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DIOPHANTINE GEOMETRY OVER GROUPS III: RIGID AND SOLID SOLUTIONS

BY

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ABSTRACT

This paper is the third in a series on the structure of sets of solutions to systems of equations in a free group, projections of such sets, and the structure of elementary sets defined over a free group. In the third paper we analyze exceptional families of solutions to a parametric system of equations. The structure of the exceptional solutions, and the global bound on the number of families of exceptional solutions we obtain, play an essential role in our approach towards quantifier elimination in the elementary theory of a free group presented in the next papers of this series. The argument used for proving the global bound is a key in proving the termination of the quantifier elimination procedure presented in the sixth paper of the series.

Introduction

In the first paper of this series on Diophantine geometry over groups, we studied sets of solutions to systems of equations defined over a free group and parametric families of such sets. With a given system of equations we associated a canonical Makanin–Razborov diagram. This Makanin–Razborov diagram encodes the entire set of solutions to the system. Later on, we studied systems of equations with parameters, and with each such system we associated a (canonical) graded Makanin–Razborov diagram which encodes the Makanin–Razborov diagrams of the systems of equations associated with each specialization of the defining parameters.

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In constructing the (ungraded) Makanin–Razborov diagram associated with a system of equations, we are guaranteed that the abelian JSJ decomposition associated with a limit group is non-trivial, and that a shortening quotient of a limit group is always a proper quotient of that limit group ([Se1], theorems 2.7 and 5.3). In studying graded limit groups and their associated (canonical) graded Makanin–Razborov diagrams (sections 9–10 in [Se1]), we were forced to study *rigid* and *solid* graded limit groups ([Se1], definition 10.2). A rigid graded limit group is a graded limit group for which its associated graded abelian JSJ decomposition is trivial, and a solid graded limit group is a graded limit group with a non-trivial graded abelian JSJ decomposition, but for which there exists a unique *maximal graded shortening quotient*, and this shortening quotient is isomorphic to the original solid graded limit group (i.e., it is not a proper quotient of the graded solid limit group).

To study rigid and solid limit groups, we have associated a (canonical) finite collection of (maximal) *flexible quotients* with either a rigid or a solid limit group, and we showed that for each specialization of the defining parameters there are at most finitely many families of solutions that do not factor through these flexible quotients ([Se1], 10.7). Flexible quotients were shown to be proper quotients of the original rigid or solid limit group with which they are associated, so they enabled us to use the ascending chain condition for limit groups to construct the graded Makanin–Razborov diagram. The families of solutions that do not factor through the flexible quotients were termed rigid solutions for a rigid limit group, and solid solutions for a solid one.

Understanding the exceptional rigid and solid solutions seems to be a necessity in any attempt to analyze elementary sets defined over a free group. In defining these families (section 10 of [Se1]), we have only shown that for each value of the defining parameters there exist at most finitely many families of either rigid or solid solutions. However, in analyzing elementary sets this finiteness is not sufficient. In fact a global bound (independent of the parameter value) on the number of families of rigid and solid solutions is required. Achieving such a global bound is the main goal of this paper.

We start this paper by studying the combinatorial structure of rigid solutions of a rigid limit group, and (strictly) solid solutions of a solid limit group that are the shortest in their (strictly) solid family. In the first section we get a "bound" on the combinatorial structure of such solutions. In the second section we use the combinatorial boundedness achieved in the first section to get the desired global bound on the number of rigid and of (strictly) solid families of solutions.

The argument used for getting this global bound plays a key role in proving the termination of the quantifier elimination procedure in the sixth paper in this sequence. In the third section, we use our basic understanding of rigid and solid solutions to analyze the singular locus of the bundle of solutions of a parametric system of equations, and show that the various strata in the basis of the bundle that correspond to the different possible degenerations which occur in the fibers of the bundle are in the Boolean algebra of AE sets. The global bound on the number of families of rigid and solid solutions and the basic analysis of the singular locus play an essential role in our approach towards quantifier elimination in the elementary theory of free groups presented in the next papers in this series. In the fourth section we present *solid configuration limit groups*, which are limit groups that are used to study a finite collection of solid solutions simultaneously. Solid configuration limit groups can be used to show that in some cases the results of section 3 can be strengthened, and the strata associated with the singular locus are in fact in the Boolean algebra of universal sets. This, in combination with the procedure presented in the next papers in the series, can be used to show that certain elementary sets are in the Boolean algebra of universal sets. However, the construction of solid configuration limit groups seems to be restricted to the case of solid limit groups with no flexible quotients, which prevents us from using them in our general procedures for analyzing elementary sets. Still, they are an important part of the structure theory and are likely to be applicable elsewhere.

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1. Rigid and solid solutions and their combinatorial types

In the first paper in this series we introduced graded limit groups, their graded Makanin–Razborov diagram, and the singular graded limit groups associated with each of its resolutions and their terminal rigid and solid graded limit groups (sections 9–11 in [Se1]). The structure of the singular locus (section 11 in [Se1]) gives us a basic understanding of the connection between the graded Makanin–Razborov diagram and the restricted diagrams for each possible specialization of the set of the defining parameters. To get a better understanding of this connection, which will be used later to analyze the structure of elementary sets of a free group, we need to look more closely into the structure of rigid and solid graded limit groups.

Our first goal in studying rigid and solid specializations is to get a bound on their combinatorial structure. In principle, the combinatorial bound we are looking for is showing that there exist finitely many (combinatorial) ways to cut the specializations of the defining parameters, and finitely many words in these pieces (*fractions*), so that the specializations of the (given) generating set of a rigid or solid limit group inherited from any rigid or (shortest) solid specialization can be presented as one of these (finitely many) words in fractions of the specializations of the defining parameters.

To prove such a combinatorial bound, we actually prove a stronger result, stated in Theorem 1.2 in the rigid case, and in Theorem 1.7 in the solid case. To state the stronger statement we need the notion of an R-AP-covered (graded) homomorphism.

Let X be the Cayley graph of a rigid graded limit group $Rgd(x,p,a)$ with respect to the generating system $Rgd(x,p,a) =< x,p,a >$, and let T be the Cayley graph (tree) of the free group F_k with respect to the generating system $F_k =< a >$. Clearly, a homomorphism h: $Rgd(x,p,a) \to F_k$ corresponds to a natural equivariant map τ: $X \to T$, where each edge in X is mapped to a (possibly degenerate) path in T.

Definition 1.1: Let B_R be the ball of radius R in the Cayley graph X of $Rgd(x,p,a)$. We say that a homomorphism h: $Rgd(x,p,a) \to F_k$ is *R-AP-covered*, if the image in T of the union of edges labeled by either an element of $\{a\}$ or an element of $\{p\}$ in B_R covers the entire image in T of the ball B_1.

To define and control the combinatorial types of rigid solutions, we need the following basic theorem which bounds combinatorially the structure of rigid solutions (specializations) of a rigid limit group.

THEOREM 1.2: *Let $Rgd(x,p,a)$ be a rigid limit group. There exists a constant R_0 so that every rigid homomorphism h: $Rgd(x,p,a) \to F_k$ is R_0-AP-covered.*

Proof: Suppose that there is no such constant R_0. Let $\{h_n | h_n : Rgd(x,p,a) \to F_k\}$ be a sequence of rigid homomorphisms, so that for every integer n, the n-th homomorphism in the sequence, h_n, is not n-AP-covered. By a theorem of Paulin ([Pa], 2.3), from the sequence of rigid homomorphisms $\{h_n\}$ one can extract a subsequence converging in the Gromov–Hausdorff topology into an action of $Rgd(x,p,a)$ on a (pointed) real tree (Y,y_0) (where y_0 is the limit of the vertex associated with the identity element in the Cayley graph (tree) T of the free group F_k).

The action of $Rgd(x,p,a)$ on the real tree Y satisfies the conclusions of lemmas 1.3 and 1.4 of [Se1], hence it is in particular a stable action. If K_∞ denotes the kernel of the action, then $L = Rgd(x,p,a)/K_\infty$ is a limit group, so by ([Se1], 4.4) it is finitely presented.

Let $Y_{AP} \subset Y$ be the convex hull of the images of the base point y_0 under the action of the subgroup $AP =< a,p >$ of our rigid graded limit group $Rgd(x,p,a)$. Clearly, either there exists a non-degenerate segment $J \subset Y$ which is not in the orbit of Y_{AP} in Y, or the orbit of every non-degenerate segment in Y contains a non-degenerate subsegment which is in the subtree Y_{AP}.

Suppose that there exists a non-degenerate segment $J \subset Y$ which does not contain a non-degenerate subsegment that is in the orbit of Y_{AP}. Suppose further that the action of L on the real tree Y is geometric, i.e., that the action of L on Y is equivariantly isometric to the action of L on some resolution of Y (recall that a resolution of an action of a f.g. group on a real tree is an action reconstructed from the restricted action of the given set of generators of the f.g. group on some finite subtree of the real tree — see Bestvina–Feighn analysis of stable actions on real trees, presented in [Be-Fe2], for a detailed discussion of these notions). Note that in this (geometric) case, there exists a canonical graph of groups with fundamental group L associated with the action of L on the real tree Y (see [Be-Fe2]). Let Λ_L be this graph of groups.

Recall that a subtree U of the tree Y is called mixing, if for every two segments $I_1, I_2 \subset U$, I_1 can be cut into finitely many subintervals so that every subinterval can be mapped into I_2 by an element of L (see [Mo] for basic properties of mixing subtrees).

If the orbit of Y_{AP} under the action of L intersects a mixing component of Y in a non-degenerate segment, then the whole mixing component is in the orbit of Y_{AP}. If the orbit of Y_{AP} under the action of L intersects an edge in the discrete part of Y in a non-degenerate segment, then the entire edge is in the orbit of Y_{AP}. Therefore, since the orbit of Y_{AP} misses a non-degenerate segment $J \subset Y$, it misses either the orbit of a mixing subtree or the orbit of an edge in the discrete part of the tree Y, so by the construction of the graph of groups Λ_L (see [Be-Fe2]), the subgroup $AP < L$ is contained in a proper connected subgraph of groups $\Lambda' \subset \Lambda_L$.

If the action of L on the real tree Y is not geometric, and the orbit of Y_{AP} misses a non-degenerate segment in the real tree Y, then there exists a resolution, (Y^r, y^r), of the action of L on (Y, y_0), equipped with an isometric action of L, and an equivariant map from (Y, y_0) to (Y^r, y^r), so that the orbit of the convex hull of the images of y^r under the action of the subgroup $AP < L$ in Y^r, denoted Y^r_{AP}, does not contain a non-degenerate segment $J^r \subset Y^r$, and no non-degenerate subsegment of J^r is contained in the orbit of the subtree Y^r_{AP}.

Since the action of L on the resolution Y^r of Y is geometric by construction, there is a canonical graph of groups Λ^r_L associated with the action of L on Y^r, and the subgroup $AP < L$ is contained in a proper connected subgraph of groups $\Lambda' \subset \Lambda^r_L$.

Now, since AP is contained in a proper connected subgraph of groups Λ' of a graph of groups Λ_L with fundamental group L, and since the real tree Y was constructed from the homomorphisms $\{h_n: Rgd(x,p,a) \to F_k\}$, for large enough n, the homomorphisms $\{h_n\}$ (that belong to the convergent subsequence) are flexible homomorphisms of the rigid limit group $Rgd(x,p,a)$, a contradiction to our assumption of the homomorphisms $\{h_n: Rgd(x,p,a) \to F_k\}$ being rigid.

By the argument given above, for the rest of the argument we may assume that the orbit of Y_{AP} in Y intersects any non-degenerate segment in the real tree (Y, y_0) in a non-degenerate subsegment. Furthermore, given any non-degenerate segment J in the real tree Y, the segment J is covered by finitely many translations (depending only on the segment J) of subsegments of the paths $(y_0, a_i(y_0))$ and $(y_0, p_j(y_0))$, where the elements a_i are the generators of the coefficient group F_k, and the elements p_j are the given generators of the parameter subgroup P. This last assertion is straightforward for geometric actions of L on real trees Y, hence it is true for a general action since it is true for every resolution of it that is reconstructed from a finite subtree of Y that contains the segment J.

In case every non-degenerate segment in the real tree Y intersects the orbit of the tree Y_{AP} in a non-degenerate subsegment, we continue inductively by analyzing the actions of each of the point and edge stabilizers on corresponding real trees. To be able to apply such an inductive argument, we first need to strengthen Paulin's theorem ([Pa], 2.3) in the context of limit groups. The argument we use to prove this strengthening is based on the construction of the analysis lattice of a limit group, presented in section 4 of [Se1]. However, note that the abelian decompositions used in the construction of the analysis lattice are the cyclic JSJ decompositions of the groups in question, whereas in analyzing a given set of homomorphisms we need to deal with general abelian decompositions that are in no way canonical.

Theorem 1.3: *Let G be a f.g. group and let $\{h_n | h_n: G \to F_k\}$ be a sequence of homomorphisms. Then there exist some integer $s \geq 1$, and a subsequence of the given sequence of homomorphisms that converges into a free action of some* *limit quotient L of G on some R^s-tree.*

Proof: First, by applying Paulin's theorem ([Pa], 2.3), and by basic properties of limit groups ([Se1], 1.3–1.4), from the sequence of homomorphisms $\{h_n\}$ it is possible to extract a subsequence (still denoted $\{h_n\}$) that converges (after rescaling) to a stable action of some limit group L on a real tree Y. If L fixes a point in Y (i.e., if the action is trivial), we can further vary the homomorphisms $\{h_n\}$ by composing them with appropriate inner automorphisms of the free (coefficient) group F_k, apply Paulin's theorem ([Pa], 2.3) once again, and get a non-trivial stable action of the limit group L on some real tree (still denoted) Y.

Suppose first that the stabilizer of every non-degenerate segment in the real tree Y is trivial and the action of L_∞ on Y contains no IET components and no axial components.

Let $V_1, \ldots, V_q$ be the (conjugacy classes of) point stabilizers in the real tree Y. Starting with the sequence of homomorphisms $\{h_n \colon G \to F_k\}$, we pass to a subsequence, and compose the homomorphisms $\{h_n\}$ with appropriate inner automorphisms (we still denote these compositions $\{h_n\}$) for which each of the sequences of actions of the subgroups V_i on the Cayley graph T of F_k via the sequence of homomorphisms $\{h_n\}$ converges (after rescaling) into a (non-trivial) action of the subgroup V_i on some real tree Y_i.

We continue inductively (assuming the actions of the subgroups V_i on the real trees Y_i contain no IET and axial components and stabilizers of non-degenerate segments are trivial) by analyzing the actions of each of the point and edge stabilizers in the real trees Y_i on corresponding real trees. Since we assumed that the stabilizer of each non-degenerate segment in the real tree Y is trivial, and the action of L on Y contains no IET components and no axial components, the rank of each of the point stabilizers V_i is strictly smaller than the rank of the ambient group L. Hence, if we assume that throughout our inductive process the action of each of the point stabilizers on the corresponding real trees satisfies the same assumptions, the inductive procedure terminates after finitely many steps. Combining the (decreasing) sequence of actions of point and edge stabilizers on corresponding real trees, we obtained a free action of the limit group L (that must be free in this case) on some R^s tree, where n is the number of layers in the inductive process.

In the general case it is not possible to obtain a non-trivial free decomposition of L from its action on the real tree Y, still it is possible to show that a similar inductive process terminates after finitely many steps.

Let Λ_L be the graph of groups associated with the action of L on the real tree Y (see section 3 in [Se3] for the graph of groups associated with an action and not with a resolution of the action). If the graph of groups Λ_L gives rise to a free decomposition of L, or if Λ_L collapses to a graph of groups with cyclic edge stabilizers and more than a single vertex group, then each factor in a free decomposition and each vertex group in a cyclic splitting with more than a single vertex has a strictly smaller first betti number than the first betti number of L. If the action of L on the real tree Y contains an axial component, or if there exists a non-cyclic free abelian subgroup of L which does not fix a point in the real tree Y, then L admits a graph of groups with a non-cyclic free abelian vertex group, so that the first betti number of the fundamental group of the subgraph that does not contain the abelian vertex group is smaller than the first betti number of L. Hence, if each of these cases applies, we continue (inductively) with either a factor in a free decomposition, a vertex group in a cyclic splitting with more than a single vertex, or the fundamental group of the proper subgraph that does not contain the corresponding abelian vertex group. If none of these cases applies to the action of L on the real tree Y, then the action of L on the real tree Y is discrete, each edge group is a f.g. free abelian subgroup of L, and there exists a (possibly degenerate) spanning subtree in the graph of groups Λ_L in which each edge group is a non-cyclic free abelian group.

With the graph of groups Λ_L associated with the discrete action of L on the real tree Y, we naturally associate the modular group $Mod(\Lambda_L)$, generated by Dehn twists along the edges of Λ_L. For some index n_0, and for every index $n > n_0$, each of the homomorphisms (from the convergent sequence) $h_n\colon Rgd(x,p,a) \to F_k$ splits through the (quotient) limit group L, i.e., it is in fact a composition of the quotient map from G to L with a homomorphism $\hat{h}_n\colon L \to F_k$. Hence, we may act on the homomorphisms $\{\hat{h}_n\colon L \to F_k\}$ by pre-composing each of them with modular automorphisms from the modular group $Mod(\Lambda_L)$, which allows us to apply the shortening procedure for the homomorphisms $\{\hat{h}_n\}$.

For each homomorphism $\hat{h}_n\colon L \to F_k$, we choose a modular automorphism $\varphi_n \in Mod(\Lambda_L)$ and compose the result with an appropriate inner automorphism τ_n of the coefficient group F_k, so that the homomorphism $h_n^1 = \tau_n \circ \hat{h}_n \circ \varphi_n$ is the shortest among all possible choices of an automorphism $\varphi \in Mod(\Lambda_L)$ and an inner automorphism $\tau \in Inn(F_k)$. Note that the modular automorphisms $\varphi_n \in Mod(\Lambda_L)$ preserve the conjugacy classes of the specializations of the various vertex groups in the graph of groups Λ_L.

From the sequence of homomorphisms $\{h_n^1\}$ we extract an infinite subsequence (still denoted $\{h_n^1\}$) that converges in the Gromov–Hausdorff topology into a faithful action of some limit group L^1 on some pointed real tree (Y_1, y_1). The limit group L^1 is naturally a quotient of the limit group L with which we started. Let Λ_{L^1} be the graph of groups associated with the action of L^1 on the real tree Y_1. Each of the vertex groups V_i in the graph of groups Λ_L is naturally embedded in the limit group L^1, so it inherits an action on the tree Y_1, and a decomposition Λ_i associated with this action, which is the same decomposition associated by the vertex group V_i from the graph of groups Λ_{L^1}.

If L^1 is a proper quotient of L, we replace L by L^1 and proceed (with the sequence $\{h_n^1\}$) inductively. If L^1 is isomorphic to L and the graph of groups Λ_{L^1} gives rise to a free decomposition of $L = L^1$, or if Λ_{L^1} collapses to a graph of groups with cyclic edge stabilizers and more than a single vertex group, then each factor in a free decomposition and each vertex group in a cyclic splitting with more than a single vertex has a strictly smaller first betti number than the first betti number of $L = L^1$. If the action of L^1 on the real tree Y_1 contains an axial component, or if there exists a non-cyclic free abelian subgroup of L^1 which does not fix a point in the real tree Y_1, then $L = L^1$ admits a graph of groups with a non-cyclic free abelian vertex group, so that the first betti number of the fundamental group of the subgraph that does not contain the abelian vertex group is smaller than the first betti number of L. Hence, if each of these cases applies, we continue (inductively) with either a factor in a free decomposition, a vertex group in a cyclic splitting with more than a single vertex, or the fundamental group of the proper subgraph that does not contain the corresponding abelian vertex group. If none of these cases applies to the action of L^1 on the real tree Y_1, then the action of L^1 on the real tree Y_1 is discrete, each edge group is a f.g. free abelian subgroup of L, and there exists a (possibly degenerate) spanning subtree in the graph of groups Λ_{L^1} in which each edge group is a non-cyclic free abelian group.

Let D be the subgroup of the limit group L which is the fundamental group of the subgraph of Λ_L containing all the edges with non-cyclic stabilizers. By our assumptions, the graph of groups Λ_L collapses to a graph of groups Λ_D with one vertex stabilized by D and (possibly) few loops with cyclic stabilizers based on this vertex. Clearly, $b_1(D) \leq b_1(L)$. Recall that with each vertex group V_i in the graph of groups Λ_L, we have associated a graph of groups Λ_i, which is the graph of groups inherited by V_i from the graph of groups Λ_{L^1}.

By construction, all the non-cyclic edge groups in the graph of groups Λ_L are elliptic in each of the graphs of groups Λ_i of the vertex groups V_i. If there exists an edge with trivial stabilizer in one of the graphs of groups Λ_i, then the graphs of groups Λ_i can be used to obtain a free decomposition of D. Clearly, the first betti number of each of the factors in this free decomposition is strictly smaller than the first betti number of the limit group D, hence from the limit group L, so we proceed our analysis with each of the factors of this free decomposition. Hence, for the rest of our analysis we may assume that none of the edge groups in the graphs of groups Λ_i is trivial.

If all the cyclic edge groups in the graph of groups Λ_L are elliptic in all the graphs of groups Λ_i as well, then we can replace each of the vertex groups V_i by its associated graph of groups Λ_i and obtain a graph of groups Δ_L which is a proper refinement of the graph of groups Λ_L, i.e., the number of edges in Δ_L is strictly bigger than the number of edges in Λ_L. Hence, if all the edge groups in Λ_L are elliptic in the graphs of groups Λ_i, we proceed our analysis by replacing the graph of groups Λ_L with its proper refinement Δ_L. If the enlarged graph of groups collapses to a cyclic splitting with more than one vertex group, we continue with each of the vertex groups in the cyclic splitting that has a smaller first betti number than the ambient group L. Otherwise, we repeat the shortening procedure with the corresponding enlarged modular group. Since by the generalized accessibility of M. Bestvina and M. Feighn [Be-Fe1] there is a global bound on the combinatorics of small graphs of groups associated with L, such a refinement procedure has to terminate in a finite time. Hence, for the rest of our analysis we may assume that none of the edge groups in each of the graphs of groups Λ_i is trivial, and that at least one of the cyclic edge groups in Λ_L is not elliptic in at least one of the graphs of groups Λ_i.

Suppose that there exists a cyclic edge group Z_e in Λ_L which is not elliptic in one of the graphs of groups Λ_i. Then Z_e is not elliptic in the graph of groups Λ_{L^1} of L^1. By our assumptions, both graphs of groups Λ_L and Λ_{L^1} contain spanning trees where all edge groups are non-cyclic abelian. Let $D^1 < L^1$ be the fundamental group of the maximal subgraph of Λ_{L^1} in which all edge groups are non-cyclic abelian, and let $M < L$ be the fundamental group of the subgraph obtained from Λ_L by taking out the edge stabilized by Z_e. By construction, the graph of groups Λ_L collapses to a graph of groups Θ_L with a single vertex stabilized by M and single edge stabilized by the cyclic group Z_e.

