, we calculate $a - 2b$ for each pure strategy pair, to obtain Table 4. Let the mixed strategies of the players be $(x,y) = ((p,1 - p),(q,1 - q))$. Then for an equilibrium point $-6p + 6(1 - p) = 3p - 2(1 - p)$, or $p^* = 8/17$, so

Payoffs (a,b) in G

0,3	3,0
0,-3	2,2

Table 3

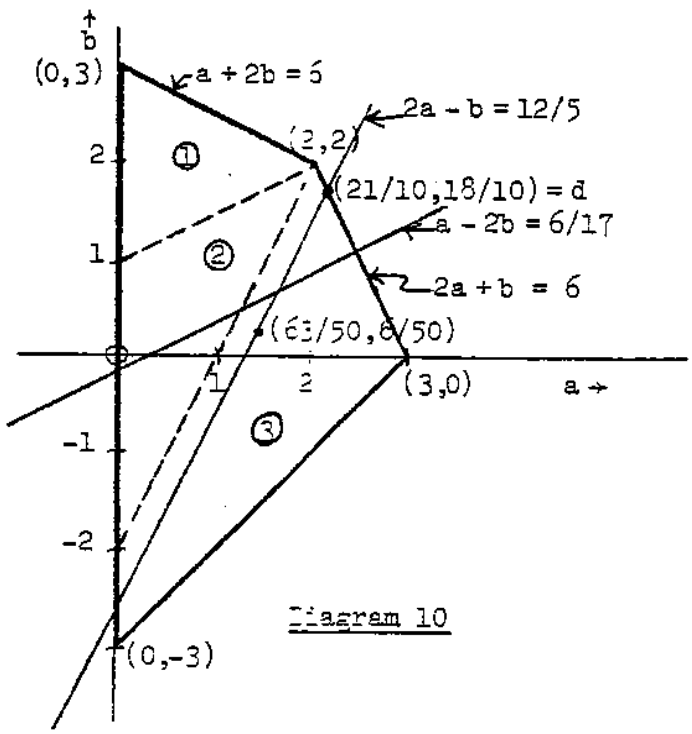


Diagram 10

a - 2b

-6	3
6	-2

Table 4

2a - b

-3	6
3	2

Table 5

the minmax value is $K^* = -6(8/17) + 6(9/17) = 6/17$. But $a - 2b = 6/17$ does not intersect region 1, so the agreement point does not lie there.

Consider region 3. The values of $2a - b$ are given in Table 5. We have $-3p + 3(1 - p) = 6p + 2(1 - p)$, so $p^* = 1/10$; hence $K^* = -3(1/10) + 3(9/10) = 12/5$. $2a - b = 12/5$ intersects region 3; the agreement point is (d_1,d_2) such that $2d_1 + d_2 = 6$ and $2d_1 - d_2 = 12/5$, so that $(d_1,d_2) = (21/10,18/10)$. Also, $q^* = 2/5$, so a pair of optimal threats is $(x_o,y_o) = ((1/10,9/10),(2/5,3/5))$, giving a nonagreement point of $H(x_o,y_o) = (63/50,6/50)$. By the uniqueness of d, we need not examine region 2.

2.18 Example: Let the payoff matrix for G be as in Table 6; C is then as in Diagram 11. We proceed as in the previous case, merely outlining the argument. Consider region 1. The values of $a - 3b$ are as in Table 7. $p^* = 13/14$, and $K^* = -1/2$; $a - 3b = -1/2$ does not intersect region 1. So consider region 3. Values of $2a - b$ are as in Table 8. $p^* = 1$, $q^* = 1$, and $K^* = 0$; $2a - b = 0$ does not intersect region 3. Hence we know that the agreement point is $(d_1,d_2) = (2,1)$. Moreover, from the argument for region 1 we know that if player 1 plays $p^* = 13/14$ he will guarantee that his payoff in the game in Table 7 is at least $-1/2$ (in fact his payoff will be $-1/2$ whatever strategy player 2 uses); i.e. he will guarantee

