

SEMISTABLE REDUCTION IN CHARACTERISTIC 0

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ABSTRACT. In 2000 Abramovich and Karu proved that any dominant morphism $f: X \rightarrow B$ of varieties of characteristic zero can be made weakly semistable by replacing B by a smooth alteration B' and replacing the proper transform of X by a modification X' . In the language of log geometry this means that $f': X' \rightarrow B'$ is log smooth and saturated for appropriate log structures. Moreover, Abramovich and Karu formulated a stronger conjecture that $f': X' \rightarrow B'$ can be even made semistable, which amounts to making X' smooth as well, and explained why this is the best resolution of f one might hope for. In this paper, we solve the semistable reduction conjecture in the larger generality of finite type morphisms of quasi-excellent schemes of characteristic zero.

1. INTRODUCTION

1.1. The semistable reduction conjecture. Resolution of singularities of an integral scheme X is a classical and difficult problem of finding a modification $X' \rightarrow X$ with a regular source. The relative analogue seeks to improve a dominant morphism $f: X \rightarrow B$ of integral schemes via a base change $B' \rightarrow B$ and a modification X' of the proper transform $(X \times_B B')^{\text{pr}}$ such that $f': X' \rightarrow B'$ is “as smooth as possible”. Probably the main credit for coining the latter problem should be given to Mumford and the groundbreaking papers [DM69] and [KKMS73], where B was a trait, $B' \rightarrow B$ was a finite cover, and one achieved that f' is *semistable*, that is, étale locally of the form $\text{Spec}(k[t_0 \dots t_n]/(t_0 \dots t_m - \pi))$. In [dJ97], de Jong proved that semistable reduction of relative curves is possible for a wide class of bases B , including varieties, once one allows alterations of the base. (Note also in a somewhat different direction, one can use works of Alexeev, Kollár and Shepherd-Barron (see [Ale94], [Ale96]) to obtain a minimal model reduction theorem in the case of relative surfaces.)

To the best of our knowledge, the case of arbitrary $\dim(B)$ and $\dim(X) - \dim(B)$ was first studied by Abramovich and Karu in [AK00]. They showed that in the characteristic zero case at the very least the notion of semistability should be relaxed as follows: X and B are regular and possess local parameters t_1, \dots, t_n and π_1, \dots, π_l such that $f^\#(\pi_i) = t_{n_i+1} \dots t_{n_{i+1}}$ for $0 = n_1 < n_2 < \dots < n_{l+1} \leq n$, see also §4.2.1. Then they proposed in [AK00, Conjecture 0.2] a best possible conjecture for projective varieties over \mathbf{C} , which we formulate in the maximal expected generality of qc schemes:

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Conjecture 1.1. *Let $f: X \rightarrow B$ denote a dominant morphism of finite type of integral qc schemes. Then there is a projective alteration $B' \rightarrow B$, and a projective modification $X' \rightarrow (X \times_B B')^{\text{pt}}$ such that $f': X' \rightarrow B'$ is semistable.*

1.2. Previous works. Abramovich and Karu proved a weak version of this conjecture for varieties of characteristic zero. First, they proved toroidal (or log smooth) reduction in [AK00, Theorem 2.1]: f' can be made toroidal. As in [KKMS73], further improvements of f' can be done by toroidal (or combinatorial) methods. In particular, the semistable reduction conjecture was reduced to a question about polyhedral subdivisions, see [AK00, Conjecture 8.4]. Finally, toroidal methods were used to make f' weakly semistable in the following sense: B' is regular and f' is flat and has geometrically reduced fibers, see [AK00, Theorem 0.3 and §0.8.2]. Thus, the gap from achieving semistable reduction was reduced to making X' regular, but this turned out to be a difficult task. In [Kar00] Karu managed to solve it when $\dim(X) - \dim(B) \leq 3$, and no further progress has been achieved until now. Nevertheless, the following aspects were improved in the meantime:

First, the language of toroidal geometry is being gradually replaced by a more robust language of log geometry, and we will use the latter in this paper. In particular, toroidal (resp. weakly semistable) morphisms were reinterpreted as log smooth (resp. and saturated) morphisms of log smooth log schemes. In addition, Molcho showed in [Mol16] that one can use non-representable modifications (and toric stacks) instead of alterations, and this allows to proceed from the step of toroidal reduction in a canonical way.