Each of the vertex groups in the graph of groups Λ_{L^1}, being a subgroup of L, inherits a graph of groups from Λ_L. We claim that at least one of these graphs of groups gives rise to a free decomposition of the corresponding vertex group of Λ_{L^1}. Suppose this is not the case, i.e., the stabilizers of all edges in the graphs of groups inherited by the vertex groups of Λ_{L^1} from the graph of groups Λ_L are non-trivial. Since the cyclic group Z_e cannot be conjugated into any of these vertex groups by assumption, each of these vertex groups can be conjugated into the vertex groups M in the graph of groups Θ_L, which implies that the subgroup $D^1 < L^1$ can be conjugated into M as well. By assumption, Λ_{L^1} collapses to a graph of groups with one vertex group stabilized by D^1 and a bouquet of circles with cyclic stabilizers based on this vertex. Since none of the cyclic stabilizers of these circles can be conjugated into the cyclic group Z_e, if D^1 can be conjugated into M then the ambient group $L = L^1$ can be conjugated into M as well, a contradiction. Hence, at least one of the vertex groups in Λ_{L^1} inherits a graph of groups containing at least one edge with a trivial stabilizer, so the graphs of groups inherited by all the vertex groups in Λ_{L^1} can be combined to a graph of groups of D^1, which gives rise to a free decomposition of D^1. Clearly, $b_1(D^1) \leq b_1(L)$, so the first betti number of each of the factors in the free decomposition of D^1, obtained from the graph of groups Λ_L, is strictly smaller than the first betti number of L. Therefore, in case there exists a cyclic edge group Z_e in Λ_L which is not elliptic in Λ_{L^1}, we proceed with each of the factors of D^1 separately.

Since at each step of our inductive procedure we either properly refine a given graph of groups of a limit group, replace a limit group by its proper quotient, or continue with limit groups which have strictly smaller betti number than the betti number of the limit group in the previous step of the procedure, the inductive procedure has to terminate.

Starting with the limit group L and its action on the real tree Y, and continuing with actions of point and edge stabilizers inherited from the actions of the groups along the inductive process, we may combine this (decreasing) sequence of actions and obtain a free action of the limit group L on some R^s tree, where s is the number of layers in the inductive process. ∎

With Theorem 1.3, we can complete the proof of Theorem 1.2 in the general case. Recall that if the set of rigid homomorphisms, from a rigid limit group $Rgd(x,p,a)$ to F_k, is not R-AP-covered for any integer R, there exists a sequence of rigid homomorphisms $\{h_n | h_n \colon Rgd(x,p,a) \to F_k\}$, so that for every integer n, the n-th homomorphism in the sequence, h_n, is not n-AP-covered. By Theorem 1.3 we can further assume that the sequence $\{h_n\}$ converges into a free action of some limit group L on some (pointed) R^s-tree (U,u_0), where L is a quotient of the rigid group $Rgd(x,p,a)$. Let $U_{AP} \subset U$ be the convex hull of the images of the base point u_0 under the action of the subgroup $AP =< a,p >$ of our rigid graded limit group $Rgd(x,p,a)$.

With the action of the limit group L on the R^s-tree U, we can naturally associate (stable) actions of corresponding limit groups (point stabilizers) on the R-trees associated with the various levels of the R^s-tree U.

We start with the highest level of the R^s-tree U, and get a stable action of the ambient limit group L on a (pointed) real tree Y. Let Λ_L be the abelian decomposition of L associated with its action on the real tree Y, and let Y_{AP} be the convex hull of y_0 under the action of the subgroup AP.

We have already argued that if there exists a non-degenerate segment $J \subset Y$ which does not contain a non-degenerate subsegment that is in the orbit of Y_{AP}, then there exists a proper subgraph of groups $\Lambda' \subset \Lambda_L$ such that the fundamental group Λ' contains the subgroup $AP =< a,p >$. This contradicts the rigidity of the homomorphisms $\{h_n\}$. Hence, we may assume that the orbit of Y_{AP} in Y intersects any non-degenerate segment in the real tree (Y, y_0) in a non-degenerate segment.

Note that in this case, the axis of a boundary element of a QH subgroup associated with an IET component in Y is contained in the orbit of the subtree U_{AP} of the R^s-tree U. Furthermore, the axis of an axial component in the real tree Y is contained in the orbit of the subtree U_{AP}, hence the axis of the (abelian) subgroup that stabilizes an axial component of Y (which is the axis of the axial component) is contained in the subtree U_{AP}.

Suppose that there exists a germ of segments in the real tree Y that does not intersect the orbit of the subtree U_{AP}, i.e., suppose that there exists a non-degenerate segment J in the R^s-tree U that is infinitesimal compared to the length of (non-degenerate) segments in the (top level) real tree Y, and the translation length of elements in point stabilizers in Y are infinitesimal compared to the length of the segment J, and the segment J does not intersect the orbit of the subtree U_{AP} in the R^s-tree U.

In case there exists a germ of segments in the real tree Y that does not intersect the orbit of the subtree Y_{AP}, we can use the fact that boundary elements of IET components and stabilizers of axial components are contained in the subtree U_{AP}, and further refine the abelian decomposition Λ_L, associated with the action of the limit group L on the real tree Y, to obtain a graph of groups Δ_L, where the new edges in the refined graph of groups are associated with the germs in Y that do not intersect the orbit of the subtree U_{AP}. Since the germs do not intersect the orbit of U_{AP}, the subgroup AP is contained in the fundamental group of a proper graph of groups Δ'_L, i.e., a connected component of the graph of groups Δ_L that does not contain the new edges associated with germs which do not intersect the orbit of U_{AP}. Clearly, for large enough n, this implies that the homomorphisms $\{h_n\}$ are flexible, a contradiction.

Hence, we may assume that the orbit of Y_{AP} in Y intersects any non-degenerate segment in the real tree (Y, y_0) in a non-degenerate segment, and it intersects any germ of segments in (Y, y_0).

Let $V_1, \ldots, V_q$ be the (conjugacy classes of) point stabilizers in the real tree Y. Since the homomorphisms $\{h_n\}$ converge into an action of the limit group L on the R^s-tree U, the restriction of the homomorphisms $\{h_n\}$ to each of the subgroups V_i converges into a (stable) action of the subgroups V_i on R^m-trees U_i, where $m \leq s-1$.

We now look at the action of each of the subgroups V_i on the top level of the R^m-trees U_i, which are R-trees that we denote Y_i. If at least one of the trees Y_i contains a non-degenerate segment that does not intersect the orbit of the subtree U_{AP} (of the ambient R^s-tree U) in a non-degenerate segment, then the graph of groups Λ_L (with fundamental group L) can be refined to a graph of groups Λ_L^1 of L so that the subgroup AP is contained in the fundamental group of a proper subgraph of the graph of groups Λ_L^1. Therefore, for large enough n, each homomorphism h_n is a flexible homomorphism, a contradiction.

If a germ in one of these R-trees Y_i does not intersect the orbit of U_{AP}, then the same conclusion holds, and we get a contradiction.

In case every non-degenerate segment in the real trees Y_i intersects the orbit of the tree U_{AP} in a non-degenerate subsegment, and every germ in the trees Y_i intersects the orbit of U_{AP}, we continue iteratively. At each step we analyze the actions of each of the point and edge stabilizers in the real trees associated with the corresponding level in the R^s-tree U. If there exists a non-degenerate segment in one of these trees that does not intersect the orbit of the tree U_{AP} in a non-degenerate segment, or there exists a germ in these trees that does not intersect U_{AP}, then we get a graph of groups from that level of the R^s-tree U that can be extended to a graph of groups with fundamental group L, and the subgroup AP is contained in a proper subgraph of the obtained graph of groups, a contradiction to the rigidity of the given set of homomorphisms $\{h_n\}$. Otherwise, if every non-degenerate segment, and every germ, intersect the orbit of the tree U_{AP}, we continue to the next level.

Since, for each index n, the homomorphism h_n is not n-AP covered, in one of the steps of the iterative process, at least in one of the trees corresponding to a point stabilizer, there must exist a non-degenerate segment containing no non-degenerate segments that are contained in the orbit of the subtree U_{AP}, or a germ of segments that does not intersect U_{AP}. Hence, at some level of the R^s-tree U, we get an abelian decomposition that contradicts the rigidity of the homomorphisms $\{h_n\}$ for large enough n, and the proof of Theorem 1.2 follows. ∎

In a similar way to rigid homomorphisms of rigid graded limit groups, we analyze solid homomorphisms of solid graded limit groups. We start by defining a **solid family** of specializations of a solid graded limit group.

Definition 1.4: Let $Sld(x,p,a)$ be a solid limit group and let Λ_S be its graded abelian JSJ decomposition (with respect to the subgroup $AP =< a,p >$). We define an equivalence relation, $h_1 \sim h_2$, on the set of homomorphisms $\{h | h: Sld(x,p,a) \to F_k\}$ that is generated by:

(i) $h_2 = h_1 \circ \varphi$, where $\varphi \in GMod(Sld(x,p,a))$ (see definition 10.1 in [Se1], for the definition of the graded modular group associated with a graded limit group).

(ii) Let E be an edge group in Λ_S, the graded abelian JSJ decomposition of $Sld(x,p,a)$, or the cyclic subgroup associated with an essential s.c.c. in a QH vertex group in Λ_S. Suppose that the abelian decomposition Λ_S (or its refinement obtained by cutting a QH vertex group along the corresponding s.c.c.) collapses to the (elementary) abelian decomposition $B *_E D$, where $AP < B$. $h_2 \sim h_1$ if the restrictions of h_1 and h_2 to the subgroup B are identical, and the restrictions of h_2 to the subgroup D is obtained from the restriction of h_1 to D by post composing it with an inner automorphism of F_k corresponding to an element in the maximal cyclic subgroup containing $h_1(E)$.

If the abelian decomposition Λ_S collapses to the elementary decomposition $B*_E$, where $AP < B$, we require the restrictions of h_1 and h_2 to B to be identical, and if $t \in Sld(x,p,a)$ is the Bass–Serre element corresponding to the given elementary (HNN) decomposition, then $h_2(t) = h_1(t)u$, where u belongs to the maximal cyclic subgroup containing $h_1(E)$.

(iii) Let Ab be an abelian vertex group in Λ_S, so that the vertex stabilized by Ab is not the distinguished vertex stabilized by the subgroup AP. Suppose that $Ab = M + B$, where the subgroup generated by the edge groups connected to the vertex stabilized by Ab is of finite index in M. $h_1 \sim h_2$ if the restrictions of h_1 and h_2 to M, to all the other vertex groups and to the Bass–Serre elements in Λ_S are identical.

We say that a homomorphism h: $Sld(x,p,a) to F_k$ is *solid* if it is non-degenerate (i.e., it does not factor through the singular locus associated with $Sld(x,p,a)$ — see definition 11.1 in [Se1]), and no homomorphism in its equivalence class factors through a flexible quotient of $Sld(x,p,a)$ (see definition 10.3 in [Se1] for a flexible quotient). We call the equivalence class of a solid homomorphism its *solid family*.

Solid families do not seem to be definable subsets. For the purposes of analyzing first order predicates, we need to introduce a slightly coarser equivalence relation on homomorphisms of solid limit groups, and replace solid families by **strictly solid families**. Each strictly solid family is a finite union of solid families, but in general not every solid family is contained in a strictly solid family.

Definition 1.5: Let $Sld(x,p,a)$ be a solid graded limit group and let

$$Flx_1(x,p,a),\ldots,Flx_s(x,p,a)$$

be the canonical collection of (maximal) flexible quotients of the solid graded limit group $Sld(x,p,a)$.

With each flexible quotient, $Flx(x,p,a)$, we associate a natural one-step resolution given by the quotient map $Sld(x,p,a) \to Flx(x,p,a)$, and the graded abelian JSJ decomposition of the solid limit group $Sld(x,p,a)$, Λ_S. Given the one-step resolution, we associate with it the entire collection of homomorphisms that factor through the one-step resolution, and the limit group corresponding to that collection which we denote $Q(x,p,a)$, which is a quotient of the solid limit group $Sld(x,p,a)$.

Clearly, the limit group $Q(x,p,a)$ inherits a graded abelian decomposition, Δ_Q, from the solid limit group $Sld(x,p,a)$, which is in general either similar to the abelian decomposition Λ_S (i.e., it has isomorphic QH and abelian vertex and edge groups, and the other (rigid) vertex groups in Δ_Q are quotients of the corresponding ones in Λ_S), or it is a degeneration of the abelian decomposition Λ_S, associated with $Sld(x,p,a)$. By construction, the one-step resolution given by the quotient map $Q(x,p,a) \to Flx(x,p,a)$, and the graded abelian decomposition Δ_Q of $Q(x,p,a)$, is a strict resolution (see definition 5.11 in [Se1] for a strict resolution).

Every strict resolution can be (tautologically) viewed as a well-structured resolution, by associating with each level a trivial free decomposition (see definition 1.11 in [Se2] for a well-structured resolution). With the one-step, strict (well-structured) resolution $Q(x,p,a) \to Flx(x,p,a)$, with the abelian decomposition Δ_Q, we associate its completion, according to the construction that appears in definition 1.12 of [Se2]. We denote the obtained completion, $Comp_{Flx}$.

By the construction of the completion, $Comp_{Flx}$, the solid limit group $Sld(x,p,a)$ is naturally mapped into it (onto the subgroup associated with its top level, that is isomorphic to $Q(x,p,a)$ — see definition 1.12 in [Se2]). Let $\tau\colon Sld(x,p,a) \to Comp_{Flx}$ be this natural map.

A specialization (x_0,p_0,a) of the solid graded limit group $Sld(x,p,a)$ is called a **strictly solid specialization** if it is non-degenerate (definition 11.1 in [Se1]), and for every index i, $1 \le i \le s$, it does not factor through the homomorphisms $\tau_i\colon Sld(x,p,a) \to Comp_{Flx_i}$, associated with the (finitely many) maximal flexible quotients of the solid limit group, $Sld(x,p,a)$. Note that a strictly solid specialization is, in particular, a solid specialization.

On the set of strictly solid specializations of $Sld(x,p,a)$ we define an equivalence relation which is coarser than the equivalence relation defined on solid specializations (Definition 1.4).

With the solid limit group, $Sld(x,p,a)$, we associate a (tautological) one-step resolution, $Sld(x,p,a) \to Sld(x,p,a)$, with the identity map between the two levels, and the abelian decomposition Λ_S associated with the top level copy of $Sld(x,p,a)$. With this tautological one-step resolution, we associate its completion ([Se2], 1.12), which we denote $Comp_T$.

The solid limit group $Sld(x,p,a)$ admits two natural embeddings into the completion, $Comp_T$, $\eta_i\colon Sld(x,p,a) \to Comp_T$, $i=1,2$, corresponding to the limit groups that appear in the top and bottom levels of the one-step resolution. We define an equivalence relation on the set of homomorphisms $h\colon Sld(x,p,a) \to F_k$. We say that two homomorphisms $h_j\colon Sld(x,p,a) \to F_k$, $j=1,2$, are in the same equivalence class if there exists a homomorphism $u\colon Comp_T \to F_k$ so that $h_1 = u \circ \eta_1$ and $h_2 = u \circ \eta_2$ (this relation is clearly reflexive, and can be easily seen to be symmetric and transitive).

Note that if h is a strictly solid homomorphism, then all the non-degenerate homomorphisms in its equivalence class are strictly solid as well. We call the equivalence class of a strictly solid homomorphism a *strictly solid family*.

In the case of rigid limit groups, we were able to combinatorially bound all its rigid homomorphisms, by showing they are all R_0-AP-covered, for some integer R_0. In studying solid limit groups, it is not possible to get a similar combinatorial bound for all solid or even strictly solid homomorphisms. However, in Theorem 1.7 we prove the existence of such a combinatorial bound for strictly solid homomorphisms that are the shortest in their solid family.

Definition 1.6: Let $Sld(x,p,a)$ be a solid graded limit group, let X be the Cayley graph of the solid graded limit group $Sld(x,p,a)$ with respect to the generating set $Sld(x,p,a) =< x,p,a >$, and let T be the Cayley graph (tree) of the free group F_k with respect to the generating set $F_k =< a >$.

With each abelian vertex group A_i in the graded abelian JSJ decomposition of the solid limit group $Sld(x,p,a)$, we associate a fixed retraction $r_i\colon A_i \to A_i^1$, where A_i^1 is the direct summand of A_i that contains the subgroup generated by the edge groups connected to the vertex stabilized by A_i as a subgroup of finite index. Clearly, the collection of retractions $\{r_i\}$ extends to a retraction $r\colon Sld(x,p,a) \to Sld(x,p,a)$ which restricts to the identity on all the non-abelian vertex groups, and to the retraction r_i on the abelian vertex group A_i.

We say that a solid homomorphism $h\colon Sld(x,p,a) \to F_k$ is among the shortest in its solid family if $h = h \circ r$, and the image of the generating set of $Sld(x,p,a)$ in the free F_k is the shortest among all homomorphisms $h_1\colon Sld(x,p,a) \to F_k$ that are in the same solid family of the homomorphism h, and for which $h_1 = h_1 \circ r$.

In the case of a solid limit group we are able to bound the combinatorial types of strictly solid solutions which are among the shortest in their solid families.

THEOREM 1.7: *Let $Sld(x,p,a)$ be a solid graded limit group, and let $r\colon Sld(x,p,a) \to Sld(x,p,a)$ be a retraction composed from retractions of the abelian vertex groups in the graded abelian JSJ decomposition of $Sld(x,p,a)$ as described in Definition 1.6.*

Then there exists a constant R_0 so that every strictly solid homomorphism $h\colon Sld(x,p,a) \to F_k$, which is among the shortest in its solid family, is R_0-AP-covered.

Proof: Our argument is a modification of the one used to study rigid solutions of rigid limit groups (Theorem 1.2). Suppose that there is no such constant R_0. Let $\{h_n | h_n\colon Sld(x,p,a) \to F_k\}$ be a sequence of strictly solid homomorphisms, so that for every index n, h_n is one of the shortest homomorphisms in its corresponding solid family, and so that for every index n, the n-th homomorphism in the sequence, h_n, is not n-AP-covered. By Theorem 1.3, by passing to a subsequence, we can assume that the sequence of homomorphisms (still denoted) $\{h_n\}$ converges into a free action of some limit group L (a quotient of the solid limit group $Sld(x,p,a)$) on an R^s-tree (U,u_0), for some integer s. Let U_{AP} be the subtree of the R^s-tree, U, which is the convex hull of the orbit of the base point u_0 under the action of the subgroup $AP =< a, p >$.

Since for every index n the homomorphism h_n is not n-AP-covered, by analyzing the levels of the R^s-tree (U, u_0) iteratively from top to bottom, exactly as we did in the proof of Theorem 1.2, from the action of the limit group L on the R^s-tree U we get a decomposition (graph of groups), Λ_L, with fundamental group L, so that the subgroup $AP =< a, p >$ is contained in the fundamental group of a proper subgraph $\Lambda' \subset \Lambda_L$.

Since the homomorphisms $\{h_n\}$, from which the action of the limit group L on the R^s-tree U is constructed, are assumed to be strictly solid homomorphisms, each of the homomorphisms h_n does not factor through an epimorphism onto a free group of the form $F_k* < v >$, an epimorphism in which the subgroup AP is mapped into F_k, and the limit group L is mapped onto $F_k* < v >$ ([Se1], 10.3). Hence, by the analysis of the R^s-tree U presented in the proof of Theorem 1.2, that eventually constructs the decomposition Λ_L of the limit group L, every vertex group which is in Λ_L and not in Λ' corresponds to either a point stabilizer in one of the real trees associated with a level in the R^s-tree U, an IET component or an axial component that does not intersect the orbit of the subtree U_{AP} in a non-degenerate segment.

Every edge in Λ_L that is not in Λ' and is not connected to an IET or an axial vertex group in Λ_L is either an edge in the discrete part of some real tree associated with a level of the R^s-tree U, that does not intersect the subtree U_{AP} and its stabilizer is necessarily a non-trivial abelian subgroup of L, or it is a germ with non-trivial (abelian) stabilizer in such a real tree that does not intersect the subtree U_{AP}.

Let Δ_L be the graph of groups obtained from Λ_L by collapsing Λ' to a vertex. In Δ_L the subgroup AP is contained in a vertex stabilizer, each edge group is a non-trivial abelian subgroup of L and each vertex group that does not contain a conjugate of AP corresponds to either a point stabilizer, an IET component, or an axial component in some real tree associated with a level in the R^s-tree U.

Let η: $Sld(x, p, a) \to L$ be the canonical quotient map. Suppose that the quotient map η maps every edge group and every vertex group that is not quadratically hanging in the graded abelian JSJ decomposition of $Sld(x, p, a)$, into a conjugate of an edge group or into a conjugate of a vertex group which does not correspond to an IET or an axial component in the graph of group Δ_L. Then, by the analysis of the R^s-tree U and the construction of the decompositions Λ_L and Δ_L, for large enough n, either the homomorphism h_n factors through a free group of the form $F_k * < v >$, where the subgroup AP is mapped into the subgroup F_k, a possibility that contradicts our assumption that the homomorphisms h_n are strictly solid, or there exists a homomorphism $\hat{h}_n: Sld(x,p,a) \to F_k$ which is in the same solid family as the homomorphism h_n, $\hat{h}_n \circ r = \hat{h}_n$, and $\hat{h}_n$ is strictly shorter than h_n, a contradiction to the assumption that the homomorphisms h_n are among the shortest homomorphisms in their solid families.

Therefore, we may assume that there exists either an edge group or a non-QH vertex group in the graded abelian JSJ decomposition of $Sld(x,p,a)$, which is not mapped by the quotient map η either into a conjugate of an edge group or into a conjugate of a non-abelian, non-QH vertex group in Δ_L. In this case we argue that for large enough n, the homomorphisms h_n are not strictly solid, which contradicts our assumptions.

Definition 1.8: Let $G(x,p,a)$ be a graded limit group, and let Θ be a graded decomposition of $G(x,p,a)$ (i.e., an abelian graph of groups in which the subgroup $AP =< a,p >$ is contained in a vertex group). We say that Θ^{ess} is the **essential** decomposition associated with Θ, if Θ^{ess} is obtained from Θ by first applying the following folding operations:

(i) If e is an edge between two non-QH, non-abelian vertex groups in Θ, we fold Θ so that the edge group associated with the edge e in Θ^{ess} is the maximal abelian subgroup of $G(x,p,a)$ containing the edge group associated with e in Θ.

(ii) If e is an edge between a non-QH vertex group and an abelian vertex group Ab in Θ, we fold Θ so that the edge group associated with the edge e in Θ^{ess} is the summand in Ab that contains the subgroup generated by the edge groups connected to it as a subgroup of finite index.

Finally, after applying the above folding operations, we change the obtained decomposition to be reduced (i.e., we contract an edge connecting an abelian vertex group and a non-QH vertex group if the edge group is the same as the abelian vertex group).

We say that a graded decomposition Θ of a graded limit group $G(x,p,a)$ is **essential** if $\Theta^{ess} = \Theta$.

PROPOSITION 1.9: *Let $Sld(x,p,a)$ be a solid graded limit group, let Λ_S be the graded abelian JSJ decomposition of $Sld(x,p,a)$, and let*

$$\{v_n | v_n: Sld(x,p,a) \to F_k\}$$

be a sequence of homomorphisms that converge to a limit group L. Let ν: $Sld(x,p,a) \to L$ be the natural quotient map.

Suppose that there exists an essential graded abelian decomposition Δ_L of L, for which there exists either an edge group or a non-abelian, non-QH vertex group in the graded abelian JSJ decomposition of $Sld(x,p,a)$, Λ_S, which is not mapped by the quotient map ν either into a conjugate of an edge group or into a conjugate of a non-abelian, non-QH vertex group in Δ_L, and if it is mapped by ν into a conjugate of an abelian vertex group in Δ_L then it is not mapped into a conjugate of the direct summand of this abelian vertex group that contains the subgroup generated by the edge groups connected to the abelian vertex group as a subgroup of finite index.