Payoffs (a,b) in G

0,0	2,1
-1,2	3,-1

Table 6

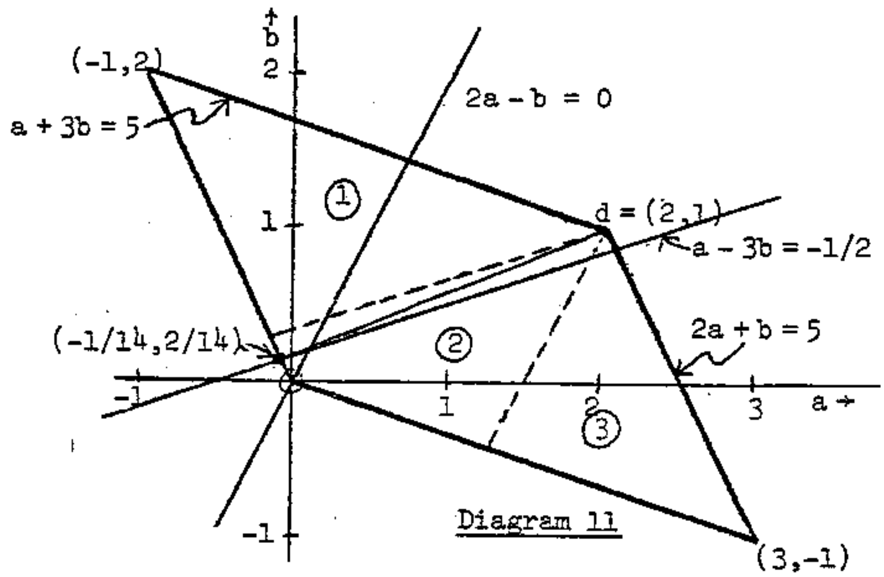


Diagram 11

a - 3b

0	-1
-7	6

Table 7

2a - b

0	3
-4	7

Table 8

that the nonagreement point satisfies $a - 3b = -1/2$. So by using this strategy he can ensure that the nonagreement point is in region 2 or region 3, so that his agreement payoff is at least 2. Similarly we know that if player 2 plays $q^* = 1$ his payoff in the game in Table 8 is at least 0, so that he can guarantee that the nonagreement point satisfies $2a - b \leqq 0$, and that his agreement payoff is at least 1. Hence $(x_o, y_o) = ((13/14, 1/14), (1,0))$ satisfies the condition for a pair of optimal threats; $H(x_o, y_o) = (-1/14, 2/14)$. Note that if the equilibrium strategies for Table 8 were mixed we could deduce that the disagreement point would be <u>on</u> $2a - b = 0$, in which case it would be precisely the intersection of $2a - b = 0$ and $a - 3b = -1/2$. Note also that $(x_o', y_o') = ((1,0),(1,0))$ is a pair of optimal threats too (with $H(x_o, y_o) = (0,0)$): in this example the optimal threats are not unique.

The following examples further elucidate the nature of the Nash solution; we will only sketch the argument in each case.

2.19 Example: Let G have the payoffs given in Table 9; C is shown in Diagram 12. Consider region 1. Values of $a - 3b$ are given in Table 10. $p^* = 1$, $q^* = 1$, so $(x_o,y_o) = ((1,0),(1,0))$ and $H(x_o,y_o) = (2,1) = (d_1,d_2)$ (which are just the payoffs to the Nash equilibrium of G).

2.20 Example: Let G have the payoffs given in Table 11; C is shown in Diagram 13. Consider region 1. Values of $a - 3b$ are given in Table 12. $p^* = 4/21$; $K^* = -82/21$; $a - 3b = -82/21$ does not intersect region 1. So consider region 3. Values of $a - b$ are given in Table 13. $p^* = 2/11$, $K^* = 10/11$. $a - b = 10/11$ intersects region 3, giving $(d_1,d_2) = (38/11, 28/11)$; $(x_o,y_o) = ((2/11,9/11),(6/11,5/11))$, $H(x_o,y_o) = (402/121,292/121)$.