Second, log smooth reduction was extended in [Tem17, Theorem 4.3.1] to a wide class of qc schemes at cost of allowing $X' \rightarrow (X \times_B B')^{\text{pt}}$ to be an alteration whose degree is only divisible by primes non-invertible on B (see also [IT14b, Section 3]).

Third, if $B_0 \subseteq B$ is open, $X_0 = X \times_{B_0} B \rightarrow B_0$ is smooth, and the characteristic is zero, then log smooth reduction can be achieved by modifications $B' \rightarrow B$ and $X' \rightarrow X$ that restrict to isomorphisms over B_0 and X_0 , see [ATW19].

1.3. Main results. Our main new ingredient is Theorem 2.7 resolving the combinatorial conjecture of Abramovich and Karu. Lifting it to log schemes we obtain Theorem 4.4 about resolution of monoidal structure of morphisms of log schemes. In particular, it resolves log smooth morphisms to semistable ones, see Theorem 4.5. Similarly to [Mol16], we use root stacks and non-representable modifications $B' \rightarrow B$ instead of alterations to obtain the sharpest result. Using Kawamata trick one can then deduce the usual formulation with $B' \rightarrow B$ an alteration. Finally, we apply Theorem 4.5 to improve the results on resolution of morphisms we have mentioned earlier. In particular, Theorem 4.7 resolves a strong version of Conjecture 1.1 in characteristic zero, which also addresses divisors $Z \hookrightarrow X$ and controls the modification locus.

Finally, let us say a couple of words about our proof of Theorem 2.7. It can be viewed as a relative analogue of a difficult theorem of [KKMS73] on polyhedral subdivisions, so its resistance to attacks in the past is not so surprising. A breakthrough was obtained in the recent work [ALPT18] of the three authors and Pak, where a local version was established, and the current paper builds on the methods of [ALPT18] to obtain a complete solution. We would like to mention that a recent work [HPPS14] of Haase, Paffenholz, Piechnik and Santos provided a relatively simple new proof of the subdivision theorem, and its constructions with

Cayley polytopes were used as a starting point for the methods of [ALPT18] and this paper.

2. CONICAL COMPLEXES

2.1. The combinatorial conjecture of Abramovich and Karu. In this subsection we recall the combinatorial semistable reduction conjecture from [AK00] in the slightly increased generality of maps with non-trivial vertical parts. In addition, it will be important that our complexes do not have to be connected.

2.1.1. Terminology. We follow the notation and terminology on rational cones, polyhedral complexes, etc., from [AK00]. Recall that a *rational conical polyhedral complex*, or *conical complex*, is an (abstract) polyhedral complex formed by gluing together finitely many polyhedral cells σ , each of which is equipped with a lattice $N_\sigma \cong \mathbf{Z}^{d_\sigma}$ such that σ is a full-dimensional, rational, strictly convex polyhedral cone in $N_\sigma \otimes \mathbf{R}$, and such that if τ is a face of σ , then $N_\tau = N_\sigma|_{\text{Span}(\tau)}$. We will denote such a conical complex by $\{(\sigma, N_\sigma)\}$. A *map* $f: X \rightarrow Y$ of conical complexes $X = \{(\sigma, N_\sigma)\}$ and $Y = \{(\rho, N_\rho)\}$ is a collection of group homomorphisms $f_\sigma: N_\sigma \rightarrow N_\rho$, one for each $\sigma \in X$, such that the extension $f_\sigma: N_\sigma \otimes \mathbf{R} \rightarrow N_\rho \otimes \mathbf{R}$ maps σ into ρ , and such that if τ is a face of σ , then $f_\tau = f_\sigma|_\tau$. A conical complex is *regular* (or *nonsingular*) if each of its cells σ is generated by a lattice basis of N_σ . A subdivision $X' \rightarrow X$ with a regular X' will be called a *resolution* of X .

Remark 2.1. The category of conical complexes is equivalent to the category of saturated Kato fans, in particular, it has fiber products. One can easily describe them explicitly, and we skip the details. In this section we prefer to work with cones because they are much more geometric, but we will switch to the language of fans in Section 4.

Definition 2.2 ([AK00, Definition 8.1]). A map of conical complexes $f: X \rightarrow B$ is *weakly semistable* if the following conditions hold.

- (1) For every $\sigma \in X$, we have $f(\sigma) \in B$.
- (2) For every $\sigma \in X$, we have $f(N_\sigma) = N_{f(\sigma)}$.
- (3) B is regular.

If, in addition, X is regular, then f is called *semistable*.