Then for large enough n, the homomorphisms v_n: $Sld(x,p,a) \to F_k$ are not strictly solid.

Proof: With the (graded) limit quotient L of the solid limit group $Sld(x,p,a)$, and its essential graded abelian decomposition Δ_L, we associate a natural one-step resolution given by the identity map $L(x,p,a) \to L(x,p,a)$, and the graded abelian decomposition Δ_L (associated with the top level). We denote the limit group associated with the top level in this resolution, L_t, and with the bottom level, L_b.

This one-step resolution is clearly a strict resolution (definition 5.11 in [Se1]). Every strict resolution can be (tautologically) viewed as a well-structured resolution, by associating with each level a trivial free decomposition (see definition 1.11 in [Se2] for a well-structured resolution). With the one-step, strict (well-structured) resolution $L_t \to L_b$, with the abelian decomposition Δ_L associated with its top level, we associate its completion, according to the construction that appears in definition 1.12 of [Se2]. We denote the obtained completion, $Comp_L$. By the basic properties of the construction of the completion ([Se2], 1.12), the limit groups associated with the two levels of the resolution with which we started are embedded into the completion. We denote their images, L_t and L_b as well.

With the one-step resolution, $L_t \to L_b$, and its completion $Comp_L$, we associate the collection of (graded) test sequences (see definitions 1.20 and 3.1 in [Se2] for the conditions test sequences satisfy), $\{h_n\colon Comp_L \to F_k\}$, for which for every index n, the restriction of h_n to the image of the bottom limit group in the completion $Comp_L$, $h_n(L_b)$, is the given homomorphism v_n.

The (graded) limit group L is a quotient of the solid limit group $Sld(x,p,a)$. The quotient map ν: $Sld(x,p,a) \to L$ naturally gives two maps of $Sld(x,p,a)$

into the completion, $Comp_L$, ν_1: $Sld \to L_t$ and ν_2: $Sld \to L_b$. Given a test sequence $\{h_n\colon Comp_L \to F_k\}$, we look at the sequence of specializations of $Sld(x,p,a)$, $\{h_n \circ \nu_1\colon Sld \to F_k\}$. Since, by the assumptions of the lemma, there exists either an edge group or a non-abelian, non-QH vertex group in the graded abelian JSJ decomposition of $Sld(x,p,a)$, Λ_S, which is not mapped by the quotient map ν either into a conjugate of an edge group or into a conjugate of a non-abelian, non-QH vertex group in Δ_L, and if it is mapped into an abelian vertex group in Δ_L then it is not mapped into the direct summand that contains the subgroup generated by the edge groups connected to that vertex group as a subgroup of finite index, the conditions a graded test sequence satisfy ([Se2], 3.1) imply that for large enough n, the translation length of the image (under $h_n \circ \nu_1$) of an element in either an edge group or a non-abelian, non-QH vertex group in Λ_S is much bigger than the translation lengths of the (image of) elements in a (given) generating set of the defining parameters. Hence, for large enough n, the specializations $h_n \circ \nu_1(Sld)$ are not strictly solid.

Since for large enough n, the specializations $\{h_n \circ \nu_1\}$ are not strictly solid, they have to satisfy one of the (finitely many) Diophantine conditions associated with the (finitely many) maximal flexible quotients of $Sld(x,p,a)$ that are specified in Definition 1.5.

At this point we are finally ready to apply the techniques used for generalizing Merzlyakov's theorem and construct formal solutions in [Se2], to analyze the entire collection of test sequences and their algebraic envelopes. We look at the collection of test sequences $\{h_n\colon Sld \to F_k\}$, for which (as before) $h_n \circ \nu_2 = v_n$, and $h_n \circ \nu_1$ satisfies a fixed Diophantine condition (one from the finite possible list specified in Definition 1.5), which demonstrates that these specializations of the solid limit group $Sld(x,p,a)$ are not strictly solid. Hence, each homomorphism from a given test sequence, $h_n\colon Comp_L \to F_k$, can be extended to (an algebraic envelope) homomorphism $g_n\colon Comp_L *_{AP} Comp_{Flx} \to F_k$. As we did in [Se2], we set the extension g_n of h_n so that the specialization of $g_n(Comp_{Flx})$ is the shortest possible (with respect to a fixed generated set).

As we did in analyzing graded formal limit groups in section 3 of [Se2], we look at the collection of all the convergent sequences $\{g_n\}$. Each such sequence converges into some limit group (a quotient of $Comp_L *_{AP} Comp_{Flx}$), and by the ascending chain condition for limit quotients, with the collection of these limit groups we can associate (canonically) a finite collection of maximal limit groups $M_1, \dots, M_r$. By the techniques used in analyzing graded formal limit groups (see theorem 3.7 in [Se2]), each maximal limit group M_i, $1 \leq i \leq r$, has the structure $M_i = E_i *_{Term} GCl_i(Comp_L)$, where $GCl_i(Comp_L)$ is a graded formal closure of the completion, $Comp_L$, and L_b is embedded in the subgroup $Term$.

Suppose that there exists an infinite sequence of indices (still denoted n) for which the specializations v_n of $Sld(x,p,a)$ are strictly solid. Since there are only finitely many maximal limit groups through which all the convergent sequences $\{g_n\}$ factor, there are only finitely many coefficients in the Diophantine equations (associated with the corresponding graded closures) that represent elements in abelian vertex groups in $Comp_L$ in terms of corresponding (extended) abelian vertex groups in $GCl_i(Comp_L)$. Hence, the least common multiple of these coefficients is also bounded. Therefore, we may look at finitely many test sequences, $\{h_n^1,\dots,h_n^d\}$, in which the orders of fixed sets of generators of the various abelian groups in $Comp_L$ go over all the possibilities for classes of values modulo the least common multiple of the coefficients in the (finitely many) Diophantine equations associated with $M_1,\dots,M_r$.

By passing to a subsequence, we may assume that the associated sequences $\{g_n^1,\dots,g_n^d\}$ converge and factor through the maximal limit groups $M_{j(1)},\dots,M_{j(d)}$. Now, since the test sequences, $\{h_n^1,\dots,h_n^d\}$, were chosen to go over all the classes modulo the least common multiple of the coefficients in the finitely many Diophantine equations, for each index n, the specializations of the abelian vertex groups associated with the given homomorphism v_n are in the same class modulo the least common multiple as the specializations $h_n^{i(n)} \circ \nu_1$, for some index $i(n)$, $1 \leq i(n) \leq d$. Therefore, for each index n, we can vary the homomorphism $g_n^{i(n)}$ to a homomorphism $\hat{g}_n^{i(n)}\colon M_{j(i(n))} \to F_k$, so that

$$v_n = \hat{g}_n^{i(n)} \circ \nu_2 = \hat{g}_n^{i(n)} \circ \nu_1;$$

but by construction $\hat{g}_n^{i(n)} \circ \nu_1$ is not strictly solid (since it satisfies one of the Diophantine conditions associated with the flexible quotients of $Sld(x,p,a)$ specified in Definition 1.5), a contradiction to the assumption that the homomorphisms v_n are strictly solid. ∎

Since the abelian decomposition Δ_L, obtained from the action of the limit group L on the R^s-tree U is essential by construction, Proposition 1.9 implies that for large enough n, the homomorphisms in the sequence $\{h_n\}$, that were assumed to be not n-AP-covered and strictly solid, are not strictly solid homomorphisms of the solid limit group $Sld(x,p,a)$, a contradiction, that finally completes the proof of Theorem 1.7. ∎

2. A bound on the number of rigid and solid solutions

In the previous section we have shown that it is possible to combinatorially bound the entire collection of rigid homomorphisms of a rigid limit group, and strictly solid homomorphisms of a solid limit group that are the shortest in their solid families. For a rigid (solid) graded limit group we have shown the existence of a constant R_0, so that all the rigid homomorphisms of the rigid limit group (strictly solid homomorphisms of the solid limit group that are the shortest in their solid families) are R_0-AP-covered (Theorems 1.2 and 1.7). In this section we use this combinatorial bound to get a global bound on the number of possible distinct rigid homomorphisms of a rigid limit group for which the specializations of the defining parameters are identical, and a global bound on the number of distinct strictly solid families of homomorphisms of a solid limit group for which the specializations of the defining parameters are identical.

The existence of a global bound on the number of rigid (strictly solid families of) homomorphisms plays an essential role in our approach to the quantifier elimination of predicates defined over a free group, and the argument we use to prove the existence of the bound plays an essential role in proving the termination of the quantifier elimination procedure in the sixth paper in the sequence. We first consider rigid limit groups and their rigid specializations and then generalize the argument to study strictly solid homomorphisms. Mainly for presentation purposes, we first prove the existence of a bound on the number of rigid solutions assuming a bound on the *periodicity* of the specialization p_0 of the defining parameters p. Later on, we modify the argument to get a global bound on the number of rigid solutions without the periodicity assumption. Note that the removal of the periodicity assumption in studying the set of solutions to a system of equations over a free group was a major technical difficulty in Makanin's work [Ma], and became possible only with Razborov's thesis [Ra].

Definition 2.1: Let $w \in F_k$. The maximal integer s for which w can be written graphically (that is without cancellations) as $w = u_1 \alpha^s u_2$ for some $u_1, u_2, \alpha \in F_k$, is called the **periodicity** of the word w.

THEOREM 2.2: *Let $Rgd(x,p,a)$ be a rigid graded limit group. For every integer s there exists a bound $b_s(Rgd(x,p,a))$ for which for any particular value of the defining parameters p_0 with periodicity bounded by s, there are at most b_s distinct rigid homomorphisms h: $Rgd(x,p,a) \to F_k$ satisfying $h(p) = p_0$.*

Proof: Suppose that for some given integer s, there is no bound on the number of distinct rigid specializations x (with the same specializations of the defining parameters) for all possible specializations p_0 of the defining parameters p with periodicity bounded by s. Then there must exist some sequence $\{p_n\}$ of specializations of the defining parameters p with periodicity bounded by s, so that for every index n, the rigid graded limit group $Rgd(x,p,a)$ admits n distinct rigid specializations $(x_1^n,p_n,a),\dots,(x_n^n,p_n,a)$.

Let B be a ball in the Cayley graph of $Rgd(x,p,a)$, so that the ball B contains all the defining relations of the rigid graded limit group $Rgd(x,p,a)$. By Theorem 1.2 there exists some constant $\hat{R}$, so that for any rigid specialization (x_0,p_0,a) the union of the paths in the tree T (the Cayley graph of the free group F_k), that are the images of the edges in the ball B (the Cayley graph of $Rgd(x,p,a)$), are $\hat{R}$-AP covered (i.e., the images in F_k of edges associated with the defining parameters and with the coefficients in the ball of radius $\hat{R}$ in the Cayley graph of $Rgd(x,p,a)$, cover the image of the ball B in F_k).

We set the constant R to be $R = 2\cdot\hat{R}$. For each rigid homomorphism $h\colon Rgd(x,p,a)\to F_k$, we look at the ball of radius R in the Cayley graph of $Rgd(x,p,a)$, which we denote B_R, and its image, $h(B_R)$, in the Cayley graph of the free group F_k, the tree T.

Definition 2.3: We call the finite subtree of T, which is the union of the images of the edges in B_R under a homomorphism $h\colon Rgd(x,p,a)\to F_k$, an R-**state**. The **combinatorial type** of an R-state is the combinatorial (not metric!) finite tree corresponding to the R-state, together with notation for the image of the corresponding vertices and edges of the ball B_R.

Since the number of edges and vertices in the ball B_R is fixed, there exist only finitely many combinatorial types of R-states. Hence, by passing to a subsequence, we may further assume that for every index n, the combinatorial types of the R-states associated with the specializations $(x_1^n,p_n,a),\dots,(x_n^n,p_n,a)$ are identical.

Furthermore, by passing to a subsequence (and still keeping the same notation), we may also assume that for the metric obtained after rescaling the metric on the R-states associated with the rigid specializations $\{(x_i^n,p_n,a)\}$ (*rescaled* R-states), by dilatation constants depending only on the specializations of the defining parameters p_n (say, by rescaling the maximal length of a generator of the defining parameter subgroup to be 1), for every sequence of indices $\{i_n|1\le i_n\le n\}$, the (rescaled) R-states associated with the sequence of specializations $\{(x_{i_n}^n,p_n,a)\}$ converge in the Gromov–Hausdorff topology on metric spaces, and the limit (finite, non-trivial) tree Y is independent of the particular choice of indices $\{i_n\}$.

We have chosen the sequence of specializations $\{(x^n_{i_n}, p_n, a)\}$ to have the same combinatorial type of R-state. We denote by $Mp^n_{i_n}$ the maximal length of a segment between branching points in the rescaled R-state corresponding to the specialization $(x^n_{i_n}, p_n, a)$ which is mapped to a point in the (finite) limit tree Y (note that this length may be 0). For every two specializations $(x^n_{i_{n_1}}, p_n, a), (x^n_{i_{n_2}}, p_n, a)$ we have associated rescaled R-states $State_1$ and $State_2$ in correspondence. Given a branching point in these states and a copy of one of the defining parameters that passes through this branching point, we can measure the difference in the placement of the branching point on the copy of the defining parameter in the two different rescaled R-states $State_1$ and $State_2$. We denote by $Mfluct^n_{i_{n_1}, i_{n_2}}$ the maximum among all these (finite set of) differences. Note that by construction both $Mp^n_{i_n}$ and $Mfluct^n_{i_{n_1}, i_{n_2}}$ grow asymptotically as $o(1)$.

Definition 2.4: We say that the sequence of specializations $\{(x^n_{i_n}, p_n, a)\}$ is **tame** with respect to segments in the rescaled R-state that correspond to edges in the real tree Y, if there exists a constant c and a subsequence of indices $\{n_m\}_{m=1}^{\infty}$, so that for every index m there are m distinct indices $1 \leq i_1 < \cdots < i_m \leq n_m$ for which

$$c \cdot Mp^{n_m}_{i_{j_1}} > Mfluct^{n_m}_{i_{j_2}, i_{j_3}}$$

for every possible indices j_1, j_2, j_3, $1 \leq j_1, j_2, j_3 \leq m$.

If the sequence of specializations $\{x^n_{i_n}, p_n, a)\}$ is tame, we continue with the tame subsequence of specializations. By possibly passing to a further subsequence, we may assume that the maximum $Mp^n_{i_{n_m}}$ is always obtained for the same segment in the corresponding rescaled R-state, that the maximal fluctuation $Mfluct^{n_m}_{i_1, i_2}$ grows asymptotically as $o(Mp^{n_m}_{i_j})$, and that the ratio between the segments in the R-state that grow asymptotically as $O(Mp^{n_m}_{i_j})$ converge.

If the original sequence of specializations is tame with respect to the segments in the rescaled R-state corresponding to non-degenerate segments in the limit tree Y (the "uncovered" segments), we check if the obtained subsequence is tame with respect to segments in the rescaled R-state corresponding to both non-degenerate segments in Y, and segments for which their length grow as the growth of $Mp^{n_m}_{i_j}$ (i.e., their length is approximately a fixed non-zero constant times the length of $Mp^{n_m}_{i_j}$). Note that in doing that we consider the limiting object to be an R^s-tree, and not just the R-tree Y. Since there are only finitely many segments in the (rescaled) R-state, and since there are m distinct solutions for each step m in our sequence, after finitely many "uncoverings" we obtain a sequence which is not tame with respect to the previously "uncovered" segments in the (rescaled) R-state.

Once we obtain a sequence of rigid specializations $\{(x^n_{i_n}, p_n, a)\}$ with no tame subsequences with respect to the previously "uncovered" segments in the corresponding (rescaled) R-states, we can choose a subsequence of indices $\{n_m\}_{m=1}^{\infty}$, so that for every index m there are m distinct indices $1 \leq i_1 < \cdots < i_m \leq n_m$ for which

$$m \cdot Mp^{n_m}_{i_{j_1}} < Mfluct^{n_m}_{i_{j_2}, i_{j_3}}$$

for all possible indices j_2, j_3, and an index j_1 for which the minimal $Mp^{n_m}_{i_j}$ is obtained. We call such a sequence of specializations a **perturbed** sequence.

Given a perturbed sequence, for each index m we construct the R-state corresponding to the specialization $(x^{n_m}_{i_{j(m)}}, p_{n_m}, a)$ for which $Mp^{n_m}_{i_{j(m)}}$ is minimal among all possible indices j. Clearly, given an R-state we can present each of the variables x in terms of *fractions* of (the given generating set of) the defining parameters. Hence, for every integer m we can present the specialization $(x^{n_m}_{i_{j(m)}}, p_{n_m}, a)$ in terms of fractions of the parameters p_{n_m} that appear in the R-state corresponding to this specialization, and for any other possible index j, we can represent the specialization $(x^{n_m}_{i_j}, p_{n_m}, a)$ in terms of the same fractions of the parameters p_{n_m} that appear in the R-state corresponding to the specialization $(x^{n_m}_{i_{j(m)}}, p_{n_m}, a)$, together with elements $u^m_r(j)$ that encode the fluctuations in the corresponding branching points along the variables x between the R-state associated with the specialization $(x^{n_m}_{i_j}, p_{n_m}, a)$ and the "reference" R-state associated with the specialization $(x^{n_m}_{i_{j(m)}}, p_{n_m}, a)$.

Each relation from the defining relations of the rigid graded limit group $Rgd(x, p, a)$ corresponds to a closed loop on the R-state, and since the combinatorial types of the R-states associated with all specializations from the perturbed sequence, $\{(x^{n_m}_{i_j}, p_{n_m}, a)\}$, are assumed to be identical, the "combinatorial type" of the loop corresponding to a given relation is identical for all the specializations from the perturbed sequence.

Furthermore, since we assume that the periodicity of the defining parameters is bounded, every part of a loop that starts at some point on any of the "uncovered" segments of the defining parameters, and gets back to the same point, represents the identity element in all R-states corresponding to the specializations, $(x^{n_m}_{i_j}, p_{n_m}, a)$, from our perturbed sequence.

Since for every index m such a part of a loop of a defining relation represents the identity element in the R-state associated with the (distinguished) specialization $(x^{n_m}_{i_{j(m)}}, p_{n_m}, a)$, and the same holds for every other R-state associated with a specialization $(x_{i_j}^{n_m}, p_{n_m}, a)$, every such part of a loop of one of the defining relations corresponds to a (parametric) equation involving (possibly) coefficients from the free group $F_k =< a >$, fractions of the given set of generators of the parameter subgroup that correspond to segments in the R-state associated with the (distinguished) specialization, $(x_{i_{j(m)}}^{n_m}, p_{n_m}, a)$, which have not been uncovered (that we denote $\hat{p}$), and variables corresponding to the "fluctuations" (that we denote u). Since the combinatorics of the R-states associated with all the specializations from our perturbed sequence is identical, the set of parametric equations corresponding to different parts of (closed) partial loops is identical for all specializations from the perturbed sequence. We denote this system of equations by $\Sigma_1(u, \hat{p}, a) = 1$.

To the system of equations $\Sigma_1(u, \hat{p}, a) = 1$ which correspond to the loops around each of the branching points, we add equations that guarantee partial equivariance, i.e., the additional equations guarantee that the images in F_k of all the edges, associated with the given set of generators x_i, in the ball B in the Cayley graph of $Rgd(x, p, a)$ (recall that the ball B was chosen so that it contains the loops associated with the defining relations of $Rgd(x, p, a)$), are identical. These equations can also be written in terms of the same fractions of the given generating set of the defining parameters (that we denoted $\hat{p}$), and the variables corresponding to the various fluctuations u that were used in formulating the equations corresponding to the loops around the branching points. We denote the system of equations that combine the equations corresponding to the loops around branching points and the equations coming from the partial equivariance condition by $\Sigma(u, \hat{p}, a) = 1$.

By our construction of maximal graded limit groups, there are finitely many maximal graded limit groups corresponding to all the specializations derived from the perturbed sequence $\{(x_{i_{j_m}}^{n_m}, p_{n_m}, a)\}$. Since for every index m and any possible j_1, j_2

$$m \cdot Mp_{i_{j(m)}}^{n_m} < Mfluct_{i_{j_1}, i_{j_2}}^{n_m},$$

at least one of these maximal graded limit groups is not rigid (i.e., it admits a non-trivial graded JSJ decomposition, where the grading is with respect to the subgroup $< a, \hat{p} >$). We denote this non-rigid maximal graded limit group $MLim(u, \hat{p}, a)$. Since the limit group $MLim(u, \hat{p}, a)$ is not rigid, either it admits a non-trivial graded free decomposition, i.e., a non-trivial free decomposition in which the subgroup $< \hat{p}, a >$ can be conjugated into one of the factors, or it admits a non-trivial graded abelian JSJ decomposition, which implies that there is a non-trivial graded modular group associated with it.

The limit group $MLim(u,\hat{p},a)$ was obtained as a limit group from a sequence of specializations $\{(u_m,\hat{p}_m,a)\}$, which were derived from a sequence $\{(x^{n_m}_{i_{j_m}},p_{n_m},a)\}$ of rigid specializations of the rigid limit group $Rgd(x,p,a)$ with which we started. If the limit group $MLim(u,\hat{p},a)$ admits a non-trivial graded free decomposition, then all the specializations in the sequence $\{(u_m,\hat{p}_m,a)\}$ split through epimorphisms η_m: $MLim(u,\hat{p},a) \to F_k * F_s$, where F_s is a non-trivial free group, and so that the epimorphisms $\{\eta_m\}$ map the subgroup $< \hat{p},a >$ onto the factor F_k. Hence, all the specializations in the sequence $\{(x^{n_m}_{i_{j_m}},p_{n_m},a)\}$ split through epimorphisms ν_m: $Rgd(x,p,a) \to F_k * F_t$, where F_t is a non-trivial free group, and so that the epimorphisms $\{\nu_m\}$ map the subgroup $<p,a>$ onto the free group F_k. This clearly contradicts our assumption that the specializations $\{(x^{n_m}_{i_j},p_{n_m},a)\}$ are rigid. Hence, we may assume that the limit group $MLim(u,\hat{p},a)$ admits no non-trivial graded free decomposition.

Suppose that $MLim(u,\hat{p},a)$ does not admit a non-trivial graded free decomposition, and let m be an index for which specialization $(u_m,\hat{p}_m,a)$ associated with the specialization $(x^{n_m}_{i_{j_m}},p_{n_m},a)$ is not degenerate; hence, in particular, the action of the graded modular group associated with the limit group $MLim(u,\hat{p},a)$ on the specialization $(u_m,\hat{p}_m,a)$ is non-trivial.

Since the action of the graded modular group associated with $MLim(u,\hat{p},a)$ on the specialization $(u_m,\hat{p}_m,a)$ is non-trivial, we can act with various elements of this graded modular group on the specialization $(u_m,\hat{p}_m,a)$ to obtain a sequence of specializations $(u'_i,\hat{p}_m,a)$, so that if $|u'_i|$ denotes the length (in the free group F_k) of the element u'_i, then $|u'_1| < |u'_2| < |u'_3| < \cdots$.