Payoffs (a,b) in G

2,1	0,-2
-1,2	3,0

Table 9

a - 3b

-1	6
-7	3

Table 10

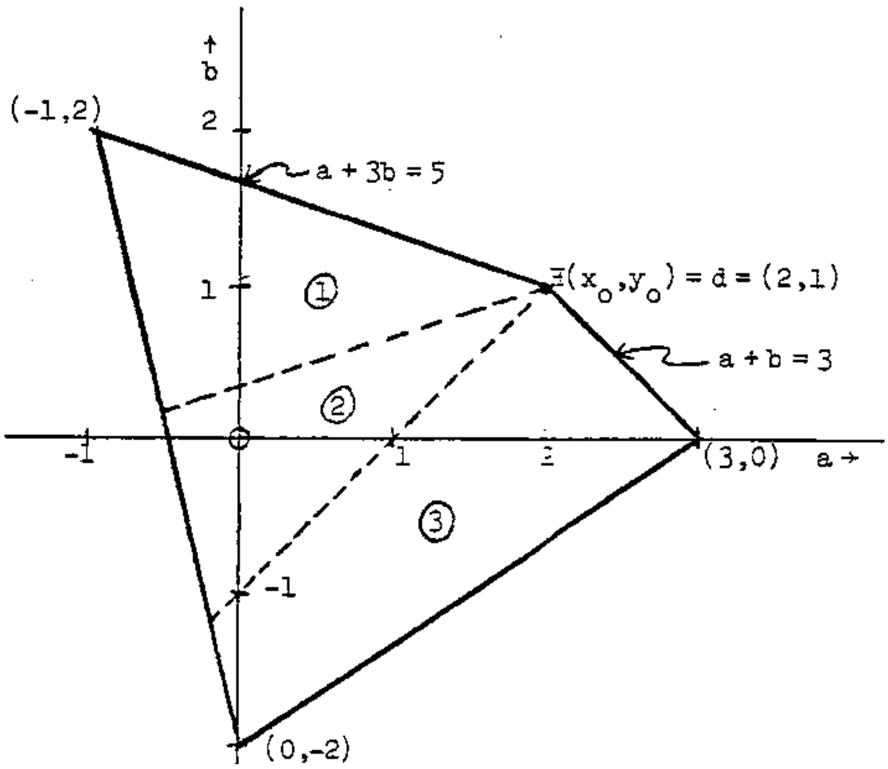


Diagram 12

Payoffs (a,b) in G

5,0	0,4
3,3	4,2

Table 11

a - 3b

5	-12
-6	-2

Table 12

a - b

5	-4
0	2

Table 13

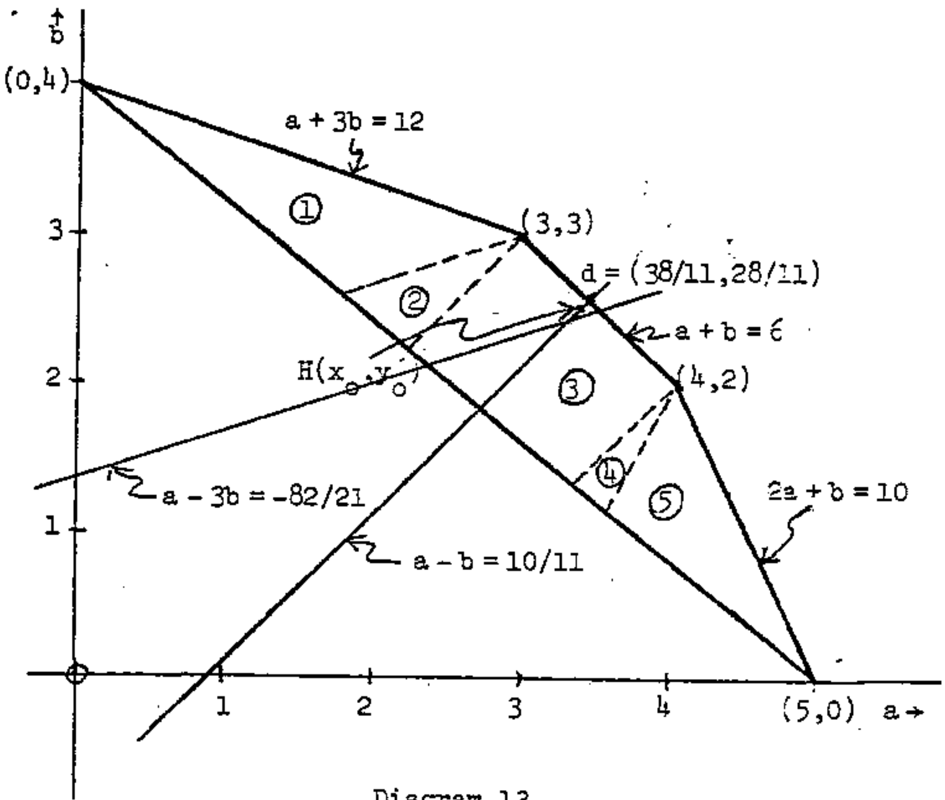


Diagram 13

2.21 Example: Let G have the payoffs given in Table 14; C is shown in Diagram 14. Consider region 1. Values of a - 2b are given in Table 15. The equilibrium strategies in this strictly competitive game are (2/7,5/7) for player 1 and (0,6/7,1/7) for player 2; K* = -9/7, and a - 2b = -9/7 does not intersect region 1. So consider region 3. Values of 2a - b are given in Table 16. Equilibrium strategies are (1/2,1/2) for player 1, and either (2/5,3/5,0) or (2/3,0,1/3) for player 2; K* = 0, and 2a - b = 0 does not intersect region 3. Hence (d_1,d_2) = (2,2). Also, the argument for region 1 establishes that if player 1 plays x = (2/7,5/7), the disagreement point will lie on or below a - 2b = -9/7, while the argument for region 3 establishes that if player 2 plays y = (2/5,3/5,0)