The condition $f^{-1}(0) = 0$ in [AK00] means that the fibers are connected and have the trivial vertical part, so we do not include it.

Definition 2.3 ([AK00, Definition 8.2]). Let X be a conical complex. A *lattice alteration* is a map $X' \rightarrow X$ where X' is a conical complex of the form $\{(\sigma, N'_\sigma): \sigma \in X\}$ and each N'_σ is a sublattice of N_σ . An *alteration* is a composition $X' \rightarrow X_1 \rightarrow X$ of a lattice alteration $X' \rightarrow X_1$ with a subdivision $X_1 \rightarrow X$. The alteration is *projective* if the subdivision $X_1 \rightarrow X$ is projective. Finally, a subdivision is projective if there exists a piecewise linear function to the reals that is convex on each simplex of X and whose domains of linearity are the faces of X_1 [AK00].

Fix a map $f: X \rightarrow Y$ of conical complexes. For any alteration $g: Y' \rightarrow Y$, there is an alteration $g': X' \rightarrow X$ with

$$X' = \{(\sigma \cap f^{-1}(g(\tau)), N_\sigma \cap f^{-1}(g(N_\tau))): \sigma \in X, \tau \in Y'\}$$

which is the unique minimal alteration admitting a map $X' \rightarrow Y'$. It is easy to see that $X' = Y' \times_Y X$ in the category of conical complexes, so we say that g' is the *base change* of g (with respect to f). If g is projective, then g' is projective as well.

We can now state the polyhedral conjecture of Abramovich and Karu.

Conjecture 2.4. *Let $f: X \rightarrow B$ be a map of conical complexes. Then there exists a projective alteration $b: B' \rightarrow B$ and a projective subdivision $a: X' \rightarrow X \times_B B'$ such that $f': X' \rightarrow B'$ is semistable.*

2.2. Statement of the main result. Any f' as above will be called a *resolution* of f , and we will construct a canonical resolution. This is convenient for the proof and important for applications, but should be spelled out carefully. Ideally, we would like the construction to be local, but the choice of b must be affected by the whole fibers over cones of B . So, we propose the following technical solution.

Definition 2.5. (i) A map $h: Y \rightarrow X$ of conical complexes is called a *local isomorphism* if $f: \sigma \xrightarrow{\sim} \rho$ and $f_\sigma: N_\sigma \xrightarrow{\sim} N_\rho$ for any $\sigma \in Y$.

(ii) Loosely speaking, a construction is local if it is compatible with (or functorial with respect to) local isomorphisms. Constructions compatible with surjective local isomorphisms will be called *quasi-local*.

Example 2.6. (i) Barycentric subdivision $X \mapsto X^b$ is the most basic local construction. It outputs a simplicial conical complex X^b . Moreover, the cones of X^b are provided with a canonical *order*: the edges of each simplicial cone σ are ordered by dimension of the cone they are mapped to in X . This induces an order on any further simplicial subdivision of X .

(ii) It was shown in [KKMS73] that any X possesses a resolution X^r . It was later proved that the construction can be made local. For example, see [IT14a, Theorem 3.2.20], though there also are purely combinatorial constructions. We fix any such local (or just quasi-local) construction, since it will be used later.

Now we can formulate a functorial strengthening of Conjecture 2.4. Its proof will occupy Sections 2 and 3.

Theorem 2.7. *There exists a construction which associates to each map of conical complexes $f: X \rightarrow B$ a resolution $b: B' \rightarrow B$, $a: X' \rightarrow X \times_B B'$, $f': X' \rightarrow B'$ in a quasi-local way: if $g: Y \rightarrow C$ is another map and $\alpha: Y \rightarrow X$, $\beta: C \rightarrow B$ are surjective local isomorphisms such that $\beta \circ g = f \circ \alpha$, then the resolution g' of g is the base change of f' in the sense that $C' = C \times_B B'$ and $Y' = Y \times_X X'$.*

2.3. Some reductions. As often happens with resolution problems, the solution will be constructed by composing few steps that gradually improve f . We start with a few standard reductions. (Their analogues)

2.3.1. Quasi-localization. Now, we are going to essentially use non-connected complexes and a typical descent argument. For a conical complex X let X_0 denote the disjoint union of cones of X and let $X_1 = X_0 \times_X X_0$, in particular, $X_0 \rightarrow X$ and both projections $X_1 \rightrightarrows X_0$ are surjective local isomorphisms. Any quasi-local resolution of $f: X \rightarrow B$ is compatible with the resolution of $f_0: X_0 \rightarrow B_0$, and hence is determined by the latter. Moreover, any quasi-local resolution on the category \mathcal{C}_0 of disjoint unions of cones automatically extends to the category \mathcal{C} of all conical complexes. Indeed, $f_1: X_1 \rightarrow B_1$ also lies in \mathcal{C}_0 , hence both pullbacks to f_1 of the

resolution f'_0 of f_0 are isomorphic to the resolution f'_1 of f_1 by the quasi-locality. This implies that f'_0 descends to a resolution f' of f .