Now, the specializations $\{u'_i\}$ denote the fluctuations in the intersection points between segments of the defining parameters on the R-state corresponding to a specialization of the tuple (x,p,a). Since the length of the specializations $\{u'_i\}$ grows to infinity, the length of the corresponding specializations of the tuple (x,p,a), (x'_i,p_m,a), grows to infinity as well. Hence, by possibly taking a subsequence (still denoted $\{(x'_i,p_m,a)\}$), the sequence of specializations $\{(x'_i,p_m,a)\}$ converges into a flexible quotient of the rigid limit group $Rgd(x,p,a)$ with which we started. Since the specializations (x'_i,p_m,a) were obtained from the specialization $(x^{n_m}_{i_{j_m}},p_{n_m},a)$ using elements from the modular group associated with the limit group $MLim(u,\hat{p},a)$, if all the elements $\{(x'_i,p_m,a)\}$ factor through a flexible quotient of $Rgd(x,p,a)$, so does the specialization $(x^{n_m}_{i_j},p_{n_m},a)$ we started with, which clearly contradicts our assumption that the specializations $\{(x^{n_m}_{i_j},p_{n_m},a)\}$ are rigid. ∎

Theorem 2.2 proves the existence of a bound on the number of rigid solutions assuming a bound on the periodicity (Definition 2.1) of the specialization of the (given set of generators of the) defining parameters. To get a global bound on the number of rigid solutions, regardless of the periodicity of the defining parameters, a slight modification of the argument is required. In fact, the only places a bound on the periodicity was used is the derivation of an equation from any partial closed loop of any of the loops corresponding to one of the defining relations on a R-state, and in deriving the equations that guarantee that the specializations of a fixed variable x_i in all places it appears along the defining relations are indeed identical (the partial equivariance condition). The way these equations are derived is no longer valid in the presence of high periodicity of the defining parameters. However, a modification of the argument used to prove Theorem 2.2, that takes into account the length of periods of the defining parameters, is suitable for proving the existence of a global bound omitting the periodicity assumption.

THEOREM 2.5: *Let $Rgd(x,p,a)$ be a rigid graded limit group. There exists a global bound $b_R(Rgd(x,p,a))$ for which, for any particular value of the defining parameters p_0, there are at most b_R distinct rigid homomorphisms $h: Rgd(x,p,a) \to F_k$ satisfying $h(p) = p_0$.*

Proof: The argument we use is a modification of the argument used to prove Theorem 2.2. Suppose that there is no bound on the number of rigid solutions x (as the theorem claims) for all specializations p_0 of the defining parameters p. Then there must exist some sequence $\{p_n\}$ of specializations of the defining parameters p, so that for every index n, the rigid graded limit group $Rgd(x,p,a)$ admits n distinct rigid specializations $(x_1^n, p_n, a), \ldots, (x_n^n, p_n, a)$.

As in the proof of Theorem 2.2, let B be a ball in the Cayley graph of $Rgd(x,p,a)$ that contains the paths associated with all the defining relations of $Rgd(x,p,a)$. By Theorem 1.2 there exists some constant $\hat{R}$, so that for any rigid specialization (x_0, p_0, a) the ball B is $\hat{R}$-AP covered (Definition 1.1). Let $R = 2 \cdot \hat{R}$ (i.e., the images in F_k of edges associated with the defining parameters and with the coefficients in the ball of radius $\hat{R}$ in the Cayley graph of $Rgd(x,p,a)$, cover the the image of the ball B in F_k). We set $R = 2 \cdot \hat{R}$.

Recall that with a homomorphism $h: Rgd(x,p,a) \to F_k$, and the ball B_R in the Cayley graph of $Rgd(x,p,a)$, we have associated an R-state (Definition 2.3). Since the number of edges and vertices in the ball B_R is fixed, there exist only finitely many combinatorial types of R-states (Definition 2.3). Hence, by passing to a subsequence, we may further assume that for every index

n, the combinatorial types of the R-states associated with the specializations $(x_1^n, p_n, a), \ldots, (x_n^n, p_n, a)$ are identical.

Furthermore, we may also assume that after rescaling the metric on the R-states associated with the specializations $\{(x_i^n, p_n, a)\}$, by dilatation constants depending only on the specializations of the defining parameters p_n, for every sequence of indices $\{i_n | 1 \leq i_n \leq n\}$, the rescaled R-states associated with the sequence of specializations $\{(x_{i_n}^n, p_n, a)\}$ converge in the Gromov–Hausdorff topology on metric spaces, and the limit (finite) tree Y is independent of the particular choice of indices $\{i_n\}$.

Definition 2.6: Let $p_r(i_n, n)$ be a segment in the R-state corresponding to the specialization $(x_{i_n}^n, p_n, a)$, that is mapped to the non-degenerate segment e_r in the finite tree Y ($p_r(i_n, n)$ is a subword of a specialization of one of the defining parameters). We say that $p_r(i_n, n)$ is **pseudo-periodic** if:

(i) $p_r(i_n, n) = v_1 \alpha^s v_2$ and the equality is graphical (i.e., with no cancellations).

(ii) $s > 10$ and the length of α^s is at least three quarters of the length of $p_r(i_n, n)$.

It follows from the convergence of the rescaled R-states corresponding to the specializations $\{(x_{i_n}^n, p_n, a)\}$ that for large enough n, if $p_r(i_n, n)$ is pseudo-periodic for some index $1 \leq i_n \leq n$, then $p_r(i_n, n)$ is pseudo-periodic for any $1 \leq i_n \leq n$ (perhaps for slightly smaller pseudo-periodicity constants). If $p_r(i_n, n)$ is pseudo-periodic, we set $period(p_r(i_n, n))$ to be the minimal length of an element $\alpha \in F_k$ that satisfies condition (i) above. We call $period(p_r(i_n, n))$ the **pseudo-period** of the pseudo-periodic segment $p_r(i_n, n)$. Clearly, for large enough n, the pseudo-period does not depend on the specific index $1 \leq i_n \leq n$. We call the maximal possible s that satisfies condition (i) above, the **pseudo-periodicity** of $p_r(i_n, n)$.

We continue by going over the non-degenerate segments in the tree Y. If for such a non-degenerate segment e_r, there exists a subsequence $\{p_r(i_n, n)\}_{n=1}^{\infty}$ which is either:

(i) not pseudo-periodic, or

(ii) with bounded pseudo-periodicity, or

(iii) there exists a constant $c > 0$ (independent of the index n) for which

$$length(v_1(n)) + length(v_2(n)) > c \cdot length((\alpha_n)^{s_n}),$$

we restrict our attention to this subsequence and declare the particular segment as **not truly periodic**. After possibly restricting our attention to some subsequence, we may assume that the sequence of specializations corresponding to each non-degenerate segment is either not truly periodic, or it is pseudo-periodic with no bounded subsequence of pseudo-periods. We call these last segments **truly periodic**.

We have chosen the sequence of specializations $\{(x^n_{i_n}, p_n, a)\}$ to have the same combinatorial type of R-states. We denote by $Mper^n$ the maximum among all pseudo-periods of truly periodic segments in the R-state corresponding to the specialization $(x^n_{i_n}, p_n, a)$ (note that this maximum is independent of the index $1 \leq i_n \leq n$). We denote by $Mp^n_{i_n}$ the maximum between $Mper^n$ and the maximum length of a segment between branching points in the rescaled R-state, corresponding to the specialization $(x^n_{i_n}, p_n, a)$, which is mapped to a point in the limit tree Y. For each two specializations $(x^n_{i_{n_1}}, p_n, a), (x^n_{i_{n_2}}, p_n, a)$ we have associated (rescaled) R-states $State_1$ and $State_2$ in correspondence. Given a branching point in these states and a copy of one of the defining parameters that passes through this branching point, we can measure the difference in the placement of the branching point on the copy of the defining parameter in the two different rescaled R-states, $State_1$ and $State_2$. We denote by $Mfluct^n_{i_{n_1}, i_{n_2}}$ the maximum among all these (finite set of) differences. Note that by construction both $Mp^n_{i_n}$ and $Mfluct^n_{i_{n_1}, i_{n_2}}$ grow asymptotically as $o(1)$.

We define a sequence of specializations $\{(x^n_{i_n}, p_n, a)\}$ to be tame exactly as we did in Definition 2.4. If the sequence of specializations $\{(x^n_{i_n}, p_n, a)\}$ is tame, we continue with the tame subsequence of specializations. By possibly passing to a further subsequence, we may assume that the maximum $Mp^n_{i_{n_m}}$ is always obtained for the same segment in the corresponding rescaled R-state or for a pseudo-period of the same truly periodic segment, that the maximal fluctuation $Mfluct^{n_m}_{i_1, i_2}$ grows asymptotically as $o(Mp^{n_m}_{i_j})$, and that the ratio between the length of segments between branching points and pseudo-periods of truly periodic segments in the R-state that grow asymptotically as a (non-zero) constant times $Mp^{n_m}_{i_j}$, do converge.

If the original sequence of specializations is tame with respect to the segments in the R-state corresponding to non-degenerate segments in the tree Y, we declare each of the original segments in Y that are truly periodic, and for which their pseudo-period grows asymptotically as a constant multiple of $Mp^{n_m}_{i_j}$, as *not truly periodic* for the continuation. We also check which of the "newly uncovered" segments are truly periodic according to Definition 2.6, and pass to an appropriate subsequence. We check if the obtained subsequence is tame with respect to segments in the rescaled R-state corresponding to both non-degenerate segments in Y and segments and pseudo-periods of truly periodic segments for which their length grow asymptotically as a constant multiple of $Mp_{i_j}^{n_m}$. Since there are only finitely many segments in the R-state, and since there are m distinct solutions for each index m in our sequence, after finitely many "uncoverings" we obtain a sequence which is not tame with respect to the previously "uncovered" segments, and pseudo-periods of truly periodic segments in the R-state.

Once we obtain a sequence of rigid specializations $\{(x_{i_n}^n, p_n, a)\}$ with no tame subsequences with respect to the previously "uncovered" segments, and pseudo-periods of truly periodic segments in the corresponding R-states, we can choose a subsequence of indices $\{n_m\}_{m=1}^{\infty}$, so that for every index m there are m distinct indices $1 \leq i_1 < \cdots < i_m \leq n_m$ for which

$$m \cdot Mp_{i_{j_1}}^{n_m} < Mfluct_{i_{j_2}, i_{j_3}}^{n_m}$$

for all possible indices j_2, j_3, and an index j_1 for which the minimal Mp_{i_j} is obtained. We call such a sequence of specializations a **perturbed** sequence.

Given a perturbed sequence, for each index m we construct the (reference) R-state corresponding to the specialization $(x_{i_{j(m)}}^{n_m}, p_{n_m}, a)$ for which $Mp_{i_{j(m)}}^{n_m}$ is minimal among all possible indices j. Since we assume that the image in F_k of the ball B (in $Rgd(x,p,a)$) is covered by the union of the paths in T, the Cayley graph of F_k, which are the images of edges associated with the coefficients and the defining parameters in the ball of radius $R/2$ in $Rgd(x,p,a)$, given an R-state we can present each of the variables x in terms of *fractions* of (the given generating set of the) defining parameters. Hence, for every integer m we can present the specialization $(x_{i_{j(m)}}^{n_m}, p_{n_m}, a)$ in terms of fractions of the parameters p_{n_m} that appear in the R-state corresponding to this specialization, and for any other possible index j, we can present the specialization $(x_{i_j}^{n_m}, p_{n_m}, a)$ in terms of the same fractions of the parameters p_{n_m} that appear in the (reference) R-state corresponding to the specialization $(x_{i_{j(m)}}^{n_m}, p_{n_m}, a)$, together with elements $u_r^m(j)$ that encode the fluctuations in the corresponding branching points along the variables x between the R-state associated with the specialization $(x_{i_j}^{n_m}, p_{n_m}, a)$ and the reference R-state associated with the specialization $(x_{i_{j(m)}}^{n_m}, p_{n_m}, a)$.

Each relation from the defining relations of the rigid graded limit group $Rgd(x,p,a)$ corresponds to a closed loop on the R-state, and since the combinatorial types of the R-states associated with all specializations from the perturbed sequence $\{(x_{i_j}^{n_m}, p_{n_m}, a)\}$ are assumed to be identical, the "combinatorial type" of the loop corresponding to a given relation is identical for all the specializations from the perturbed sequence.

Furthermore, by our "uncovering" procedure, since every "uncovered" segment is much bigger than the maximal length of a fluctuation, and since every "uncovered" segment that is not truly periodic is either not pseudo-periodic, and if it is pseudo-periodic either the remaining parts at the ends of the segment that are outside the periodic part are much larger than the maximal fluctuation, or the period is much larger than the maximal fluctuation, every part of a loop that starts at some point on any of the "uncovered" segments of the defining parameters, which are not truly periodic, and get back to the same point, represent the identity element in all R-states corresponding to specializations $(x_{i_j}^{n_m}, p_{n_m}, a)$ from our perturbed sequence.

Also, every part of a loop that starts at some point on any of the "uncovered" segments of the defining parameters, which are truly periodic, and get back to the same point, represents an element that shifts the truly periodic element in an element that is much smaller than its length (though it may be much larger than its pseudo-period). Hence, such a loop must represent an element that commutes with a pseudo-period of that truly periodic segment in all the R-states corresponding to specializations $(x_{i_j}^{n_m}, p_{n_m}, a)$ from our perturbed sequence.

Since for every index m such a part of a loop of a defining relation represents either the identity element or an element that commutes with a pseudo-period of a truly periodic "uncovered" segment in the R-state associated with the (reference) specialization $(x_{i_{j(m)}}^{n_m}, p_{n_m}, a)$, and the same holds for every other R-state associated with a specialization $(x_{i_j}^{n_m}, p_{n_m}, a)$, every such part of a loop of one of the defining relations corresponds to a (parametric) equation involving (possibly) coefficients from the free group $F_k =< a >$, fractions of parameters corresponding to segments in the R-state associated with the (reference) specialization $(x_{i_{j(m)}}^{n_m}, p_{n_m}, a)$ which have not been uncovered (that we denote $\hat{p}$), and pseudo-periods of truly periodic "uncovered" segments (that are assumed to be part of the fractions $\hat{p}$), and variables corresponding to the "fluctuations" that we denote u. Since the combinatorics of the R-states associated with all the specializations from our perturbed sequence is identical, the set of parametric equations corresponding to different parts of (closed) partial loops corresponding to the defining relations is identical for all specializations from the perturbed sequence. We denote this system of equations by $\Sigma_1(u, \hat{p}, a) = 1$.

To the system of equations $\Sigma_1(u, \hat{p}, a) = 1$ that involve the loops around each of the branching points, we add equations that guarantee partial equivariance, i.e., the additional equations guarantee that the images in F_k of the edges associated with the variables x_i in the ball B (that is assumed to contain the paths corresponding to the defining relations) do all represent the same elements in the free group F_k. These equations can also be written in terms of the same fractions of the defining parameters $\hat{p}$ (that include pseudo-periods of truly periodic uncovered segments), and the variables corresponding to the various fluctuations u that were used in formulating the equations that involve the loops around the branching points. We denote the system of equations that combine the equations corresponding to the loops around branching points and the equations coming from the partial equivariance condition by $\Sigma(u,\hat{p},a)=1$. Given this (modified) system of equations, the rest of the argument is identical with the one described in the proof of Theorem 2.2. ∎

Theorem 2.5 bounds the number of rigid homomorphisms of a rigid graded limit group for any possible value of the defining parameters. We continue by proving the existence of a similar bound for families of strictly solid homomorphisms of solid graded limit groups. For presentation purposes we first prove the existence of such a bound for solid limit groups for which their associated graded abelian decomposition contains no QH vertex group, and afterwards gradually generalize the argument to all solid limit groups.

THEOREM 2.7: *Let $Sld(x,p,a)$ be a solid graded limit group, and suppose that the graded abelian decomposition associated with $Sld(x,p,a)$ contains no QH vertex groups. Then there exists a global bound $b_S(Sld(x,p,a))$ for which, for any particular value of the defining parameters p_0, there are at most b_S distinct solid families of strictly solid homomorphisms, $h\colon Sld(x,p,a)\to F_k$, satisfying $h(p)=p_0$ (see Definitions 1.4 and 1.5 for solid and strictly solid families of homomorphisms).*

Proof: The argument we use is a modification of the argument used to prove Theorem 2.5. Before presenting the argument, it is convenient to work with homomorphisms that are not necessarily the shortest in their corresponding solid families of homomorphisms, but rather with *almost shortest homomorphisms*.

Definition 2.8: Let $Sld(x,p,a)$ be a solid graded limit group and let Λ be its associated abelian decomposition. Let W be a rooted maximal tree in Λ, where the root of W is the distinguished vertex (the one containing the subgroup $AP=<a,p>$ in Λ). We fix a generating set for each of the vertex groups in Λ, and Bass–Serre generators that correspond to the graph of groups Λ and its maximal subtree W. We assume that a generating set of an edge group is contained in each vertex group adjacent to the corresponding edge. With each abelian vertex group in Λ we associate a fixed retraction from the abelian vertex group onto the direct summand that contains the subgroup generated by the edges connected to that abelian vertex group as a subgroup of finite index.

We say that a specialization (x_0, p_0, a) of the solid graded limit group $Sld(x,p,a)$ is **almost shortest** if the following hold:

(i) The specialization of each abelian vertex group in Λ factors through the retraction associated with this abelian vertex group.

(ii) Let v_0 be the distinguished vertex in the tree W (the one stabilized by $AP =< a,p >$). With each edge e_i in the tree of groups $W \subset \Lambda$ that connects the distinguished vertex v_0 to its neighbor v_i, we associate a base point b_i on the axis of the specialization of the edge group associated with the edge e_i. If the specialization of the (distinguished) vertex group V_0 is abelian, we set the base point b_i to be the point in the axis of the specialization of the (abelian) edge group E_i, associated with the edge e_i, that is the closest to the identity vertex in the tree T, the Cayley graph of F_k. If the specialization of the vertex group V_0 is not abelian, there exists a convex bounded set U_0 in the Cayley graph T of F_k that contains the set of points in T which are moved not more than any other point in T by the fixed set of generators of the group V_0. We set $u_0 \in U_0$ to be the point that is closest to the identity in U_0. In this case we set the base point b_i to be the point in the axis of the specialization of the (abelian) edge group E_i, associated with the edge e_i, that is the closest to u_0 in T, the Cayley graph of F_k.

For an almost shortest specialization we assume that the specializations of the fixed set of generators of the vertex group V_i, associated with the vertex v_i adjacent to the edge e_i, moves the base point b_i the smallest possible distance among all the homomorphisms that are in the same solid family, and factor through the given retraction of the solid limit group, $Sld(x,p,a)$.

Furthermore, suppose that the specialization of the vertex group V_i is not abelian, and let U_i be the convex bounded set of points in T which are moved not more than any other point in T by the specializations of the given set of generators of the vertex group V_i. For an almost shortest specialization the maximal distance from a point in the set U_i to b_i is the minimal possible among all the homomorphisms in its solid family for which the given set of generators moves the point b_i the minimal possible distance and so that the homomorphisms factor through the given retraction of the solid limit group, $Sld(x,p,a)$.

We repeat this construction for the vertices adjacent to the vertices v_i, i.e., with each edge e_j adjacent to a vertex v_i that is adjacent to v_0 in W, we associate a base point b_j on the axis of the specialization of the edge group associated with e_j. If the specialization of the vertex group V_i is abelian, we set the base point b_j to be the base point b_i. If the specialization of the vertex group V_i is not abelian, there is a (unique) point u_i in T that is moved not more than any other point in T by the fixed set of generators of V_i, and is the closest to b_i among all such points. In this case we set b_j to be closest point to u_i on the axis of the edge group E_j in T.

For an almost shortest specialization we assume that the specialization of the fixed set of generators of the vertex group V_j, associated with the vertex group v_j that is adjacent to the edge e_j, moves the base point b_j the smallest possible distance among all the homomorphisms that are in the same solid family, factor through the given retraction of $Sld(x,p,a)$, and for which the specializations of the vertex group V_0 and the vertex groups adjacent to it in the tree W are already given.

Suppose that the specialization of the vertex group V_j is not abelian, and let U_j be the convex bounded set of points in T which are moved not more than any other point in T by the specializations of the given set of generators of the vertex group V_j. For an almost shortest specialization the maximal distance from a point in the set U_j to b_j is the minimal possible among all the homomorphisms in its solid family for which the given set of generators move the point b_j the minimal possible distance and so that the homomorphisms factor through the given retraction of the solid limit group, $Sld(x,p,a)$, and for which the specializations of the vertex group V_0 and the vertex groups adjacent to it in the tree W are already given.

We repeat this construction until we exhaust all the vertices in the tree of groups W. Then we set the specialization of each of the fixed set of Bass–Serre generators, so that the distance between the base point on the axis of the specialization of the edge they connect to the image of the other base point (associated with the other side of that edge) under the action of the Bass–Serre generator, is the minimal possible among all the homomorphisms in the same solid family that factor through the retraction of

$Sld(x,p,a)$, and for which the specialization of the fundamental group of the tree of groups W is already given.

In Theorem 1.7 we have shown that there exists a global constant R, so that each strictly solid homomorphism $h: Sld(x,p,a) \to F_k$ that is among the shortest in its solid family of homomorphisms is R-AP-covered. The argument used to prove Theorem 1.7 works without any changes to almost shortest homomorphisms. Hence, if we fix a retraction of $Sld(x,p,a)$, a maximal tree in its graded abelian JSJ decomposition Λ, a generating set for each of its edge and vertex groups, and a finite set of Bass–Serre generators, there exists some constant (still denoted R) so that every strictly solid almost shortest homomorphism $h: Sld(x,p,a) \to F_k$, that factors through the given retraction and is among the shortest in its solid family, is R-AP-covered.

Suppose there is no bound on the number of solid families of strictly solid homomorphisms that is valid for all specializations p_0 of the defining parameters p. Then there must exist some sequence $\{p_n\}$ of specializations of the defining parameters p, so that for every index n, the solid graded limit group $Sld(x,p,a)$ admits n distinct solid families of strictly solid homomorphisms. For each index n, and each solid family of strictly solid homomorphisms associated with the specialization p_n, we associate an almost shortest specialization $(x^n_{i_n}, p_n, a)$ that factors through the given retraction.

Given the sequence of parameters $\{p_n\}$, and their associated collections of almost shortest specializations, we apply the argument used to prove Theorems 2.2 and 2.5 to obtain an action of a maximal limit group $MLim(u,\hat{p},a)$ associated with the system of equations $\Sigma(u,\hat{p},a)=1$ on some real tree Y. We may assume that this limit group is not rigid with respect to the parameter subgroup $<\hat{p}>$, that the fluctuations associated with a subsequence, still denoted $\{(x^n_{i_n}, p_n, a) | 1 \leq i_n \leq n\}$, factor through $MLim(u,\hat{p},a)$, and that for any index n and any fluctuation u_{i_n}, the length of the maximal fluctuation among the u_{i_n}'s is at least n times longer than the maximal length of the fractions (that have not been "uncovered"), and the pseudo-periods of truly periodic segments, $\hat{p}_n$.

With the limit group $MLim(u,\hat{p},a)$ we associate a limit group $MPLim(u,\hat{p},p',a)$ generated by the limit group $MLim(u,\hat{p},a)$, and the fractions p' that were "uncovered" through the process presented in the proof of Theorems 2.2 and 2.5 (i.e., the fractions $\hat{p}$ that have not been "uncovered", together with the fractions p', cover the entire (reference) R-state, that is the entire image in F_k of the ball B in the Cayley graph of $Sld(x,p,a)$). Note that $MPLim$ is obtained from $MLim$ by "enlarging" the distinguished vertex group in its graded abelian JSJ decomposition (with respect to the parameter subgroup $\hat{p}$), by adding the elements p' to it (and the subgroup they generate). This structure of the group $MPLim$ can be studied by passing to a subsequence of the given set of homomorphisms that converges to some free action of the group $MPLim$ on an R^s-tree (using Theorem 1.3), and then analyzing the structure of the group $MPLim$ using the decompositions of the various point stabilizers associated with the various levels of that R^s tree until the level in which the fluctuations "occur".