Payoffs (a,b) in G

0,3	2,2	3,0
0,-3	-1,0	-3,0

Table 14

a - 2b

-6	-2	3
6	-1	-3

Table 15

2a - b

-3	2	6
3	-2	-6

Table 16

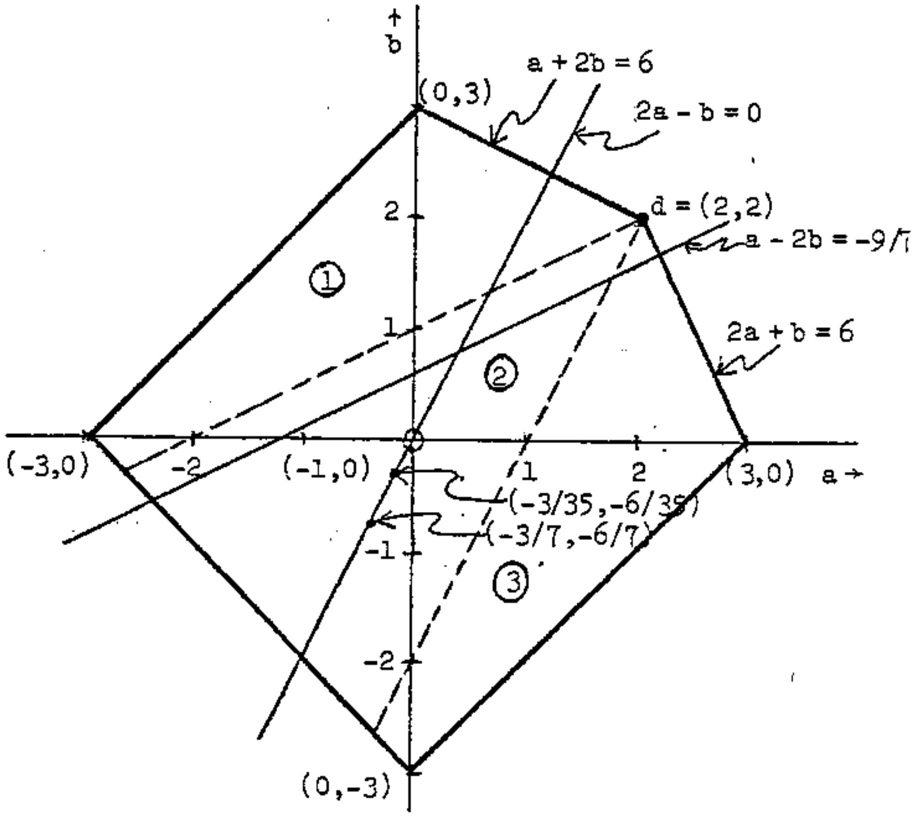


Diagram 14

or $y' = (2/3,0,1/3)$ he is assured that the disagreement point will lie on $2a - b = 0$. Hence $(x_o,y_o) = ((2/7,5/7),(2/5,3/5,0))$ and (x_o',y_o') $= ((2/7,5/7),(2/3,0,1/3))$ are both pairs of optimal threats; $H(x_o,y_o)$ $= (-3/35,-6/35)$, $H(x_o',y_o') = (-3/7,-6/7)$.

2.22 Example: Let G have the payoffs given in Table 17 (note their similarity with those of Example 2.21). C is shown in Diagram 15. Consider region 1. Values of $a - 2b$ are given in Table 18. The equilibrium strategies are $(9/13,4/13)$ for player 1 and $(1/13,12/13,0)$ for player 2; $K^* = -30/13$, and $a - 2b = -30/13$ intersects region 1. We find (d_1,d_2) $= (24/13,27/13)$; $(x_o,y_o) = ((9/13,4/13),(1/13,12/13,0))$ and $H(x_o,y_o)$ $= (72/169,231/169)$.

Payoffs (a,b) in G

0,3	2,2	3,0
0,-3	-3,0	-3,0

Table 17

a - 2b

-6	-2	3
6	-3	-3

Table 18

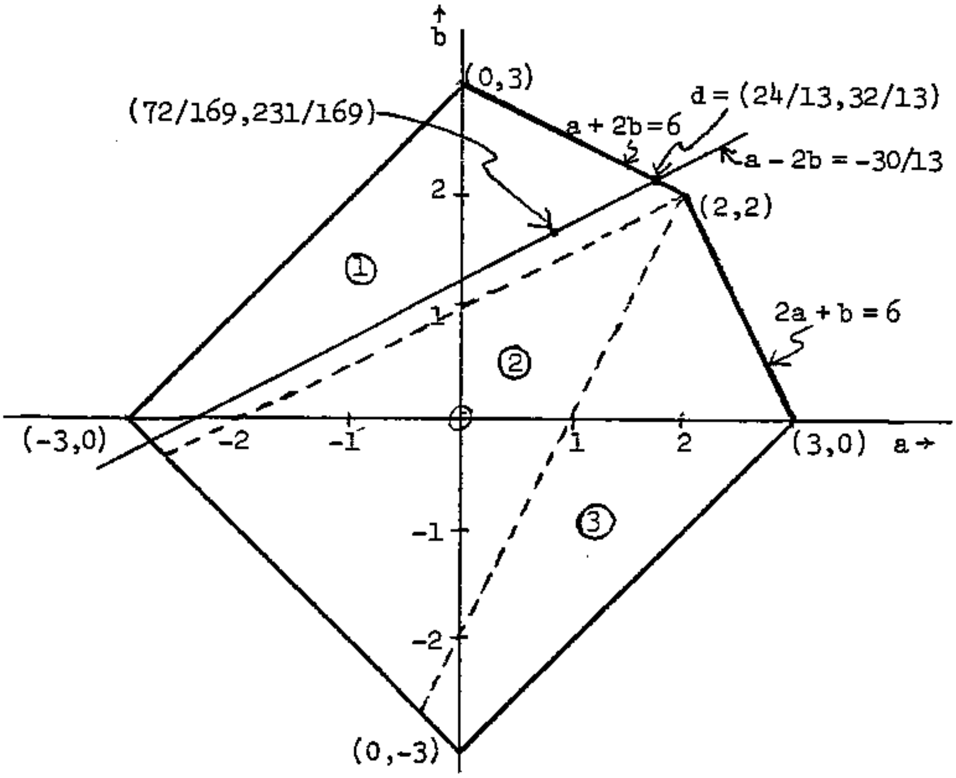


Diagram 15

2.23 Example: Let G have the payoffs given in Table 19 (note their similarity with those of Examples 2.21 and 2.22). C is shown in Diagram 16. Consider region 1. Values of $a - 2b$ are given in Table 20. The equilibrium strategies are (5/7,2/7) for player 1 and (1/7,6/7,0) for player 2; $K^* = -36/7$, and $a - 2b = -36/7$ intersects region 1. We find $(d_1,d_2) = (24/7,30/7)$, $(x_o,y_o) = ((5/7,2/7),(1/7,6/7,0))$ and $H(x_o,y_o) = (144/49,198/49)$.