To summarize, it suffices to prove Theorem 2.7 in the case when B and X are disjoint unions of cones. We cannot work all the time within the category \mathcal{C}_0 because it is not closed under subdivisions. However, each (quasi-local) step of the construction can be made under the assumption that the input is in \mathcal{C}_0 . We will use this additional localization a couple of times during the initial reduction process and each time it will be mentioned explicitly.

2.3.2. Eliminating vertical components. First step of our construction of f' is just to resolve the source. To simplify the notation we replace f by $f^r: X^r \rightarrow B$, where X^r is the resolution from Example 2.6, and assume now that X is regular. By quasi-localization, we can assume further that X is a disjoint union of regular simplicial cones X_i . Let $X_i = X_i^v \times X_i^h$, where the vertical part X_i^v is mapped by f to a point and f is injective on the edges of the horizontal part X_i^h . It suffices to construct a resolution $\coprod (X_i^h)' \rightarrow B'$ of $\coprod X_i^h \rightarrow B$ because then $\coprod (X_i^h)' \times X_i^v \rightarrow B'$ will be a resolution of f . Absence of vertical components is preserved by subdivisions and base changes, so we can assume in the sequel that it holds for f . (Notice that the regularity of X will be destroyed by the next step.)

2.3.3. Ordering the complexes. Replace B by the barycentric subdivision B^b and update X via the base change $X \times_B B^b$. By 2.6(i), from this stage the simplices of B and any its subdivision acquire a canonical ordering. In the same way, replacing X by X^b , we provide X and its further subdivisions with an order.

2.3.4. Regularizing the base. As a next step, we replace B by its resolution B^r and update X via the base change $X \times_B B^r$. By the quasi-localization we can further assume that $B = \coprod B_i$, where B_i are regular simplicial cones. From this point on, any further alteration of B will be of the form $B_c \rightarrow B$, where $c \geq 1$, one replaces each N_i by cN_i , and subdivides each B_i into $(\dim B_i)^c$ regular cones in a standard way, see Theorem 3.5 below. (We perform both operations together for convenience of the exposition – at some stages one of them would suffice.) In particular, the assumption that B is regular will be maintained until the end of the proof.

2.3.5. Making f weakly semistable. If c is such that all edges of X are mapped to edges of B_c , then replacing B by B_c and replacing f by the base change $f_c: X \times_B B_c \rightarrow B_c$ we achieve that $f(\sigma)$ is a cone for any $\sigma \in X$. So, choosing the minimal such c , we obtain a quasi-local construction which improves f so that B is regular and f takes cones to cones. Next, choose the minimal c such that $cN_{f(\sigma)} \subseteq f(N_\sigma)$. Replacing f by f_c we also achieve that $N_{f(\sigma)} = f(N_\sigma)$, and hence f becomes weakly semistable.

The weak semistability might be destroyed by subdivisions of X , but after each such step we will perform the above step to restore it. That is, we will simply replace f by the weakly semistable f_c with the minimal possible c .

2.3.6. Reduction to polytopes. Finally, as in [KKMS73] it will be convenient to perform deeper geometric constructions on the level of lattice polytopes. We will do so in detail in the next section: Each component B_i of B has a canonical section Δ_i passing through the minimal lattice points on the edges. Since there are no vertical components, the preimage Z of $\Delta = \coprod_i \Delta_i$ in X has a natural structure of a disjoint union of lattice polytopes. As in [KKMS73], the problem now reduces

to finding an appropriate c and subdividing the preimage $Z(c)$ of $c\Delta$ into regular lattice simplices. The precise formulation and the construction are in Section 3.