By construction, there is a natural map η: $Sld(x,p,a) \to MPLim(u,\hat{p},p',a)$. Let $L(x,p,a)$ be the image of $Sld(x,p,a)$ in $MPLim(u,\hat{p},p',a)$, $L(x,p,a) = \eta(Sld(x,p,a))$. Let Θ_L be the graded abelian decomposition inherited by L from the graded abelian decomposition of $MPLim(u,\hat{p},p',a)$ (with respect to the parameter subgroup $<\hat{p},p',a>$).

The limit groups $MLim(u,\hat{p},a)$, $MPLim(u,\hat{p},p',a)$ and $L(x,p,a)$ are obtained from a sequence of almost shortest specializations (that factor through the given retraction) in their solid families, still denoted $\{(x^n_{i_n},p_n,a)\}$, which are all assumed to be strictly solid.

Let Λ be the graded abelian decomposition of the solid limit group $Sld(x,p,a)$. Since the specializations $\{(x^n_{i_n},p_n,a)\}$ are all strictly solid, and they converge into the limit group $L(x,p,a)$, and since the abelian decomposition Θ_L, the graded abelian decomposition inherited by $L(x,p,a)$ from the graded abelian decomposition of $MPLim$ (with respect to the parameter subgroup $<\hat{p},p'>$), is an essential graded abelian decomposition by construction (see Definition 1.8), Proposition 1.9 implies that the map η: $Sld(x,p,a) \to L(x,p,a)$ maps every non-abelian vertex group and every edge group in Λ into either an edge group or a non-QH vertex group in Θ_L, and if it is mapped into an abelian vertex group, then it must be mapped into the summand containing the edge groups connected to this vertex group as a subgroup of finite index.

Since the abelian decompositions, Λ of $Sld(x,p,a)$ and Θ_L of $L(x,p,a)$, are compatible, i.e., every vertex group in Λ is mapped by η into either an edge group or into a non-QH, non-abelian vertex group or into the summand of an abelian vertex group that contains the subgroup generated by the edge group as a subgroup of finite index in Θ_L, the fluctuations in the positions of the branching points along the given set of generators of the (distinguished) vertex group V_0, as well as in the positions of the points b_i, associated with the vertices v_i connected with the distinguished vertex v_0, all grow as $o(Mfluct^{n_m}_{i_{j_2},i_{j_3}})$, i.e., the fluctuations in the positions of the branching points and the points b_i are much smaller than the maximal fluctuation.

Since the specializations $\{(x^n_{i_n}, p_n, a)\}$ are all almost shortest strictly solid specializations, and the abelian decompositions, Λ of $Sld(x,p,a)$ and Θ_L of $L(x,p,a)$, are compatible, by repeating this observation inductively through the (finite) subtree (of groups) $W \subset \Lambda$, the fluctuations in the positions of the branching points along the given sets of generators of the vertex groups V_j, as well as the positions of the base points b_j associated with each of the vertex groups v_j in the subtree of groups W, grow as $o(Mfluct^{n_m}_{i_{j_2},i_{j_3}})$. And again, since the specializations $\{(x^n_{i_n}, p_n, a)\}$ are all almost shortest strictly solid specializations, and the abelian decompositions, Λ of $Sld(x,p,a)$ and Θ_L of $L(x,p,a)$, are compatible, and the fluctuations in the positions of the branching points along the given set of generators of the fundamental group of the tree of groups W grow as $o(Mfluct^{n_m}_{i_{j_2},i_{j_3}})$, the fluctuations in the positions of branching points along the (pre-chosen) Bass–Serre generators grow as $o(Mfluct^{n_m}_{i_{j_2},i_{j_3}})$ as well. This means that the fluctuations in (the position of branching points along) a whole set of generators of the limit group $L(x,p,a)$ grow as $o(Mfluct^{n_m}_{i_{j_2},i_{j_3}})$. But by construction, the fluctuation in the position of a branching point along at least one of these generators must grow as a non-zero multiple of $Mfluct^{n_m}_{i_{j_2},i_{j_3}}$, a contradiction. ∎

Proving the existence of a bound on the number of solid families of strictly solid specializations of a solid limit group with no QH vertex group in its associated graded abelian decomposition (Theorem 2.7), we further generalize the argument used to prove the existence of a bound on the number of rigid solutions of a rigid limit group (Theorem 2.5) to get a bound on the number of strictly solid families of specializations (Definition 1.5) of a general solid limit group.

THEOREM 2.9: *Let $Sld(x,p,a)$ be a solid graded limit group. For any particular value p_0 of the defining parameters p, there are at most b_S distinct strictly solid families of homomorphisms $h\colon Sld(x,p,a) \to F_k$ satisfying $h(p) = p_0$.*

Proof: The argument we use is based on the arguments used to prove Theorems 2.5 and 2.7. Suppose that there is no bound on the number of strictly solid families of homomorphisms for all specializations p_0 of the defining parameters p. Then there must exist some sequence $\{p_n\}$ of specializations of the defining parameters p, so that for every index n, the solid graded limit group $Sld(x,p,a)$ admits n distinct strictly solid families of specializations. For each index n, and each strictly solid family of homomorphisms associated with the specialization p_n, we associate an almost shortest specialization (see Definition 2.8), $(x^n_{i_n}, p_n, a)$.

Given the sequence of parameters $\{p_n\}$, and their associated collections of almost shortest specializations, we apply the argument used to prove Theorems 2.2 and 2.5, and obtain an action of a maximal limit group $MLim(u, \hat{p}, a)$ associated with the system of equations $\Sigma(u, \hat{p}, a) = 1$ on some real tree Y, where $\hat{p}$ are variables associated with fractions of the given set of generators of the defining parameters p that haven't been "uncovered", and periods of truly periodic uncovered segments, and u are variables associated with the fluctuations of branching points in an R-state (see the proof of Theorem 2.5 for these notions and notation). We may assume that this limit group is not rigid with respect to the parameter subgroup $< \hat{p} >$, that the fluctuations associated with a subsequence, still denoted $\{(x^n_{i_n}, p_n, a) | 1 \leq i_n \leq n\}$, factor through $MLim(u, \hat{p}, a)$, and that for any index n and any fluctuation u_{i_n}, the length of the maximal fluctuation among the u_{i_n}'s is at least n times longer than the maximal length of the fractions $\hat{p}_n$.

As we did in proving Theorem 2.7, with the limit group $MLim(u, \hat{p}, a)$ we associate a limit group $MPLim(u, \hat{p}, p', a)$ generated by the limit group $MLim(u, \hat{p}, a)$ and the fractions p' that were "uncovered" through the process presented in the proof of Theorems 2.2 and 2.5. Recall that by construction, the limit group $MPLim$ is obtained from the limit group $MLim$ by adding the elements p' (and the subgroup they generate) to the distinguished vertex group in the graded abelian decomposition (with respect to the parameter subgroup $\hat{p}$) of $MLim(u, \hat{p}, a)$. Furthermore, there exists a natural map η: $Sld(x, p, a) \to MPLim(u, \hat{p}, p', a)$.

Since the limit group $MLim(u, \hat{p}, a)$ is not rigid, either it admits a non-trivial graded free decomposition, i.e., a non-trivial free decomposition in which the subgroup $< \hat{p}, a >$ can be conjugated into one of the factors, or it admits a non-trivial graded abelian JSJ decomposition, which implies that there is a non-trivial graded modular group associated with it.

The limit group $MLim(u, \hat{p}, a)$ was obtained as a limit group from a sequence of specializations $\{(u_{i_m}, \hat{p}_{n_m}, a)\}$ that were derived from a sequence $\{(x^{n_m}_{i_{j_m}}, p_{n_m}, a)\}$ of strictly solid, almost shortest specializations (in their solid families) of the solid limit group $Sld(x, p, a)$ with which we started. If the limit group $MLim(u, \hat{p}, a)$ admits a non-trivial graded free decomposition, then its associated limit group $MPLim(u, \hat{p}, p', a)$ admits a non-trivial graded free decomposition (with respect to the parameter subgroup $< \hat{p}, p', a >$), which implies that all the specializations in the sequence $\{(x_{i_{j_m}}^{n_m}, p_{n_m}, a)\}$ split through epimorphisms λ_m: $Sld(x,p,a) \to F_k * F_t$, where F_t is a non-trivial free group, and so that the epimorphisms $\{\lambda_m\}$ map the subgroup $< p, a >$ onto the free group F_k. This clearly contradicts our assumption that the specializations $\{(x_{i_j}^{n_m}, p_{n_m}, a)\}$ are strictly solid. Hence, we may assume that the limit group $MLim(u,\hat{p},a)$ admits no non-trivial graded free decomposition.

The solid limit group $Sld(x,p,a)$ is mapped into the limit group $MPLim(u,\hat{p},p',a)$ by the homomorphism η: $Sld(x,p,a) \to MPLim(u,\hat{p},p',a)$. Let Λ be the graded abelian JSJ decomposition associated with $Sld(x,p,a)$, and let Γ be the essential graded abelian JSJ decomposition associated with $MPLim(u,\hat{p},p',a)$ (see Definition 1.8 for an essential graded decomposition). By the construction of the limit group $MPLim(u,\hat{p},p',a)$, the image of the original solid limit group $Sld(x,p,a)$, $L = \eta(Sld(x,p,a))$, cannot be conjugated into the fundamental group of a proper subgraph of the graph of groups Γ. Since the limit group $MPLim$ was derived from a sequence of strictly solid homomorphisms, Proposition 1.9 implies that the map η: $Sld(x,p,a) \to MPLim$ maps every non-abelian, non-QH vertex group and every edge group in Λ, into either an edge group or a non-QH vertex group in Γ, and if it is mapped into an abelian vertex group in Γ, then it must be mapped into the summand containing the edge groups connected to this vertex group as a subgroup of finite index.

With the limit group $MPLim(u,\hat{p},p',a)$ we associate its (canonical) strict graded Makanin–Razborov diagram with respect to the parameter subgroup $< \hat{p}, p' >$ (see proposition 1.10 in [Se2] for the construction of the strict (graded) Makanin–Razborov diagram). The entire sequence of specializations $\{(u_{i_m}, \hat{p}_{n_m}, p'_{n_m}, a) | 1 \leq i_m \leq m\}$ factors through the limit group $MPLim(u,\hat{p},p',a)$, hence, we may assume that a subsequence, still denoted $\{(u_{i_m}, \hat{p}_{n_m}, p'_{n_m}, a) | 1 \leq i_m \leq m\}$, factors through a (fixed) graded resolution $GRes(u,\hat{p},p',a)$ in the strict graded Makanin–Razborov resolution of $MPLim(u,\hat{p},p',a)$.

With the i-th level of the graded resolution $GRes(u,\hat{p},p',a)$ there is an associated graded limit group, a quotient of $MPLim(u,\hat{p},p',a)$, that we denote $MPLim_i(u,\hat{p},p',a)$. The natural map η: $Sld(x,p,a) \to MPLim(u,\hat{p},p',a)$ composed with the quotient map τ_i: $MPLim(u,\hat{p},p',a) \to MPLim_i(u,\hat{p},p',a)$ gives rise to a natural map η_i: $Sld(x,p,a) \to MPLim_i(u,\hat{p},p',a)$. By the argument used to prove Proposition 1.9, if for some index i, the map η_i factors through a map of $Sld(x,p,a)$ onto a limit group $U(x,p,a)$ which admits a non-trivial graded free decomposition, or the map η_i does not map every non-QH, non-abelian vertex group and every edge group, in the graded abelian decomposition Λ associated with $Sld(x,p,a)$, either into a non-QH vertex group or into an edge group in the essential graded abelian JSJ decomposition Γ_i associated with $MPLim_i(u,\hat{p},p',a)$ (see Definition 1.8 for an essential abelian splitting), and if it maps it into an abelian vertex group in Γ_i then it is mapped into the summand containing the edge groups connected to this vertex group as a subgroup of finite index, then for large enough m, the specializations (x_{i_m}, p_{n_m}, a) are not strictly solid, a contradiction to our assumptions. Furthermore, by construction, the image of the original limit group $Sld(x,p,a)$ into a limit group $MPLim_i(u,\hat{p},a)$ cannot be conjugated into the fundamental group of a proper subgraph of the graphs of groups Γ_i.

The terminal limit group of the graded resolution $GRes(u,\hat{p},p',a)$, $MPLim_f(u,\hat{p},p',a)$, is by definition either rigid or solid (with respect to the parameter subgroup $<\hat{p},p'>$). If the terminal limit group $MPLim_f(u,\hat{p},p',a)$ is rigid, then the assumption that there is no bound on the number of strictly solid families that factor through the resolution $GRes(u,\hat{p},p',a)$ implies that there is no bound on the number of the rigid specializations of $MPLim_f$, a contradiction to Theorem 2.5. Hence, the terminal limit group of the graded resolution $GRes(u,\hat{p},p',a)$, $MPLim_f(u,\hat{p},p',a)$, has to be solid.

The sequence of specializations $\{(x_{i_m}, p_{n_m}, a) | 1 \leq i_m \leq m\}$ of the solid limit group $Sld(x,p,a)$ with which we started is assumed to be strictly solid and almost shortest in its corresponding solid families. The corresponding sequence of specializations $\{(u_{i_m}, \hat{p}_{n_m}, p'_{n_m}, a) | 1 \leq i_m \leq m\}$ of the terminal graded limit group of the graded resolution $GRes(u,\hat{p},p',a)$, $MPlim_f(u,\hat{p},p',a)$, is necessarily solid specializations. If there is no subsequence of specializations (still denoted $\{(u_{i_m}, \hat{p}_{n_m}, p'_{n_m}, a) | 1 \leq i_m \leq m\}$) which are strictly solid specializations of the terminal solid limit group $MPLim_f(u,\hat{p},p',a)$ of the graded resolution $GRes(u,\hat{p},p',a)$, we extend the graded resolution $GRes(u,\hat{p},p',a)$ by continuing from the terminal graded limit group $MPLim_f(u,\hat{p},p',a)$ to a flexible quotient that is associated with a subsequence of non-strictly solid specializations (still denoted $\{(u_{i_m}, \hat{p}_{n_m}, p'_{n_m}, a) | 1 \leq i_m \leq m\}$). By repeatedly extending the graded resolution $GRes(u,\hat{p},p',a)$ in case a terminal solid limit group does not admit a subsequence of strictly solid specializations, we obtain an extended graded resolution (still denoted $GRes(u,\hat{p},p',a)$) for which the associated sequence of specializations of its terminal solid limit group (still denoted $MPLim_f(u,\hat{p},p',a)$) are all strictly solid. Clearly, all the properties of the graded limit groups and the maps $\{\eta_i\}$ associated with the various levels of the graded resolution $GRes(u,\hat{p},p',a)$ remain valid for the graded limit groups in the extended graded resolution, and the maps from the original solid limit group $Sld(x,p,a)$ into these limit groups, which we still denote $\{\eta_i\}$.

The properties of the maps η_i, from the solid limit group $Sld(x,p,a)$ we have started with into the graded limit groups $MPLim_i(u,\hat{p},p',a)$, allow us to bound the *reduced complexities* of the graded abelian decompositions Γ_i of $MPLim_i(u,\hat{p},p',a)$, in terms of the reduced complexity of the graded abelian decomposition Λ of $Sld(x,p,a)$.

Definition 2.10: Let $Glim(y,p,a)$ be a graded limit group with respect to the subgroup $P =< p >$, and suppose that $Glim(y,p,a)$ admits no non-trivial graded free decomposition. Let Δ be its graded abelian JSJ decomposition.

Let $Q_1,\dots,Q_n$ be the QH vertex groups in the graded abelian decomposition Δ and let $S_1,\dots,S_n$ be their corresponding surfaces. We set the complexity of a QH vertex group Q_j to be the ordered couple $(|\chi(S_j)|, genus(S_j))$. We set the *reduced complexity* of the graded abelian decomposition Δ, denoted $Rcomp(\Delta)$, to be the tuple

$$((\chi(S_1)|, genus(S_1)),\dots,(|\chi(S_n)|, genus(S_n)), d),$$

where the sequence of couples $(\chi(S_1)|, genus(S_1)),\dots,(|\chi(S_n)|, genus(S_n))$ is ordered in a decreasing lexicographical order, and the integer d is set to be $d=e-a$, where e is the number of edges between two non-QH vertex groups in Δ, and a is the number of abelian vertex groups adjacent to the collection of these edges.

Let Δ_1 and Δ_2 be two graded abelian decompositions of the graded limit groups $Glim_1(y,p,a)$ and $Glim_2(y,p,a)$ in correspondence, and $Glim_1(y,p,a)$, $Glim_2(y,p,a)$ admit no non-trivial graded free decompositions. We say that $Rcomp(\Delta_1) \leq Rcomp(\Delta_2)$ if the sequence associated with Δ_1 is not greater than the sequence associated with Δ_2 in the natural lexicographical order on the above sequences of (non-negative) integers. Clearly, the reduced complexity of any two such graded decompositions can be compared.

Finally, note that if Δ is the graded abelian JSJ decomposition of a graded abelian group $Glim$, and Δ^{ess} is its associated essential decomposition (Definition 1.8), then by the construction of the associated essential decomposition, $Rcomp(\Delta) = Rcomp(\Delta^{ess})$.

At this point the properties of the maps η_i enable us to strictly bound the reduced complexity of the graded abelian JSJ decomposition of the terminating solid limit group, $MPLim_f$, of the graded resolution $GRes(u,\hat{p},a)$, in terms of the reduced complexity of the graded abelian JSJ decomposition of the original solid graded limit group $Sld(x,p,a)$.

PROPOSITION 2.11: *Let $MPLim_f(u,\hat{p},p',a)$ be the terminating solid limit group of the (extended) graded resolution $GRes(u,\hat{p},p',a)$. Let Γ_f be the essential graded abelian JSJ decomposition of the solid limit group $MPLim_f(u,\hat{p},p',a)$, and let Λ be the graded abelian JSJ decomposition of the original solid limit group $Sld(x,p,a)$. Then*

$$Rcomp(\Gamma_f) < Rcomp(\Lambda).$$

Proof: By Theorem 2.7 we may assume that the graded abelian JSJ decomposition of the original solid limit group $Sld(x,p,a)$ contains a QH vertex group. The map η_i: $Sld(x,p,a) \to MPLim_i(u,\hat{p},p',a)$ maps every non-QH vertex group and every edge group in the graded abelian JSJ decomposition Λ of $Sld(x,p,a)$ into a non-QH vertex group or an edge group in Γ_i, the essential graded abelian JSJ decomposition of $MPLim_i(u,\hat{p},p',a)$ (see Definition 1.8 for the essential graded decomposition), and if it is mapped into an abelian vertex group in Γ_i then it is mapped into the summand of the abelian vertex group which contains the subgroup generated by the edge groups connected to it as a subgroup of finite index. Furthermore, by construction, the image of $Sld(x,p,a)$, $\eta_i(Sld(x,p,a))$, cannot be conjugated into a proper subgraph of the graph of groups Γ_i. Hence, every QH vertex group in the graded abelian decomposition of $Sld(x,p,a)$, Λ, inherits a small splitting from its image into the limit group $MPLim_i(u,\hat{p},p',a)$, i.e., from the essential graded abelian decomposition Γ_i. Also, the pre-image of every QH vertex group in Γ_i contains conjugates of subgroups of QH vertex groups in Λ that correspond to subsurfaces of the surfaces associated with the QH vertex groups in the graded abelian decomposition Λ, and each such subgroup of a QH vertex group in Λ is mapped into a finite index subgroup of the corresponding QH vertex group in Γ_i. Since every QH vertex group in Γ_i is "covered" by a subsurface of the surfaces associated with the QH vertex groups in Λ, since the maps η_i map every non-QH vertex group and every edge group in Λ into a non-QH vertex group or into an edge group in Γ_i, and since $\eta_i(Sld(x,p,a))$ cannot be conjugated into a proper subgraph of the graph of groups Γ_i, the reduced complexity of each of the graded abelian decompositions Γ_i is bounded above by the reduced complexity of the graded abelian decomposition Λ, i.e.,

$$Rcomp(\Gamma_i) \leq Rcomp(\Lambda).$$

Suppose that for some level i, $Rcomp(\Gamma_i) = Rcomp(\Lambda)$. In this case our aim is to show that equality holds also for the essential graded abelian decomposition associated with the top level of the graded resolution $GRes(u,\hat{p},p',a)$: $Rcomp(\Gamma_1) = Rcomp(\Lambda)$. Hence, in this part of the argument we assume that $i > 1$, and i is the minimal index for which, $Rcomp(\Gamma_i) = Rcomp(\Lambda)$.

Since $Rcomp(\Gamma_i) = Rcomp(\Lambda)$, every QH vertex group in Λ is mapped by η_i isomorphically onto a QH vertex group in the essential graded abelian decomposition Γ_i, and the non-QH vertex groups in Λ are mapped by η_i into distinct non-QH vertex groups in Γ_i. Hence, the edge groups in Λ are mapped by η_i into distinct edge groups in Γ_i, and every edge group in Γ_i contains the image of one of the edge groups in Λ.

For every level j in the graded resolution of $MPLim$, $GRes$, the pre-image of a QH vertex group in the essential graded abelian decomposition Γ_j contains subgroups corresponding to subsurfaces of the surfaces associated with the QH vertex groups in the graded abelian decomposition Λ of $Sld(x,p,a)$, and the fundamental group of each such subsurface is mapped into a finite index subgroup of the corresponding QH vertex group in Γ_j.

The map η_i: $Sld(x,p,a) \to MPLim_i(u,\hat{p},p',a)$ factors as $\eta_i = \nu_{i-1} \circ \eta_{i-1}$, where ν_{i-1} is the canonical quotient map

$$\nu_{i-1}\colon MPLim_{i-1}(u,\hat{p},p',a) \to MPLim_i(u,\hat{p},p',a).$$

Since the map η_i maps the QH vertex groups in Λ isomorphically onto the QH vertex groups in Γ_i, the map η_{i-1} maps distinct subsurfaces of the surfaces associated with the QH vertex groups in Λ isomorphically onto the QH vertex groups in Γ_{i-1}, and the quotient map ν_{i-1} maps the QH vertex groups in Γ_{i-1} isomorphically onto subsurfaces of the surfaces associated with the QH vertex groups in the essential graded abelian decomposition Γ_i.

Since the image of $Sld(x,p,a)$ in $MPLim(u,\hat{p},p',a)$, $\eta_{i-1}(Sld(x,p,a))$, cannot be conjugated into the fundamental group of a proper subgraph of the graph of groups Γ_{i-1}, and for each QH vertex group in Γ_{i-1} there exists a subsurface of a surface associated with a QH vertex group in Λ that is mapped isomorphically onto it by η_{i-1}, each edge group in Γ_{i-1} either contains the image of an edge group in Λ or it contains the image of a non-boundary parallel s.c.c. in one of the surfaces associated with a QH vertex group in Λ.