2.24 Example: Let G have the payoffs given in Table 21. C is shown in Diagram 17. Consider region 1. Values of $a - 2b$ are given in Table 22. $p^* = 0$, and $q^* = 0$, so $(x_o,y_o) = ((0,1),(0,1))$ and $H(x_o,y_o) = (3,3) = (d_1,d_2)$ (which are just the payoffs at the Nash equilibrium of G; cf. Example 2.19).

Payoffs (a,b) in G

0,6	4,4	6,0
0,-6	2,5	-6,0

Table 19

a - 2b

-12	-4	6
12	-8	-6

Table 20

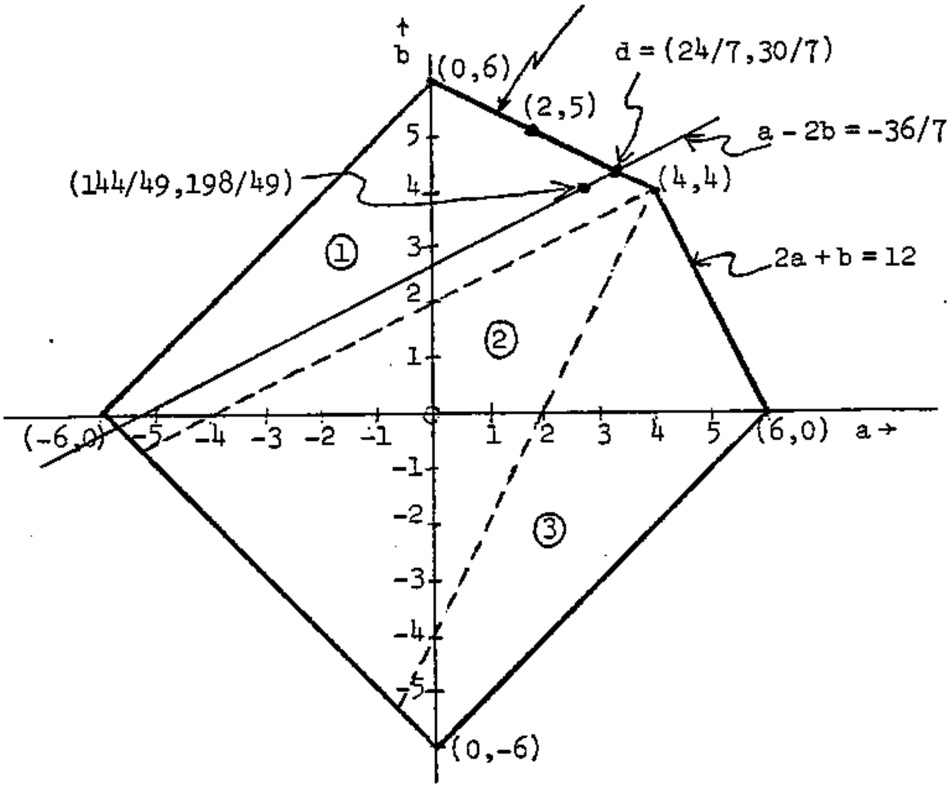


Diagram 16

Payoffs (a,b) in G

0,0	1,4
4,0	3,3

Table 21

a - 2b

0	-7
4	-3

Table 22

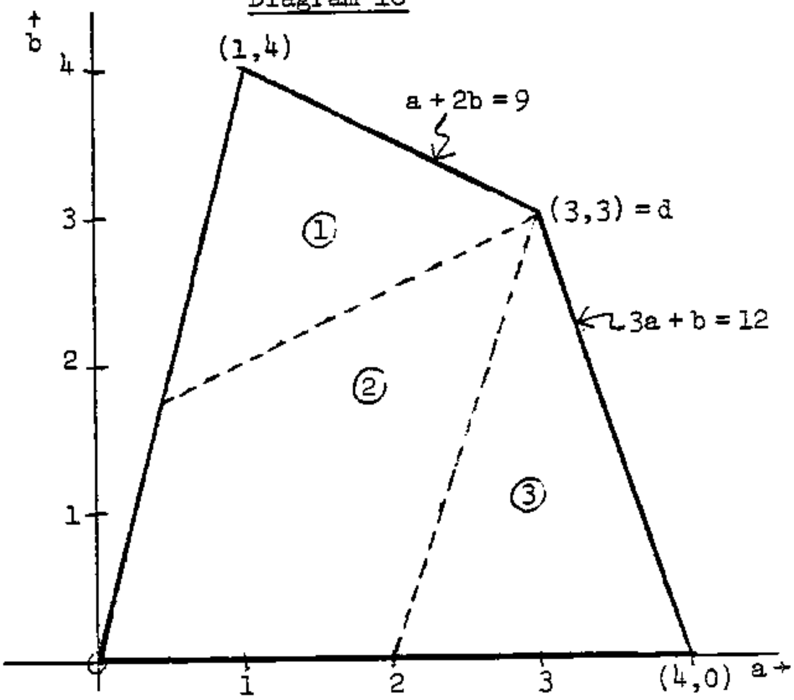


Diagram 17