3. PROOF OF THE MAIN THEOREM

A *lattice polytope* is a pair (P, N_P) (usually denoted as just P) where $N_P \cong \mathbf{Z}^{d_P}$ is an affine lattice and P is a polytope in $N_P \otimes \mathbf{R}$ with vertices in N_P . A *face* of a lattice polytope (P, N_P) is a lattice polytope (F, N_F) where F is a face of P and $N_F = N_P|_{\text{AffSpan}(F)}$. A *lattice polytopal complex*, or *polytopal complex*, is a polyhedral complex formed by gluing together finitely many lattice polytopes (with the usual gluing conditions of polyhedral complexes) such that lattices agree on common faces and each polytope P is full-dimensional in N_P . Maps between polytopal complexes are defined analogously to the conical case.

Let (P, N_P) be a lattice polytope. We let L_P denote the affine sublattice of N_P spanned by the vertices of P . We say that the *index* of P is the index

$$[N_P \cap \text{AffSpan}(P) : L_P].$$

A *unimodular simplex* is a lattice simplex of index 1. A *unimodular triangulation* is a polytopal complex all of whose elements are unimodular simplices. If the vertices of P are contained in some sublattice N' of N_P , then we say that the *index of P with respect to N'* is the index of (P, N') . We define unimodularity with respect to N' accordingly.

Usually, we will specify an origin in N_P , which allows us to define dilations of lattice polytopes and Minkowski sums of polytopes with the same lattice. Since all of our results will be invariant under translation, the choice of origin will not matter. If $X = \{(P, N_P)\}$ is a polytopal complex, then we can define a polytopal complex $cX = \{(cP, N_P)\}$. Given a map $f : X \rightarrow Y$ of polytopal complexes, there is the obvious induced map $cX \rightarrow cY$; we denote this map by cf .

We define lattice alterations and alterations of polytopal complexes analogously to the conical case. For induced alterations, we need to proceed more carefully. Let $f : X \rightarrow Y$ be a map of polytopal complexes and $g : Y_1 \rightarrow Y$ an alteration. If the rational subdivision

$$X_1 := \{(P \cap f^{-1}(g(Q)), N_P \cap f^{-1}(g(N_Q))) : P \in X, Q \in Y_1\} \quad (1)$$

is a lattice polytopal complex, then $X_1 \rightarrow X$ is the unique minimal alteration admitting a map $X_1 \rightarrow Y_1$, and we say that $X_1 \rightarrow X$ is *induced* by g (with respect to f).

We say a map $f : X \rightarrow Y$ of polytopal complexes is *good* if for every $P \in X$, we have $f(P) \in Y$. We have the following.

Proposition 3.1. *Let $f : X \rightarrow Y$ be a map of polytopal complexes. Then there exists a positive integer c and a projective subdivision $Y' \rightarrow cY$ which induces with respect to cf an alteration $X' \rightarrow cX$ such that $X' \rightarrow Y'$ is good.*

Proof. It is easy to see that there is a rational projective subdivision \tilde{Y} of Y such that for every $P \in X$, we have that $f(P)$ is a union of cells of \tilde{Y} . This induces as in (1) a rational subdivision \tilde{X} of X . For some c , we have that $c\tilde{X}$ and $c\tilde{Y}$ are lattice subdivisions of cX and cY , and the map $c\tilde{X} \rightarrow c\tilde{Y}$ gives the result. \square

Proposition 3.2. *Let $f : X \rightarrow Y$ be a good map of polytopal complexes and $Y_1 \rightarrow Y$ an alteration. Then there exists a positive integer c such that if $Y_1 \rightarrow cY_1$ is the*

lattice alteration given by $(P, N_P) \cong (cP, cN_P) \mapsto (cP, N_P)$ and g is the alteration $Y_1 \rightarrow cY_1 \rightarrow cY$, then g induces with respect to cf an alteration $X_1 \rightarrow cX$.

Proof. As in the previous proof, there is c such that $cY_1 \rightarrow cY$ induces a lattice subdivision $X' \rightarrow cX$. Then the lattice alteration $Y_1 \rightarrow cY_1$ induces an alteration $X_1 \rightarrow X'$, and $X_1 \rightarrow X' \rightarrow cX$ is the desired alteration. \square

Our goal now is to prove the following theorem:

Theorem 3.3. *Let $f : X \rightarrow B$ be a map of polytopal complexes. Then there exists a positive integer c , a projective alteration $B_1 \rightarrow cB$ which induces with respect to cf an alteration $X_1 \rightarrow cX$, and a projective subdivision $Y \rightarrow X_1$ such that Y and B_1 are both unimodular triangulations. The construction is quasi-local.*

The quasi-locality falls out naturally as a product of the proof, relying on local improvements of the index we detail in the following. Additionally, observe that the proof of the semistable reduction conjecture, and more generally Theorem 2.7 follows as discussed in Section 2.3. Indeed, as observed, it even suffices to use a particular case of Theorem 3.3, where X and B are ordered, B is a disjoint union of unimodular simplices, etc. We stated it here for its combinatorial beauty in full generality nevertheless.