Since every edge group in the graph of groups Γ_i contains the image of an edge group in Λ, and since every edge group in Θ_{i-1} contains the image of either an edge group or a non-boundary parallel s.c.c. on one of the surfaces associated with a QH vertex groups in Λ, the graph of groups Γ_i can be further refined by cutting some of the surfaces associated with its QH vertex groups along non-boundary parallel s.c.c. to give an essential graph of groups Γ_i^{ref} of the graded limit group $MPLim_i(u,\hat{p},p',a)$, so that the quotient map ν_{i-1} maps every edge group in the graph of groups Γ_{i-1} into an elliptic subgroup in the refined graph of groups Θ_i^{ref}.

Since the (extended) graded resolution $GRes(u,\hat{p},p',a)$ is assumed to be strict, every non-QH, non-abelian vertex group in Γ_{i-1} is mapped monomorphically by the quotient map ν_{i-1}. If the reduced complexity of the graph of groups Γ_{i-1}, $Rcomp(\Gamma_{i-1})$, is strictly smaller than the reduced complexity of Γ_i, $Rcomp(\Gamma_i)$, then the image of at least one of the non-QH, non-abelian vertex groups in Γ_{i-1} under the map ν_{i-1}, say $\nu_{i-1}(V_j)$, inherits a non-trivial abelian decomposition from the (essential) refined graph of groups Γ_i^{ref}. Since every edge group in Γ_{i-1} is mapped by ν_{i-1} into an elliptic subgroup in Γ_i^{ref}, the graph of groups inherited by $\nu_{i-1}(V_j)$ from Γ_i^{ref} is compatible with the graph of groups Γ_{i-1}. Hence, the graph of groups inherited by the non-QH, non-abelian vertex group V_j from Γ_i^{ref} can be used to further refine the graph of groups Γ_{i-1}, so that the subgroup $(\hat{p},p',a)$ remains elliptic, and the obtained graded graph of groups is essential. Since the graph of groups Γ_{i-1} is assumed to be the essential graded abelian JSJ decomposition of $MPLim_{i-1}(u,\hat{p},p',a)$, a further refinement certainly contradicts its canonical properties. Hence, necessarily, $Rcomp(\Gamma_{i-1}) = RComp(\Lambda)$, and by finite induction $Rcomp(\Gamma_1) = RComp(\Lambda)$.

Suppose that the reduced complexity of the essential graded abelian decomposition Γ_1 of the graded limit group $MPLim_1(u,\hat{p},p',a)$, associated with the top level of the graded resolution $GRes(u,\hat{p},p',a)$, is equal to the reduced complexity of the graded abelian decomposition Λ associated with the original solid limit group $Sld(x,p,a)$. Since the natural map, η_1: $Sld(x,p,a) \to MPLim_1(u,\hat{p},p',a)$, maps every non-$QH$ vertex group, and every edge group in the abelian decomposition Λ, into either a non-QH, non-abelian vertex group or an edge group in the abelian decomposition Γ_1, or into the direct summand of an abelian vertex group that contains the subgroup generated by the edge groups connected to it as a subgroup of finite index, and the complexities of Λ and Γ_1 are equal, every QH vertex group in Λ is mapped by η_1 isomorphically onto a QH vertex group in the abelian decomposition Γ_1.

The graded limit group $MPLim_1(u,\hat{p},p',a)$ is generated by the limit groups $MLim(u,\hat{p},a)$, and the fractions $\{p'\}$ of the defining parameters (see the argument used to prove Theorem 2.5), that are all added to the distinguished vertex group of $MLim$. Since η_1 maps each QH vertex group in the abelian decomposition Λ isomorphically onto a QH vertex group in the graded abelian decomposition Γ_1, the map η_1 maps every QH vertex group in Λ into the subgroup $MLim(u,\hat{p},a)$ of $MPLim_1(u,\hat{p},p',a)$, and each boundary element of such a QH vertex group (in Λ) is mapped into a (conjugate of an) elliptic element in the graded abelian JSJ decomposition of $MLim$ (with respect to the parameter subgroup $\hat{p}$).

The graded limit group $MLim_1(u,\hat{p},a)$ was constructed from a sequence of specializations $\{(u^n_{i_n},\hat{p}_n,a) | 1 \leq i_n \leq n\}$ that converge to an action of the limit group $MLim(u,\hat{p},a)$ on some real tree Y, where the specializations $u^n_{i_n}$ represent fluctuations in the positions of branching points in corresponding R-states in comparison with the reference R-state (see the construction that appears in the proofs of Theorems 2.2 and 2.5). Since the abelian decompositions, Λ of $Sld(x,p,a)$ and Γ_1 of $MPLim_1$, are compatible, i.e., every vertex group in Λ is mapped by η into either an edge group or into a non-QH, non-abelian vertex group or into the summand of an abelian vertex group that contains the subgroup generated by the edge group as a subgroup of finite index in $MPLim_1$, the fluctuations in the positions of the branching points along the given set of generators of the (distinguished) vertex group V_0, as well as in the positions of the points b_i, associated with the vertices v_i connected with the distinguished vertex v_0, all grow as $o(Mfluct^{n_m}_{i_{j_2},i_{j_3}})$, i.e., the fluctuations in the positions of the branching points and the points b_i are much smaller than the maximal fluctuation.

Since every QH vertex group in Λ, the graded abelian JSJ decomposition of $Sld(x,p,a)$, is mapped by η_1 into a conjugate of $MLim(u,\hat{p},a)$, and since every such QH vertex group is mapped isomorphically onto a QH vertex group in the (essential) abelian decomposition Γ_1 of $MPLim_1$, the translation lengths of the specializations of all the boundary elements of the QH vertex groups in Λ grow as $o(Mfluct^{n_m}_{i_{j_2},i_{j_3}})$, i.e., these translation lengths are much smaller than the maximal fluctuation. Hence, since the specializations $\{(x^n_{i_n},p_n,a)\}$ are all almost shortest strictly solid specializations, every QH vertex group in Γ_1 is mapped into a conjugate of a stabilizer of a point in the limit tree Y. In other words, the specializations of the given set of generators of a QH vertex group in Λ move its associated base point in a distance which grows as $o(Mfluct^{n_m}_{i_{j_2},i_{j_3}})$.

Since the specializations $\{(x^n_{i_n},p_n,a)\}$ are all almost shortest strictly solid specializations, and the abelian decompositions, Λ of $Sld(x,p,a)$ and Γ_1 of $MPLim$, are compatible, by repeating this observation inductively through the (finite) subtree (of groups) $W \subset \Lambda$, the fluctuations in the positions of the branching points along the given sets of generators of the vertex groups V_j, as well as the positions of the base points b_j associated with each of the vertex groups v_j in the subtree of groups W, and the translation lengths of the given sets of generators of the QH vertex groups in Λ all grow as $o(Mfluct^{n_m}_{i_{j_2},i_{j_3}})$. And again, since the specializations $\{(x^n_{i_n}, p_n, a)\}$ are all almost shortest strictly solid specializations, and the abelian decompositions, Λ of $Sld(x,p,a)$ and Γ_1 of $MPLim$, are compatible, and the fluctuations in the positions of the branching points along the given set of generators of the fundamental group of the tree of groups W grow as $o(Mfluct^{n_m}_{i_{j_2},i_{j_3}})$, the fluctuations in the positions of branching points along the (pre-chosen) Bass–Serre generators in Λ (and the subtree of groups W) grow as $o(Mfluct^{n_m}_{i_{j_2},i_{j_3}})$ as well. This means that the fluctuations in (the position of branching points along) a whole set of generators of the limit group $Sld(x,p,a)$ grow as $o(Mfluct^{n_m}_{i_{j_2},i_{j_3}})$. But by construction, the fluctuation in the position of a branching point along at least one of these generators must grow as a non-zero multiple of $Mfluct^{n_m}_{i_{j_2},i_{j_3}}$, a contradiction. ∎

By Proposition 2.11 the reduced complexity of the essential graded abelian JSJ decomposition Γ_f, associated with the terminal solid limit group $MPLim_f(u,\hat{p},p',a)$, is strictly smaller than the reduced complexity of the graded abelian JSJ decomposition Λ associated with the solid limit group $Sld(x,p,a)$ with which we started.

The solid limit group $MPLim_f(u,\hat{p},p',a)$ was constructed from (shortenings of) a sequence of strictly solid specializations $\{(x^n_{i_n}, p_n, a) | 1 \leq i_n \leq n\}$ of the solid limit group $Sld(x,p,a)$ we started with, that give rise to a sequence of strictly solid specializations $\{(u^n_{i_n}, \hat{p}_n, p'_n, a) | 1 \leq i_n \leq n\}$ of the terminal solid limit group of the (extended) graded resolution $GRes(u,\hat{p},p',a)$, $MPLim_f(u,\hat{p},p',a)$.

Therefore, if there is no bound on the number of strictly solid families of specializations of a solid limit group $Sld(x,p,a)$ having the same specialization of the defining parameters, then there exists another solid limit group, $MPLim_f(u,\hat{p},p',a)$, for which:

(1) The associated (essential) graded abelian JSJ decomposition of $MPLim_f$ (with respect to the parameter subgroup $< \hat{p}, p' >$) has strictly smaller reduced complexity than the reduced complexity of the (essential) graded abelian JSJ decomposition associated with the original solid limit group $Sld(x,p,a)$.

(2) There is no bound on the number of strictly solid families of specializations

of the solid limit group $MPLim_f(u,\hat{p},p',a)$ having the same specialization of the defining parameters.

Since a strictly decreasing sequence of reduced complexities of graded abelian decompositions must be finite, a finite induction argument (on the reduced complexity of the graded abelian JSJ decompositions associated with the corresponding solid limit groups) finally completes the proof of Theorem 2.9. ∎

Strictly solid families of specializations of a solid limit group, $Sld(x,p,a)$, were defined as those families of specializations that do not factor through completions of one-step resolutions associated with the maximal flexible quotients of the solid limit group, $Sld(x,p,a)$ (see Definition 1.5). Theorem 2.9 bounds the number of strictly solid families of specializations of a solid limit group $Sld(x,p,a)$ for any possible value p_0 of the defining parameters p.

In the sequel, we will need a similar global bound on the number of families of specializations of a solid limit group $Sld(x,p,a)$ that do not factor through a given set of closures (definition 1.15 in [Se2]) of the resolutions that appear in the graded Makanin–Razborov diagram of the solid limit group $Sld(x,p,a)$, for any possible value of the defining parameters p (for which the given set of closures forms a covering closure).

Definition 2.12: Let $Sld(x,p,a)$ be a solid graded limit group and let

$$GRes_1(x,p,a),\dots,GRes_r(x,p,a)$$

be the resolutions in its strict graded Makanin–Razborov diagram (see proposition 1.10 in [Se2] for the strict MR diagram). Let

$$Cl_1^1(s,x,p,a),\dots,Cl_{m_1}^1(s,x,p,a),\dots,Cl_1^r(s,x,p,a),\dots,Cl_{m_r}^r(s,x,p,a)$$

be a finite collection of graded closures of the graded resolutions

$$GRes_1(x,p,a),\dots,GRes_r(x,p,a).$$

A specialization (x_0,p_0,a) of the solid graded limit group $Sld(x,p,a)$ is called a **strictly solid specialization** with respect to the given set of closures if:

(i) For $p=p_0$ the given set of closures forms a covering closure of the (ungraded) resolutions associated with the specialization p_0, and the graded resolutions in the Makanin–Razborov diagram of $Sld(x,p,a)$ (see definition 1.16 in [Se2] for a covering closure).

(ii) The specialization (x_0,p_0,a) is a non-degenerate specialization of the solid limit group, $Sld(x,p,a)$.

(iii) The specialization (x_0, p_0, a) does not factor through any of the given closures.

Note that, by definition (Definition 1.5), a strictly solid specialization of $Sld(x,p,a)$ is strictly solid with respect to the completion of the resolutions in the strict Makanin–Razborov diagram of the solid limit group, $Sld(x,p,a)$. Hence, a strictly solid specialization with respect to the given set of closures is not necessarily a strictly solid specialization of the solid limit group, $Sld(x,p,a)$, but every strictly solid specialization of $Sld(x,p,a)$ is strictly solid with respect to any given set of closures of the resolutions in the graded strict Makanin–Razborov diagram of the solid limit group, assuming the given set of (graded) closures is a covering closure for the (given) specialization of the defining parameters.

On the set of strictly solid specializations of the solid limit group, $Sld(x,p,a)$, with respect to a given set of closures, we define an equivalence relation, identical to the one we used to define strictly solid families in Definition 1.5 (i.e., using the tautological one-step resolution, $Sld(x,p,a) \to Sld(x,p,a)$, where the map between the levels is the identity map — see Definition 1.5). We call each equivalence class with respect to that equivalence relation a **strictly solid family** with respect to the given set of closures.

In parallel with Theorem 2.9, given a solid limit group, $Sld(x,p,a)$, and a set of closures of the resolutions that appear in its graded strict Makanin–Razborov diagram, one can globally bound the number of strictly solid families of the solid limit group $Sld(x,p,a)$ with respect to the given set of closures.

THEOREM 2.13: *Let $Sld(x,p,a)$ be a solid graded limit group and let*

$$GRes_1(x,p,a), \ldots, GRes_r(x,p,a)$$

be the resolutions in its graded strict Makanin–Razborov diagram. Let

$$Cl_1^1(s,x,p,a), \ldots, Cl_{m_1}^1(s,x,p,a), \ldots, Cl_1^r(s,x,p,a), \ldots, Cl_{m_r}^r(s,x,p,a)$$

be a finite collection of graded closures of the graded resolutions

$$GRes_1(x,p,a), \ldots, GRes_r(x,p,a).$$

There exists a constant b_S so that, for any particular value p_0 of the defining parameters p for which the given set of closures form a covering closure, there are at most b_S distinct families of homomorphisms, $h\colon Sld(x,p,a) \to F_k$, that are strictly solid with respect to the given set of closures, and satisfy $h(p) = p_0$.

Proof: To prove Theorem 2.13, we use a modification of the argument used to prove Theorem 2.9. As in the argument that was used in proving Theorem 2.9, we will prove the existence of a bound on the number of strictly solid families with respect to a given closure, by looking at sequences of almost shortest strictly solid specializations with respect to the given set of closures (see Definition 2.8 for almost shortest specializations).

In Theorem 1.7, we have shown that there exists a global constant R so that each strictly solid homomorphism $h\colon Sld(x,p,a) \to F_k$ that is among the shortest in its solid family of homomorphisms is R-AP-covered. The argument used to prove Theorem 1.7 works without any changes to almost shortest strictly solid specializations with respect to a given set of closures, hence there exists some constant R so that every strictly solid almost shortest homomorphism $h\colon Sld(x,p,a) \to F_k$ with respect to a given set of closures is R-AP-covered.

Suppose that there is no bound on the number of strictly solid families with respect to the given set of closures, for all specializations p_0 of the defining parameters p. Then there must exist some sequence $\{p_n\}$ of specializations of the defining parameters p so that, for every index n, the given set of closures (associated with the specialization p_n) form a covering closure of the ungraded resolutions associated with the (graded) resolutions in the graded strict Makanin–Razborov diagram of $Sld(x,p,a)$ and the specialization p_n of the defining parameters, and the solid graded limit group $Sld(x,p,a)$ admits n distinct strictly solid families of specializations with respect to the given set of closures. For each index n, and each strictly solid family with respect to the given set of closures associated with the specialization p_n, we associate a specialization in the strictly solid family that does not factor through any of the given closures, and is an almost shortest specialization with respect to the given set of closures, which we denote $(x^n_{i_n}, p_n, a)$ (see Definition 2.8 for an almost shortest specialization).

Given the sequence of parameters $\{p_n\}$, and their associated collections of almost shortest specializations with respect to the given set of closures, we apply the argument used to prove Theorems 2.2 and 2.5 to obtain an action of a maximal limit group $MLim(u,\hat{p},a)$ associated with the system of equations $\Sigma(u,\hat{p},a) = 1$ on some real tree Y. With the limit group $MLim(u,\hat{p},a)$ we associate a limit group $MPLim(u,\hat{p},p',a)$, generated by the limit group $MLim(u,\hat{p},a)$, and the fractions p' that were "uncovered" through the process presented in the proof of Theorems 2.2 and 2.5. By construction, there is a natural map $\eta\colon Sld(x,p,a) \to MPLim(u,\hat{p},p',a)$. Let $L(x,p,a)$ be the image of $Sld(x,p,a)$ in $MPLim(u,\hat{p},p',a)$, $L(x,p,a)=\eta(Sld(x,p,a))$.

Since the limit group $MLim(u,\hat{p},a)$ is not rigid, either it admits a non-trivial graded free decomposition, i.e., a non-trivial free decomposition in which the subgroup $<\hat{p},a>$ can be conjugated into one of the factors, or it admits a non-trivial, essential (see Definition 1.8) graded abelian JSJ decomposition, which implies that there is a non-trivial graded modular group associated with it.

The limit group $MLim(u,\hat{p},a)$ was obtained as a limit group from a sequence of specializations $\{(u_{i_m},\hat{p}_{n_m},a)\}$ that were derived from a sequence $\{(x^{n_m}_{i_{j_m}},p_{n_m},a)\}$ of almost shortest strictly solid specializations of the solid limit group $Sld(x,p,a)$ with respect to the given closure. If the limit group $MLim(u,\hat{p},a)$ admits a non-trivial graded free decomposition, then its associated limit group $MPLim(u,\hat{p},p',a)$ admits a non-trivial graded free decomposition (with respect to the parameter subgroup $<\hat{p},p'>$), which implies that all the specializations in the sequence $\{(x^{n_m}_{i_{j_m}},p_{n_m},a)\}$ split through epimorphisms λ_m: $Sld(x,p,a)\to F_k*F_t$, where F_t is a non-trivial free group, and so that the epimorphisms $\{\lambda_m\}$ map the subgroup $<p,a>$ onto the free group F_k.

By the construction of the the maximal flexible quotients of a solid limit group, and more generally the graded strict Makanin–Razborov diagram of a graded limit group (see section 10 in [Se1]), all the specializations of the solid limit group, $Sld(x,p,a)$, that factor through a free product of the form F_k*F_t, where the subgroup $<p,a>$ is mapped onto F_k, factor through some of the (graded) resolutions in the graded strict Makanin–Razborov diagram of $Sld(x,p,a)$ (cf. definition 10.6 in [Se1]). Hence, for every specialization p_n from our given sequence, all such homomorphisms (those that factor through a (graded) free product) factor through the given set of closures which form a covering closure of the ungraded resolutions associated with the graded resolutions in the strict Makanin–Razborov diagram of $Sld(x,p,a)$, and the specialization p_n of the defining parameters. In particular, homomorphisms of $Sld(x,p,a)$ that factor through a free product are not strictly solid with respect to (any) given set of closures (that form a covering closure). Therefore, we may assume that the limit groups, $MLim(u,\hat{p},a)$ and $MPLim(u,\hat{p},p',a)$, admit no non-trivial graded free decomposition.

The solid limit group $Sld(x,p,a)$ is mapped into the limit group $MPLim(u,\hat{p},p',a)$ by the homomorphism η: $Sld(x,p,a)\to MPLim(u,\hat{p},p',a)$. Let Λ be the graded abelian JSJ decomposition associated with $Sld(x,p,a)$ and let Γ be the essential graded abelian JSJ decomposition associated with

$MPLim(u,\hat{p},p',a)$. If there exists a non-QH vertex group or an edge group in Λ that is not mapped by the homomorphism η into a non-QH vertex group nor into an edge group in the essential graded abelian decomposition Γ, and if it is mapped into an abelian vertex group in Γ, if it is not mapped into the summand that contains the edge groups connected to it (in Γ) as a subgroup of finite index, then the following modification of Proposition 1.9 implies that for large enough n, the specializations $\{(x_{i_j}^{n_m}, p_{n_m}, a)\}$ do factor through the given set of closures, hence they are not strictly solid with respect to it. Therefore, we may assume that the graphs of groups, Λ of $Sld(x,p,a)$ and Γ of $MPLim$, are compatible.

Proposition 2.14: *Let $Sld(x,p,a)$ be a solid graded limit group and let*

$$GRes_1(x,p,a),\dots,GRes_r(x,p,a)$$

be the resolutions in its graded strict Makanin–Razborov diagram. Let

$$Cl_1^1(s,x,p,a),\dots,Cl_{m_1}^1(s,x,p,a),\dots,Cl_1^r(s,x,p,a),\dots,Cl_{m_r}^r(s,x,p,a)$$

be a finite collection of graded closures of the graded resolutions

$$GRes_1(x,p,a),\dots,GRes_r(x,p,a).$$

Let Λ_S be the graded abelian JSJ decomposition of $Sld(x,p,a)$ and let $\{v_n | v_n\colon Sld(x,p,a) \to F_k\}$ be a sequence of homomorphisms that converge to a limit group L. Suppose that for every index n, the given set of closures forms a covering closure of the ungraded resolutions associated with the resolutions in the graded strict Makanin–Razborov diagram of $Sld(x,p,a)$ and the specialization $v_n(p)$ of the defining parameters. Let $\nu\colon Sld(x,p,a) \to L$ be the natural quotient map.

Suppose that there exists an essential graded abelian decomposition Δ_L of L, for which there exists either an edge group or a non-abelian, non-QH vertex group in the graded abelian JSJ decomposition of $Sld(x,p,a)$, Λ_S, which is not mapped by the quotient map ν either into a conjugate of an edge group or into a non-abelian, non-QH vertex group in Δ_L, and if it is mapped by ν into a conjugate of an abelian vertex group in Δ_L then it is not mapped into a conjugate of the direct summand of this abelian vertex group which contains the subgroup generated by the edge groups connected to the abelian vertex group as a subgroup of finite index.

Then for large enough n, the homomorphisms $v_n\colon Sld(x,p,a) \to F_k$ are not strictly solid with respect to the given set of closures.

Proof: The argument is essentially identical to the proof of Proposition 1.9. In order to define flexible quotients of a solid limit group, we allowed shortenings only using the same modular group as in the ungraded case, i.e., with no additional roots of specializations of abelian edge and vertex groups, and no fractional Dehn twists (see definitions 10.1 and 10.3 in [Se1], and compare with Definition 1.4). Hence, with the notation of the proof of Theorem 1.9, since only the graded modular group as defined in ([Se1], 10.1) is used for shortenings, for any given test sequence $\{h_n: Comp_L \to F_k\}$, for large enough n, the homomorphisms $\{h_n \circ \nu_1\}$ are not only not strictly solid, they are not strictly solid with respect to the given set of closures. Therefore, the rest of the argument (used to prove Proposition 1.9) proves that for large enough n, the homomorphisms $\{v_n = h_n \circ \nu_2\}$ are not strictly solid with respect to the given set of closures as well, and the proposition follows. ∎

Finally, having proved that the abelian decompositions, Λ of $Sld(x,p,a)$ and Γ of $MPLim$, are compatible, and modifying Proposition 1.9 to the context of a covering closure (Proposition 2.14), the rest of the argument (for proving Theorem 2.13) follows precisely the argument used to prove Theorem 2.9. (Note that to prove Theorem 2.9 we used a finite induction, and indeed in applying the same argument to bound the number of strictly solid families of specializations with respect to a given set of closures (Theorem 2.13), after the first step of the induction we pass from strictly solid specializations of the solid limit group $Sld(x,p,a)$ with respect to the given set of closures, to strictly solid specializations of the (terminal) solid limit group $MPlim_f(u,\hat{p},p',a)$ (without referring to any set of closures), hence after the first step, the induction steps are identical to those used in proving Theorem 2.9). ∎

In Theorem 2.9 we found a global bound on the number of strictly solid families of homomorphisms of a solid limit group, but not on the number of solid families of strictly solid homomorphisms, which is a finer equivalence relation (see Definition 1.4 for solid families, and Definition 1.5 for strictly solid families).