2.25 <u>Example</u>: Let G have the payoffs given in Table 23. C is shown in Diagram 18. Values of a - b are given in Table 24. Any pair ((p,1 - p),(1,0)) with $p \in [0,1]$ is a pair of equilibrium strategies. Hence $(d_1,d_2) = (1,4)$ and the set of optimal threats is {((p,1 - p),(1,0)): $p \in [0,1]$}; thus every point on the line joining (-4,-1) and (1,4) is a possible disagreement payoff.

Payoffs (a,b) in G

1,4	-1,-4
-4,-1	4,1

Table 23

a - b

-3	3
-3	3

Table 24

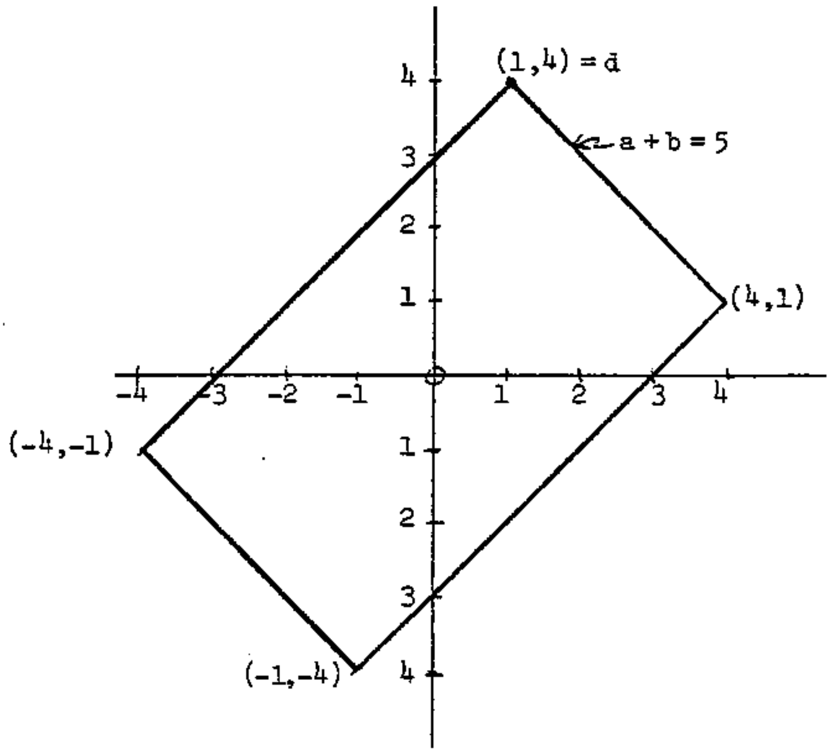


Diagram 18

[illegible] is not; this is due to the [illegible] the game itself

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