Note that in Theorem 1.7 we proved that there exists a constant R, so that every strictly solid homomorphism that is (almost) shortest in its solid family (Definition 1.6) is R-AP-covered. Still, the induction argument that we used in proving Theorem 2.9 enabled us to bound only the number of strictly solid families.

By replacing the induction argument with the argument that we use in paper no. 6 in this series, which is aimed to prove the termination of the iterative procedure that we use for obtaining quantifier elimination (which is based on the *compatible* JSJ decomposition that is defined there), it is indeed possible to get a bound on the number of solid families of strictly solid homomorphisms, and not only on strictly solid families. Since we won't need this stronger result in the sequel, we don't present its proof in detail.

THEOREM 2.15: *Let $Sld(x,p,a)$ be a solid graded limit group. For any particular value p_0 of the defining parameters p, there are at most b_S distinct solid families of strictly solid homomorphisms h: $Sld(x,p,a) \to F_k$ satisfying $h(p) = p_0$.*

A similar argument can be applied to get a global bound on the number of Nielsen equivalence classes of solutions of quadratic equations. Again, since this bound is not used in our work on Diophantine geometry over groups, we don't present its proof in detail, but it is likely to find applications elsewhere.

THEOREM 2.16: *Let F_k be a free group and let $Q(x,p)$ be a quadratic graded limit group defined by either the relation*

$$[x_1,x_2]\cdots[x_{2g-1},x_{2g}] = p_1 t_2 p_2 t_2^{-1} \cdots t_n p_n t_n^{-1}$$

or by the relation

$$x_1^2 \cdots x_{g+1}^2 = p_1 t_2 p_2 t_2^{-1} \cdots t_n p_n t_n^{-1}.$$

With the quadratic limit group $Q(x,p)$ we can naturally associate a surface which we denote S_Q. We say that a homomorphism h: $Q(x,p) \to F_k$ is **taut**, *if no s.c.c. on the surface S_Q is mapped to the identity by the homomorphism h.*

There exists a constant $c = c(g,n)$, so that for any specialization p_0 of the defining parameters p, the set of taut homomorphisms h: $Q(x,p) \to F_k$ belong to at most $c(g,n)$ distinct Nielsen equivalence classes.

We refer the interested reader to [Be-Fe4], where M. Bestvina and M. Feighn give a lower bound on the maximal number of distinct Nielsen equivalence classes of solutions to a quadratic equation, i.e., a lower bound on the function $c(g,n)$ (as a function of g and n).

3. AE sets and the singular locus

Given a parameterized system of equations $\Sigma(x,p,a) = 1$, we have used the graded Makanin–Razborov diagram of it to obtain a "weak stratification" of the corresponding family of varieties in section 11 of [Se1]. In sections 1–3 of this paper we studied the basic combinatorial structure of rigid and solid solutions, and finally obtained a global bound on the number of rigid specializations of a rigid limit group for any possible value of the defining parameters (Theorem 2.5), a global bound on the number of strictly solid families of specializations of a solid limit group (Theorem 2.9), and a global bound on the number of strictly solid families with respect to a given closure (Theorem 2.13). To further study some of the basic properties of the graded Makanin–Razborov diagram in order to study elementary predicates and sets in the next papers, we need to define *Diophantine, co-Diophantine, universal* and *co-universal* (*existential*), as well as AE and EA sets.

Definition 3.1: A subset $D \subset (F_k)^u$ is called **Diophanitine** if for some fixed system of equations $\Sigma(y,d,a)=1$, D is defined by

$$D = \{(d_1,\ldots,d_u) | \exists y=(y_1,\ldots,y_n) \Sigma(y,d,a)=1\}.$$

A subset $CD \subset (F_k)^u$ is called **co-Diophantine** if CD is the complement of a Diophantine set, i.e., $CD = (F_k)^u \setminus D$ for some Diophantine set D.

Diophantine and co-Diophantine sets behave "nicely" under elementary Boolean operations.

LEMMA 3.2:

(i) *The union and the intersection of finitely many Diophantine sets in $(F_k)^u$ is a Diophantine set.*
(ii) *The union and the intersection of finitely many co-Diophantine sets in $(F_k)^u$ is a co-Diophantine set.*
(iii) *Let A be a subset of $(F_k)^u$ and suppose A is obtained by a finite sequence of binary operations on a finite set of Diophantine and co-Diophantine sets in $(F_k)^u$. Then A can be expressed as the disjoint (finite) union of Diophantine sets, co-Diophantine sets, and intersections of a Diophantine and a co-Diophantine set.*

Proof: Clearly, the intersection of finitely many Diophantine sets is Diophantine. By a theorem of A. I. Malcev (see [Ma], 3) the conjunction of the equations $x_1=1$ and $x_2=1$ is equivalent to the equation $x_1^2a_1x_1^2a_1^{-1} = (x_2a_2x_2a_2^{-1})^2$. By a theorem of G. A. Gurevich (see [Ma], 4) the disjunction of the equations $x_1=1$ and $x_2=1$ is equivalent to the conjunction of the four equations

$$(x_1a_1^{\epsilon}x_1a_1^{-\epsilon})(x_2a_2^{\delta}x_2a_2^{-\delta}) = (x_2a_2^{\delta}x_2a_2^{-\delta})(x_1a_1^{\epsilon}x_1a_1^{-\epsilon})$$

for $\epsilon,\delta=\pm1$. Hence, also the union of finitely many Diophantine sets is a Diophantine set and we get part (i) of the lemma.

Since the union of finitely many co-Diophantine sets is the complement of the intersection of finitely many Diophantine sets, the union of finitely many co-Diophantine sets is a co-Diophantine set, and similarly the intersection of finitely many co-Diophantine sets is a co-Diophantine set, and we get part (ii). Part (iii) clearly follows from parts (i) and (ii). ∎

Definition 3.3: A subset $CU \subset (F_k)^s$ is called **co-universal** (or **existential**) if for some fixed systems of equations $\Sigma_1 = 1, \dots, \Sigma_r = 1$ and inequations $\Psi_1 \neq 1, \dots, \Psi_r \neq 1$, CU is defined by

$$U = \{(p_1, \dots, p_s) | \exists y = (y_1, \dots, y_n) \ (\Sigma_1(y,p,a) = 1 \wedge \Psi_1(y,p,a) \neq 1) \vee \cdots$$
$$\cdots \vee (\Sigma_r(y,p,a) = 1 \wedge \Psi_r(y,p,a) \neq 1)\}.$$

A subset $U \subset (F_k)^s$ is called **universal** if U is the complement of a co-universal set, i.e., $U = (F_k)^s \setminus CU$ for some co-universal set CU.

Lemma 3.4:

(i) *The union and the intersection of finitely many co-universal sets in $(F_k)^s$ is a co-universal set.*

(ii) *The union and the intersection of finitely many universal sets in $(F_k)^s$ is a universal set.*

(iii) *Let A be a subset of $(F_k)^s$ and suppose A is obtained by a finite sequence of binary Boolean operations on a finite set of universal sets in $(F_k)^s$. Then A can be expressed as a disjoint (finite) union of universal sets, co-universal sets, and intersections of a universal and a co-universal set.*

Proof: Parts (i) and (ii) are immediate from the definition. Part (iii) clearly follows from parts (i) and (ii). ∎

Definition 3.5: A subset $V \subset (F_k)^s$ is called AE if for some fixed systems of equations $\Sigma_1 = 1, \dots, \Sigma_r = 1$ and fixed systems of inequations $\Psi_1 \neq 1, \dots, \Psi_r \neq 1$, V is defined by

$$V = \{(p_1, \dots, p_s) | \forall y \exists x \ (\Sigma_1(x,y,p,a) = 1 \wedge \Psi_1(x,y,p,a) \neq 1) \vee \cdots$$
$$\cdots \vee (\Sigma_r(x,y,p,a) = 1 \wedge \Psi_r(x,y,p,a) \neq 1)\}.$$

A subset $T \subset (F_k)^s$ is called EA if T is the complement of an AE set, i.e., $T = (F_k)^s \setminus V$ for some AE-set V. Alternatively, an EA set T is defined by a predicate of the form

$$T = \{(p_1,\dots,p_s) | \exists y \forall x \ (\Sigma_1(x,y,p,a)=1 \wedge \Psi_1(x,y,p,a) \neq 1) \vee \cdots \\ \cdots \vee (\Sigma_r(x,y,p,a)=1 \wedge \Psi_r(x,y,p,a) \neq 1)\}.$$

LEMMA 3.6:

(i) *The union and the intersection of finitely many AE sets in $(F_k)^s$ is an AE set.*

(ii) *The union and the intersection of finitely many EA sets in $(F_k)^s$ is an EA set.*

(iii) *Let B be a subset of $(F_k)^s$ and suppose B is obtained by a finite sequence of binary Boolean operations on a finite set of AE sets in $(F_k)^s$. Then B can be expressed as a disjoint (finite) union of AE sets, EA sets, and intersections of an AE and an EA set.*

Proof: Parts (i) and (ii) are immediate from the definition. Part (iii) clearly follows from parts (i) and (ii). ∎

The following proposition follows immediately from the construction of the singular locus in section 11 of [Se1]. It is essential in our approach to the elimination of quantifiers obtained in the next papers in this sequence.

PROPOSITION 3.7: *Let $Glim(x,p,a)$ be a graded limit group, and let $Rgd(x,p,a)$ be a rigid graded limit group in the graded Makanin–Razborov diagram of $Glim(x,p,a)$, and suppose that the graded resolution $GRes_i$ of $Glim(x,p,a)$ terminates in the rigid graded limit group $Rgd(x,p,a)$. Then:*

(i) *The set of specializations of the defining parameters p for which $Rgd(x,p,a)$ admits a rigid solution is a co-universal set.*

(ii) *The set of specializations of the defining parameters p for which $Rgd(x,p,a)$ admits at least s rigid solutions is a co-universal set.*

(iii) *The set of specializations of the defining parameters p for which $Rgd(x,p,a)$ admits precisely s rigid solutions is the intersection of a universal set and a co-universal set.*

(iv) *The set of specializations of the defining parameters p for which the resolution $GRes_i$ is non-degenerate for some rigid solution is a co-universal set. The set of specializations of the defining parameters p for which a particular part of $GRes_i$ is non-degenerate for some rigid solution is a co-universal set.*

Proposition 3.7 studies rigid graded limit groups and graded resolutions pass through them. An analogous statement is valid for graded resolutions terminating in a solid graded limit group. The proof of the following proposition is immediate from the definitions and left for the interested reader.

PROPOSITION 3.8: *Let $Glim(x,p,a)$ be a graded limit group and let $Sld(x,p,a)$ be a solid graded limit group in the graded Makanin–Razborov diagram of $Glim(x,p,a)$. Suppose that the graded resolution $GRes_i$ of $Glim(x,p,a)$ terminates in a strictly solid family of the solid graded limit group $SldGlim(x,p,a)$. Then:*

- (i) *The set of specializations of the defining parameters p for which $Sld(x,p,a)$ admits a strictly solid solution is an EA set.*
- (ii) *The set of specializations of the defining parameters p for which $Sld(x,p,a)$ admits at least s strictly solid families of solutions is an EA set.*
- (iii) *The set of specializations of the defining parameters p for which $Sld(x,p,a)$ admits precisely s strictly solid families of solutions is the intersection of an EA set and an AE set.*
- (iv) *The set of specializations of the defining parameters p for which the resolution $GRes_i$ is non-degenerate for some strictly solid solution is an EA set. The set of specializations of the defining parameters p for which a particular part of $GRes_i$ is non-degenerate for some strictly solid solution is an EA set.*

Proposition 3.8 shows that the set of values of the parameters p for which there are at least s strictly solid families of solutions is an EA set, and the set of values of the parameters p for which there are precisely s strictly solid families of solutions is the intersection of an EA set and an AE set. For the purposes of our quantifier elimination procedure, we will need a modification of these results for strictly solid solutions of a solid limit group $Sld(x,p,a)$, with respect to a given set of closures of the resolutions in the Makanin–Razborov diagram of the solid limit group $Sld(x,p,a)$ (see Definition 2.12).

PROPOSITION 3.9: *Let $Sld(x,p,a)$ be a solid graded limit group and let*

$$GRes_1(x,p,a),\ldots,GRes_r(x,p,a)$$

be the resolutions in its graded strict Makanin–Razborov diagram (see proposition 1.10 in [Se2] for the strict MR diagram). Let

$$Cl_1^1(s,x,p,a),\ldots,Cl_{m_1}^1(s,x,p,a),\ldots,Cl_1^r(s,x,p,a),\ldots,Cl_{m_t}^r(s,x,p,a)$$

be a finite collection of graded closures of the graded resolutions

$$GRes_1(x,p,a),\ldots,GRes_r(x,p,a).$$

Then:

(i) *The set of specializations of the defining parameters p for which $Sld(x,p,a)$ admits precisely s strictly solid families of solutions with respect to the given set of closures (see Definition 2.12) is in the Boolean algebra of AE sets.*

(ii) *Let $Glim(x,p,a)$ be a graded limit group, and suppose that a resolution $GRes(x,p,a)$ in its graded strict Makanin–Razborov diagram terminates in the solid limit group $Sld(x,p,a)$. Suppose that $Sld(x,p,a)$ has the graded strict Makanin–Razborov diagram and the set of closures of the resolutions in it as listed above. Let $Cl_1(s,x,p,a),\dots,Cl_v(s,x,p,a)$ be a (graded) closure of the graded resolution $GRes(x,p,a)$. The set of the defining parameters p for which $Sld(x,p,a)$ admits precisely s strictly solid families of solutions, with respect to the given set of closures (of the resolutions in the graded strict Makanin–Razborov diagram of $Sld(x,p,a)$), and the given set of closures of the graded resolution $GRes$ form a covering closure for the ungraded resolutions associated with the s strictly solid families, is in the Boolean algebra of AE sets.*

Proof: First, suppose that $GRes(x,p,a)$ is a graded resolution that terminates in a solid limit group $Sld(x,p,a)$, and $Sld(x,p,a)$ admits no flexible quotients. By part (iii) of Proposition 3.8, the set of specializations of the defining parameters p, for which $Sld(x,p,a)$ admits precisely s strictly solid families of solutions, is the intersection of an EA set and an AE set.

Let $Cl_1(s,x,p,a),\dots,Cl_v(s,x,p,a)$ be the given set of (graded) closures of the graded resolution $GRes(x,p,a)$. Then the set of specializations of the defining parameters p, for which $Sld(x,p,a)$ admits precisely s strictly solid families of solutions and the given closure forms a covering closure for the ungraded resolutions associated with these s strictly solid families, is the intersection of an EA set and an AE set as well.

Now let $Sld(x,p,a)$ be a general solid limit group (possibly with flexible quotients), with a given set of graded closures of the resolutions that appear in the graded strict Makanin–Razborov diagram of $Sld(x,p,a)$. By Theorem 2.5 there is a global bound on the number of rigid specializations of a rigid limit group, and by Theorem 2.13 there is a global bound on the number of strictly solid families of specializations of a solid limit group with respect to a given set of closures of the resolutions in its graded strict Makanin–Razborov diagram. Hence, given a solid limit group $Sld(x,p,a)$, and a given set of closures of the resolutions that appear in its graded strict Makanin–Razborov diagram, for any specialization p_0 of the defining parameters p, there are finitely many possibilities for the numbers of rigid and strictly solid families of specializations of the terminal rigid and solid limit groups of the resolutions in the graded strict Makanin–Razborov diagram of $Sld(x,p,a)$ with respect to the given set of closures.

By a finite induction on the number of levels of the various resolutions in the strict Makanin–Razborov diagram, the set of the defining parameters p, for which there are precisely s strictly solid families of specializations of $Sld(x,p,a)$ with respect to the given set of closures and precisely one of the (finitely many) possibilities for the numbers of rigid and strictly solid families of specializations of the terminal rigid and solid limit groups of the resolutions in the graded strict Makanin–Razborov diagram of $Sld(x,p,a)$ with respect to the given set of closures, and the given set of closures forms a covering closure of the ungraded resolutions that are associated with the resolutions in the strict Makanin–Razborov diagram of $Sld(x,p,a)$ and the rigid and strictly solid specializations of the terminal limit groups of these resolutions, is the intersection of an EA set and an AE set. Hence, the set of the defining parameters p, for which there are precisely s strictly solid families of specializations with respect to the given set of closures, is a union of finitely many sets which are intersections of EA and AE sets, so it is in the Boolean algebra of AE sets, which proves part (i). Part (ii) follows by the same argument. ∎

4. Solid configuration limit groups

In section 3 we have shown that the set of specializations of the defining parameters p, for which there are precisely s strictly solid families of solutions of a solid limit group $Sld(x,p,a)$, is in the Boolean algebra of AE sets. In case the solid limit group, $Sld(x,p,a)$, has no flexible quotients, it is possible to improve this result and prove that the set of specializations of the defining parameters p, for which there are precisely s strictly solid families of solutions, is in the Boolean algebra of universal sets. To prove this stronger result, we need to construct *solid configuration limit groups* associated with the given solid limit group $Sld(x,p,a)$, a construction that, unfortunately, does not seem to give a similar improvement in case the solid limit group has flexible quotients.

For presentation purposes, we start by constructing these configuration groups for solid limit groups with no flexible quotients that have a graded JSJ decomposition with two vertices and a single edge.

Let $Sld(x,p,a)$ be a solid limit group with no flexible quotients and with graded JSJ decomposition that contains two vertices and a single edge. Suppose that $Sld(x,p,a) =< y,p,a > *_A < w >$, where the subgroup $< w >$ is not abelian. Since $< w >$ is not abelian and the graded JSJ decomposition of $Sld(x,p,a)$ consists of a single edge and two vertices, the abelian edge group A is necessarily cyclic (otherwise, the vertex group $< w >$, which is in particular a limit group, admits a non-trivial abelian JSJ decomposition in which the non-cyclic abelian subgroup A is abelian). By Theorem 2.9 there is a global bound on the number of strictly solid families of specializations of the subgroup $Sld(x,p,a)$, that is valid for all the specific specializations of the parameters p. Hence, by the definition of strictly solid families and the specific structure of the solid limit group $Sld(x,p,a)$, we may assume that for any specific specialization p_0 of the parameters p, there are at most d different specializations of the subgroup (distinguished vertex group) $< y,p,a >$ for which $p = p_0$, and for each specialization (y_0,p_0,a) there are at most e specializations w_0 of the subgroup $< w >$ up to conjugation by an element commuting with the specialization of the subgroup (edge group) A_0. At this point we need to present *configuration limit groups.* To construct configuration limit groups we need to define *configuration homomorphisms.*

Definition 4.1: A **configuration homomorphism** of the solid limit group $Sld(x,p,a)$ is a homomorphism

$$h\colon < cy_1, s_1, cw_1^1, \ldots, cw_e^1, \ldots, cy_d, s_d, cw_1^d, \ldots, cw_e^d, p, a > \to F_k$$

for which:

(1) $h(cy_i, cw_j^i, p, a)$ is a strictly solid specialization of the solid limit group $Sld(x,p,a) = Sld(y,w,p,a)$.
(2) $h(s_i)$ is a generator of the subgroup $h(a_i)$, i.e., $h(s_i)$ is a generator of the cyclic subgroup in F_k that contains the image of the edge group $h(A)$.
(3) For each index i, $1 \leq i \leq d$, if we fix $h(cw_1^i)$, then $h(cw_1^i, \ldots, cw_e^i)$ is the shortest possible under the (separate) action of conjugation by powers of the primitive root $h(s_i)$ on each of the variables $cw_2^i, \ldots, cw_e^i$.

A limit group obtained as the limit of a sequence of configuration homomorphisms is called a **configuration limit group**, if for every index i, $1 \leq i \leq d$, the sequences of strictly solid homomorphisms of $Sld(x,p,a)$ that are the restrictions of the configuration homomorphisms $\{h_n\}$, to the subgroups $< CY_i, CW_j^i >$, for $1 \leq j \leq e$, do not factor through finitely many proper quotients of the solid limit group $Sld(x,p,a)$.

By construction, there are only finitely many maximal configuration limit groups. We call each of this canonical collection of maximal limit groups a **solid configuration limit group**, and denote it $Conf(Sld)(s,cy,cw,p,a)$.

Having constructed solid configuration limit groups, we analyze the structure of their subgroups $V_i =< s_i, cw_1^i, \dots, cw_e^i >$ for $i = 1, \dots, d$. Clearly, the subgroup V_i is in particular an ungraded limit group, so it admits a canonical free decomposition and each factor admits a canonical (ungraded) abelian JSJ decomposition.

LEMMA 4.2: *Let $Conf(Sld)(s,cy,cw,p,a)$ be a solid configuration limit group associated with the solid limit group $Sld(x,p,a)$, let $\{h_n\}$ be the sequence of configuration homomorphisms that were used to construct the configuration limit group $Conf(Sld)$, let $V_i =< s_i, cw_1^i, \dots, cw_e^i >$ for some index i, $1 \leq i \leq d$, and let Λ_{V_i} be the abelian decomposition of V_i obtained by first factoring V_i into the most refined free decomposition, and then taking the essential abelian JSJ decomposition of each of the factors, a decomposition in which every non-cyclic abelian subgroup is elliptic (see Definition 1.8 for the essential decomposition). Suppose that:*

(i) *The sequences of strictly solid homomorphisms of $Sld(x,p,a)$ that are the restrictions of the configuration homomorphisms $\{h_n\}$ to the subgroups $< CY_i, CW_j^i >$, for $1 \leq j \leq e$, do not factor through finitely many proper quotients of the solid limit group $Sld(x,p,a)$.*

(ii) *V_i is not a surface group, i.e., Λ_{V_i} is not the trivial decomposition.*

Then every non-trivial element in the abelian subgroup $< s_i, A_i >$ is not elliptic in Λ_{V_i}. In particular, the subgroup $< s_i, A_i >$ has to be a cyclic group.

Proof: Suppose that A_i is elliptic in the decomposition Λ_{V_i}. For each index j, $1 \leq j \leq e$, the subgroup $CW_j^i =< cw_j^i >$, being a subgroup of V_i, inherits a decomposition Δ_j^i from the graph of groups Λ_{V_i}. Clearly, A_i is elliptic in the graph of groups Δ_j^i. If all the abelian decompositions Δ_j^i correspond to a (non-trivial) amalgamated product, in which one of the vertex groups is the maximal abelian subgroup that contains A_i in CW_j^i, then the sequences of strictly solid homomorphisms of $Sld(x,p,a)$ that are the restrictions of the configuration homomorphisms $\{h_n\}$, to the subgroups $< CY_i, CW_j^i >$, factor through finitely many proper quotients of the solid limit group $Sld(x,p,a)$, a contradiction to assumption (i).

The subgroup CW_j^i was constructed from a sequence of configuration homomorphisms $\{h_n\colon Conf(Sld)(s,cy,cw,p,a) \to F_k\}$. If for at least one of the indices j, the graph of groups Δ_j^i is neither the trivial graph of groups nor an amalgamated product in which one of the vertex groups is the maximal abelian subgroup that contains A_i in CW_j^i, then for large enough n the restriction of the homomorphism h_n to the subgroup $< CY_i, CW_j^i >$ does not belong to a strictly solid family of homomorphisms of $Sld(x,p,a)$.

Hence, the graph of groups Λ_{V_i} is a star of groups, with a (distinguished) vertex stabilized by a subgroup of V_i that contains A_i, and all the edge groups are subgroups of A_i. But the strictly solid homomorphisms $\{h_n\}$ were assumed to shorten the specializations of the various subgroups V_i for $i = 1, \ldots, d$, that contradicts this structure. Therefore, A_i cannot be elliptic in the graph of groups Λ_{V_i}.

Now, if a non-trivial element of the subgroup $< s_i, A_i >$ is elliptic in the decomposition Λ_{V_i}, then the subgroup $< s_i, A_i >$ is free abelian of rank 2, so the entire subgroup $< s_i, A_i >$ is elliptic in Λ_{V_i}, a contradiction. Therefore, every non-trivial element from of the (abelian) subgroup $< s_i, A_i >$ acts hyperbolically on the Bass–Serre tree corresponding to the graph of groups Λ_{V_i}, and the subgroup $< s_i, A_i >$ is cyclic. ∎

LEMMA 4.3: *With the notation and the assumptions of Lemma 4.2, let Δ_j^i be the graph of groups inherited by the subgroup CW_j^i from the graph of groups Λ_{V_i}. Then the edge group A_i is hyperbolic in the graph of groups Δ_j^i, and A_i cannot be conjugated into the fundamental group of any proper subgraph of Λ_{V_i} and any proper subgraph of Δ_j^i.*

Proof: A_i is hyperbolic in Δ_j^i since it is hyperbolic in Λ_{V_i} by Lemma 4.2. If the subgroup A_i can be conjugated into a proper subgraph of Λ_{V_i} or Δ_j^i, then for large enough n, the homomorphisms $\{h_n\}$ from which the configuration solid limit group V_i was constructed are not strictly solid homomorphisms (for at least one of the indices j, $1 \leq j \leq e$). ∎

PROPOSITION 4.4: *With the notation of Lemma 4.2, suppose that the sequences of strictly solid homomorphisms of $Sld(x,p,a)$, that are the restrictions of the configuration homomorphisms $\{h_n\}$ to the subgroups $< CY_i, CW_j^i >$, do not factor through finitely many proper quotients of the solid limit group $Sld(x,p,a)$.*

Let Γ_{V_i} be an arbitrary non-trivial essential abelian decomposition of the subgroup V_i (see Definition 1.8 for an essential abelian decomposition). Then the subgroup $< s_i, A_i >$ is not elliptic in Γ_{V_i}.

Proof: Suppose that the subgroup $< s_i, A_i >$ is elliptic in Γ_{V_i}. W.l.o.g. we may assume that Γ_{V_i} contains only one edge. If the edge group in Γ_{V_i} is trivial, i.e., Γ_{V_i} corresponds to a free decomposition of the group V_i in which the subgroup A_i can be conjugated into one of the factors, then the restrictions of the configuration homomorphisms $\{h_n\}$, from which the configuration limit group $Conf(Sld)(s, cy, cw, p, a)$ was constructed, to at least one of the subgroups $< CW^i_j >$, factor through a non-trivial (graded) free product, hence they are flexible, a contradiction to the assumption that they are strictly solid. Hence, we may assume that the edge group in the graph of groups Γ_{V_i} is a non-trivial abelian group.

By exactly the same argument, if the subgroup $< s_i, A_i >$ is elliptic in Γ_{V_i}, then the edge group in Γ_{V_i} has to include a conjugate of the cyclic subgroup A_i, and since by Lemma 4.2 the subgroup A_i is not elliptic in the canonical abelian (JSJ) decomposition Λ_{V_i} of V_i in case V_i is not a surface group, the maximal abelian group containing A_i is cyclic. Hence, the edge group in Γ_{V_i} is the maximal cyclic subgroup containing the subgroup $< s_i, A_i >$ in V_i.

Since A_i is contained in the edge group of Γ_{V_i}, and in particular A_i is elliptic in Γ_{V_i}, each of the subgroups CW^i_j has to be elliptic in Γ_{V_i} (otherwise, the configuration homomorphisms from which the configuration limit group was constructed are not all strictly solid). If V_i is a surface group, then by lemma 1.4 of [Se4], for every index j each of the configuration homomorphisms $\{h_n\}$ (from which $Conf(Sld)(s, cy, cw, p, a)$ was constructed) restricted to the subgroups CW^i_j is a flexible homomorphism, a contradiction to the homomorphisms $\{h_n\}$ being configuration homomorphisms. Hence, we may assume that V_i is not a surface group.

By Lemma 4.2, if V_i is not a surface group then each of the non-cyclic freely indecomposable factors of V_i is elliptic in Γ_{V_i}. Hence, in particular, V_i is not freely indecomposable. To conclude the proof of the proposition (for freely decomposable V_i) we need the following proposition. The proposition is proved for a f.g. free group in [Be-Fe3]. The proof for a general free product is a straightforward generalization of the argument presented in [Be-Fe3].

Proposition 4.5 ([Be-Fe3]): *Let G be a f.g. group and suppose*

$$G = B_1 * \cdots * B_n * F_s$$

is the maximal (most refined) free decomposition of G, where B_i are non-infinite cyclic, freely indecomposable and F_s is free. If $G = D_1 *_C D_2$, C is infinite cyclic, and each of the subgroups $B_1, \ldots, B_n$ can be conjugated into either D_1 or D_2, then C is a factor in a free decomposition of either D_1 or D_2. If $G = D*_C$, C is infinite cyclic and each of the subgroups $B_1, \ldots, B_n$ can be conjugated into

D, then either C or tCt^{-1} (where t is the corresponding Bass–Serre generator) is a factor in a free decomposition of D.

According to the above proposition the subgroup A_i is a factor in one of the vertex groups to which it is connected. Hence, at least for one index j the configuration homomorphisms $\{h_n\}$ (from which $Conf(Sld)(s,cy,cw,p,a)$ was constructed) restricted to the subgroup CW_j^i are flexible homomorphisms, a contradiction to the homomorphisms $\{h_n\}$ being configuration homomorphisms. ∎

Finally, note that in Lemmas 4.2–4.5 (and using the notation of Lemma 4.2) we assumed that the sequences of strictly solid homomorphisms of $Sld(x,p,a)$ that are the restrictions of the configuration homomorphisms $\{h_n\}$ to the subgroups $< CY_i, CW_j^i >$, for $1 \leq j \leq e$, do not factor through finitely many proper quotients of the solid limit group $Sld(x,p,a)$. However, in case they do factor through finitely many proper quotients of $Sld(x,p,a)$, we can replace $Sld(x,p,a)$ by those proper quotients, and replace the essential graded JSJ abelian decomposition of $Sld(x,p,a)$ by the essential graded abelian JSJ decompositions of the proper quotients. Lemmas 4.2–4.5 are valid after this replacement.

According to Proposition 3.8, the sets containing all the values of the parameters p for which there are at least s strictly solid families are EA sets, the sets for which there are at most s strictly solid families are AE sets, and the sets for which there are exactly s strictly solid families are the intersection of an EA set and an AE set. To show that these sets are in fact all in the Boolean algebra of universal sets (under our special assumptions), we need to be able to bound the possible conjugacies of the solid specialization of vertex groups by elements that commute with the corresponding specialization of edge groups. This was the purpose of constructing solid configuration limit groups. To obtain a desired bound on the possible required conjugacies we need to define the *configuration scalar* of a solid configuration limit group.

Definition 4.6: Let $Conf(Sld)(s,cy,cw,p,a)$ be a solid configuration limit group. We say that an integer C_{conf} is a **configuration scalar** of

$$Conf(Sld)(s,cy,cw,p,a),$$

if, for every sequence of homomorphisms $\{h_n\colon Conf(Sld)(s,cy,cw,p,a) \to F_k\}$ (that do not necessarily satisfy the conditions of configuration homomorphisms — see Definition 4.1 for these conditions) that converges to the solid configuration limit group $Conf(Sld)(cy,cw,p,a)$, and for which for every index n,

$h_n(s_i)$ is a generator of the cyclic subgroup of F_k that contains $h_n(A_i)$ for every index i, $1 \leq i \leq d$, and there exists some index n_0 (depending on the sequence of the configuration homomorphisms) so that for every $n > n_0$ if for some index i, $1 \leq i \leq d$, and couple of indices j_1, j_2, $1 \leq j_1 < j_2 \leq e$, there exists some element $t_n \in F_k$ for which t_n commutes with $h_n(A_i)$ and $t_n h_n(cw^i_{j_1}) t_n^{-1} = h_n(cw^i_{j_2})$, then there exists some integers m_n, $-C_{conf} \leq m_n \leq C_{conf}$, for which $h_n(s_i)^{m_n} h_n(cw^i_{j_1}) h_n(s_i)^{-m_n} = h_n(cw^i_{j_2})$.

Proposition 4.7: *For every solid configuration limit group*

$$Conf(Sld)(s, cy, cw, p, a),$$

of a solid limit group $Sld(x,p,a)$ with no flexible quotients and graded abelian JSJ decomposition consisting of a single edge, there exists a configuration scalar C_{conf}.

Proof: Suppose that there was no configuration scalar for some configuration limit group $Conf(Sld)(s, cy, cw, p, a)$ associated with a solid limit group $Sld(x, p, a)$, that satisfies the assumptions of the proposition. Then there exists a sequence of homomorphisms h_n: $Conf(Sld)(s, cy, cw, p, a) \to F_k$ that converges to the configuration limit group $Conf(Sld)$, $h_n(s_i)$ is a generator of the maximal cyclic subgroup that contains the subgroup $h_n(A_i)$ for every index n, and every i, $1 \leq i \leq d$, and there exists an index i, $1 \leq i \leq d$, and couple of indices j_1, j_2, $1 \leq j_1 < j_2 \leq e$, so that for every index n, there exists an element $t_n \in F_k$, for which t_n commutes with $h_n(A_i)$, and $t_n h_n(cw^i_{j_1}) t_n^{-1} = h_n(cw^i_{j_2})$, but for every integer m, $-n \leq m \leq n$, $h_n(s_i)^m h_n(cw^i_{j_1}) h_n(s_i)^{-m} \neq h_n(cw^i_{j_2})$.

By passing to a subsequence (still denoted $\{h_n\}$), we may assume that the sequence of homomorphisms $\{h_n|_{V_i}\}$ converges into a (stable) action of the subgroup V_i of the configuration limit group $Conf(Sld)(s, cy, cw, p, a)$ on some real tree Y. By [Be-Fe2], with the action of V_i on the real tree Y, one canonically associates a graph of groups Γ_Y with fundamental group V_i and abelian edge groups, which in our case must be essential (see Definition 1.8), and by Proposition 4.4 the element s_i is hyperbolic in the graph of groups Γ_Y, so s_i acts hyperbolically on the real tree Y. Hence, by the Gromov–Hausdorff convergence of the homomorphisms $\{h_n|_{V_i}\}$ into the action of V_i on the real tree Y, the sequence of homomorphisms $\{h_n\}$ itself does have a configuration scalar, i.e., there exists some integer ℓ_0 so that for every index n, if there exists an element t_n for which t_n commutes with $h_n(A_i)$ and $t_n h_n(cw^i_{j_1}) t_n^{-1} = h_n(cw^i_{j_2})$, then for some integer m, $-\ell_0 \le m \le \ell_0$, $h_n(s_i)^m h_n(cw^i_{j_1}) h_n(s_i)^{-m} = h_n(cw^i_{j_2})$, a contradiction to the properties of the sequence $\{h_n\}$, and the proposition follows. ∎

The homomorphisms that factor through $Conf(Sld)(s,cy,cw,p,a)$ and cannot be "scaled" by a bounded power of the generator $h(s_i)$ of the image of the edge group $h(A_i)$ are called **unscaled homomorphisms**.

Definition 4.8: Let C_{conf} be a configuration scalar of the solid configuration limit group $Conf(Sld)(s,cy,cw,p,a)$. A homomorphism

$$\mu\colon Conf(Sld)(s,cy,cw,p,a) \to F_k$$

is said to be an **unscaled configuration homomorphism** if for every index i, $1 \le i \le d$, $h(s_i)$ is a generator of the maximal cyclic subgroup that contains $h(A_i)$, and for some index i, $1 \le i \le d$, and couple of indices j_1, j_2, $1 \le j_1 < j_2 \le e$, there exists some element $t \in F_k$ for which t commutes with $\mu(A_i)$, $t\mu(cw^i_{j_1})t^{-1} = \mu(cw^i_{j_2})$, and there is no integer m, $-C_{conf} \le m \le C_{conf}$, for which $\mu(s_i)^m \mu(cw^i_{j_1}) \mu(s_i)^{-m} = \mu(cw^i_{j_2})$.

Proposition 4.9: *Let $UnScConf(s,cy,cw,p,a)$ be a limit group obtained from a converging sequence of either*

(i) *unscaled homomorphisms $\{\mu_n\colon Conf(Sld)(s,cy,cw,p,a) \to F_k\}$, or*

(ii) *homomorphisms $\{\nu_n\colon Conf(Sld) \to F_k\}$ for which, for some index i, $1 \le i \le d$, the sequence of specializations $\{\nu_n(s_i)\}$ have unbounded roots, i.e., for every index n, $\nu_n(s_i)$ has a root of order at least n.*

Then $UnScConf(s,cy,cw,p,a)$ is a proper quotient of the configuration limit group $Conf(Sld)(s,cy,cw,p,a)$.

Proof: Suppose that $UnScConf$ is a limit of unscaled homomorphisms (case (i)). If $UnScConf(s,cy,cw,p,a)$ is not a proper quotient of the configuration limit group $Conf(s,cy,cw,p,a)$, then there exists a sequence of homomorphisms $h_n\colon Conf(s,cy,cw,p,a) \to F_k$ converging to $Conf(s,cy,cw,p,a)$ for which C_{conf} in not a configuration scalar, a contradiction to the properties of the configuration scalar given in Definition 4.6.

Suppose that $UnScConf$ is a limit of specializations of $Conf(Sld)$, so that $\{\nu_n(s_i)\}$ have unbounded roots (case (ii)). We enlarge the group $Conf(Sld)$ with an element e, and for each index n, we set $\nu_n(e)$ to be the primitive root of $\nu_n(s_i)$. By doing that we get a sequence of homomorphisms (still denoted)

$\{\nu_n: Conf(Sld)* < e > \to F_k\}$, and by passing to a subsequence we may assume that the sequence converges to some limit group L.

We look at the (non-trivial) essential abelian JSJ decomposition of the limit group L, Λ_L, in which every abelian non-cyclic subgroup is elliptic. Since the sequence $\{\nu_n(s_i)\}$ has unbounded roots and $\nu_n(e)$ is the primitive root of $\nu_n(s_i)$, the subgroup $< e, s_i >$ in L is abelian of rank 2. Hence, it is elliptic in Λ_L. Therefore, if $UnScConf$ is not a proper quotient of $Conf(Sld)$, then V_i inherits a non-trivial essential abelian decomposition from Λ_L, a non-trivial essential abelian decomposition in which the subgroup $< s_i, A_i >$ is elliptic, a contradiction to Proposition 4.4. ∎

Definition 4.10: Let C_{conf} be a configuration scalar of the solid configuration limit group $Conf(Sld)(s, cy, cw, p, a)$. By the ascending chain condition for limit quotients (see lemma 5.5 in [Se1]), the collection of unscaled homomorphisms μ: $Conf(Sld)(s, cy, cw, p, a) \to F_k$ factors through a finite (canonical) collection of maximal limit groups, and so is the collection of all the sequences of homomorphisms $\{\nu_n: Conf(Sld) \to F_k\}$ for which, for some index i, $1 \leq i \leq d$, the sequence of specializations $\{\nu_n(s_i)\}$ has unbounded roots (see case (ii) in Proposition 4.9).

We call each of these maximal limit groups an **unscaled configuration quotient** of the solid configuration limit group $Conf(Sld)(s, cy, cw, p, a)$. By Proposition 4.9, every unscaled configuration quotient of a solid configuration limit group is a proper quotient of it.

Given a configuration limit group, $Conf(Sld)$, we associate with it its finite collection of unscaled configuration quotients that are all proper quotients of the configuration limit group $Conf(Sld)$. With each unscaled configuration quotient, $UnScConf$, we associate all the configuration homomorphisms that factor through it (see Definition 4.1). This collection of configuration homomorphisms factors through a finite collection of maximal limit groups, that are all configuration limit groups. Each of the obtained configuration limit groups is naturally a quotient of the unscaled configuration quotient, $UnScConf$, with which it is associated. By Proposition 4.7, each of the obtained configuration limit groups has a configuration scalar, and by Proposition 4.9, with the configuration limit group and its configuration scalar there is a canonical (finite) collection of unscaled configuration quotients that are all proper quotients of the configuration limit group with which they are associated. By the descending chain condition for limit groups ([Se1], 5.1), this iterative construction of configuration limit groups and their associated unscaled configuration quotients terminates after finitely many steps.

The construction of the configuration limit groups and their unscaled quotients are the tools needed for analyzing the solid families in case a solid limit group has no flexible quotients and its graded JSJ decomposition contains no QH vertex groups with more than one boundary component.

PROPOSITION 4.11: *Let $Sld(x,p,a)$ be a solid graded limit group with no flexible quotients and suppose its graded abelian JSJ decomposition contains a single (distinguished) non-abelian, non-QH vertex group, possibly few abelian and QH vertex groups with a single boundary component connected to the distinguished vertex group, and possibly few loops based on the distinguished vertex. Then the set of specializations of the defining parameters p for which $Sld(x,p,a)$ admits precisely s strictly solid families of solutions is in the Boolean algebra of universal sets.*

Proof: In case the abelian JSJ decomposition of the solid limit group $Sld(x,p,a)$ contains a single non-abelian, non-QH vertex group, and (possibly) few abelian vertex groups and QH vertex groups with a single boundary component connected to it, and (possibly) few loops based on the distinguished vertex, and assuming $Sld(x,p,a)$ has no flexible quotients, the number of the strictly solid families of $Sld(x,p,a)$ is precisely the number of different specializations of the distinguished vertex group (with a fixed specialization of the defining parameters). Therefore, the analysis of the set of specializations of the parameters p for which there are precisely s strictly solid families of solutions is identical to the analysis of the different number of rigid solutions of a rigid limit group with no flexible quotients (Proposition 3.7). Hence, the set of parameters p for which there are precisely s strictly solid families of solutions is in the Boolean algebra of universal sets. ∎

PROPOSITION 4.12: *Let $Sld(x,p,a)$ be a solid graded limit group with no flexible quotients and suppose its graded (abelian) JSJ decomposition contains two non-abelian, non-QH vertex groups connected by a single edge. Then the set of specializations of the defining parameters p for which $Sld(x,p,a)$ admits precisely s strictly solid families of solutions is in the Boolean algebra of universal sets.*

Proof: Suppose that the graded JSJ decomposition of $Sld(x,p,a)$ contains a single edge. To analyze the set of specializations of the parameters p for which there are precisely s strictly solid families of solutions of $Sld(x,p,a)$, we first look at the set of specializations of the parameters p for which there are precisely q specializations of the distinguished vertex that can be extended to non-degenerate solutions of $Sld(x,p,a)$. This set is defined in a similar way to the set of specializations of the parameters for which a rigid limit group admits a fixed number of rigid solutions, so it is in the Boolean algebra of universal sets.

Furthermore, the construction of the configuration limit groups

$$Conf(Sld)(s, cy, cw, p, a)$$

associated with $Sld(x,p,a)$, and their unscaled configuration quotients (Definition 4.10), implies that there exists a global constant D_{conf} so that, given a configuration limit group, $Conf(Sld)$, and a homomorphism $h\colon Conf(Sld) \to F_k$ that does not factor through the unscaled configuration quotients of the given configuration limit group, for every index i, $1 \leq i \leq d$, the order of a (possible) root of the element $h(s_i)$ is bounded by D_{conf}.

Therefore, the set of specializations of the parameters p for which:

(i) there are precisely q different specializations of the distinguished vertex group that can be extended into strictly solid specializations of $Sld(x,p,a)$,

(ii) given a configuration limit group, $Conf(Sld)$, there exists a homomorphism $h\colon Conf(Sld) \to F_k$ that does not factor through any of the unscaled configuration quotients of the given configuration limit group,

(iii) for each index i, $1 \leq i \leq q$, $h(s_i)$ is primitive,

is in the Boolean algebra of universal sets.

At this point we can use the (finitely many) configuration limit groups, their configuration scalars and unscaled configuration quotients to analyze the set of specializations of the parameters p for which there are precisely s strictly solid families of solutions of $Sld(x,p,a)$.

To construct a predicate that specifies the set of specializations of the defining parameters, for which there are at least s strictly solid families with precisely q different specializations of the distinguished vertex group, we restrict ourselves to the set of specializations of the defining parameters for which there are precisely q different specializations of the distinguished vertex group (which is in the Boolean algebra of universal set). Then we require the existence of a homomorphism of one of the associated configuration limit groups, $h\colon Conf(Sld) \to F_k$, for which:

(i) the homomorphism h does not factor through any of the unscaled configuration quotients, associated with the configuration limit group,

(ii) for each index i, $1 \leq i \leq q$, $h(s_i)$ is primitive (i.e., does not have a root of order bounded by D_{conf}),

(iii) for each index i, $1 \leq i \leq q$, and two different indices $1 \leq j_1 < j_2 \leq e_i$, the restrictions of the homomorphism h to the subgroups $cw^i_{j_1}$ and $cw^i_{j_2}$ cannot be made equal by a conjugation in $h(s_i)^m$, where $-C_{conf} \leq m \leq C_{conf}$.

Such a predicate is clearly in the Boolean algebra of universal sets, and since the entire set of specializations of the defining parameters p for which there are at least s strictly solid families is a finite union of such sets, the set of specializations of the defining parameters for which there are precisely s strictly solid families is in the Boolean algebra of universal sets. ∎

Proposition 4.13: *Let $Sld(x,p,a)$ be a solid graded limit group with no flexible quotients and suppose its graded abelian JSJ decomposition is a tree of groups with no QH vertex groups containing more than one boundary component. Then the set of specializations of the defining parameters p for which $Sld(x,p,a)$ admits precisely s strictly solid families of solutions is in the Boolean algebra of universal sets.*

Proof: The proof is essentially identical to the single edge case. With a solid limit group, $Sld(x,p,a)$, with a graded abelian JSJ decomposition which is a tree of groups with no QH vertex groups having more than a single boundary component, we associate a canonical (finite) collection of configuration limit groups and configuration scalars, precisely as in the single edge case. We define the unscaled configuration quotients of a configuration limit group as the limit of either a sequence of unscaled homomorphisms (see Definition 4.8) or homomorphisms of the given configuration limit group for which the specializations of one of the edge groups have unbounded roots.

By this construction, there exists some global constant D_{conf} so that, for every homomorphism of a configuration limit group that does not factor through its unscaled configuration quotients, if the specialization of an edge group has a non-trivial root, than the order of such a root is bounded by D_{conf}. Having this bound, the theorem follows by the argument used in the single edge case (Theorem 4.12). ∎

Configuration limit groups can be used in case the JSJ decomposition contains no QH vertex groups with more than one boundary component. We believe that configuration limit groups can be used in the presence of general QH vertex groups as well. However, flexible quotients seem to be a real obstacle for using the techniques presented in this section for analyzing general predicates over a free group.

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