3.1. Canonical subdivisions. As we will see, much of our proof relies on being able to construct “canonical” subdivisions for polytopes. We now formalize his notion.

An *ordered polytope* is a polytope along with a total order on its vertices. A *face* of an ordered polytope is a face of the underlying polytope along with the induced ordering. Let \mathcal{P} to be the category whose objects are ordered lattice polytopes and whose morphisms are $F \rightarrow P$ where F is a face of P . Let \mathcal{S} be the category whose objects are ordered subdivisions of ordered lattice polytopes and whose morphisms are $F' \rightarrow P'$ where F' is the subdivision induced on a face of the underlying ordered polytope of P' .

Let $\Gamma : \mathcal{F} \rightarrow \mathcal{P}$ be a full and faithful functor for some category \mathcal{F} . A *canonical subdivision* of Γ is a functor $\Sigma : \mathcal{F} \rightarrow \mathcal{S}$ such that $\Sigma(P)$ is a subdivision of $\Gamma(P)$ for all $P \in \text{Ob}(\mathcal{F})$. If $\Sigma(P)$ is a triangulation for all P , then we call Σ a *canonical triangulation*. If $\Sigma(P)$ is projective for all P , then we say that Σ is *projective*.

Remark 3.4. In what follows, we will generally omit justification for why certain canonical subdivisions are projective. This is standard practice, but we should note here how one would go about proving projectivity. Our subdivisions will be built up inductively, first constructing an initial subdivision, then subdividing each cell of this subdivision, and so on. At each step, each intermediate subdivision of a cell can easily be shown to be projective. However, this does not guarantee projectivity of the entire subdivision. To prove this, one should demonstrate that at each step, the intermediate subdivisions used are not only canonical (in the sense described above), but also that one can choose *height functions* to induce these subdivisions as projective subdivisions in a canonical way. In other words, one should define a category $\mathcal{S}_{\text{proj}}$ whose objects are pairs (P, f) , where P is an ordered polytope and f is a height function on the vertices of P , and whose morphisms are $(F, f|_F) \rightarrow (P, f)$ where F is a face of P . A projective canonical subdivision of Γ (where Γ is defined above) is then defined to be a functor $\Sigma : \mathcal{F} \rightarrow \mathcal{S}_{\text{proj}}$ such that for all $P \in \text{Ob}(\mathcal{F})$, we have $\Sigma(P) = (\Gamma(P), f)$ for some f . Each projective canonical subdivision gives

a canonical subdivision by replacing $(\Gamma(P), f)$ with the subdivision induced by f . We leave it to the reader to show that all subdivisions described later can be given as projective canonical subdivisions as defined here.

3.1.1. Canonical triangulations of dilated simplices. Let Δ be the category defined as follows. The objects are ordered pairs (P, c) where P is an ordered lattice simplex and c is a positive integer. The morphisms are $(P', c) \rightarrow (P, c)$ where P' is a face of P . We have a full and faithful functor $\mu : \Delta \rightarrow \mathcal{P}$ defined by $\mu(P, c) = cP$. We will assume the origin is in L_P , so that cP has vertices in L_P .

The following is a key result from Haase et al. [HPPS14].

Theorem 3.5. *There is a projective canonical triangulation Σ of μ such that for all $(P, c) \in \text{Ob}(\Delta)$, we have that $\Sigma(P, c)$ is unimodular with respect to L_P .*

Later we will prove a generalization of this to polysimplices, Lemma 3.7. For now, we will state a modified version of this Theorem. Let Δ' be the full subcategory of Δ whose objects are $(P, c) \in \text{Ob}(\Delta)$ with $c \geq \dim(P) + 1$. Let μ' be the restriction of μ to Δ' . Then we have the following.

Lemma 3.6. *There is a projective canonical triangulation Σ' of μ' such that for all $(P, c) \in \text{Ob}(\Delta')$ and all full-dimensional simplices Q of $\Sigma'(P, c)$, we have the following:*

- (1) $L_P = L_Q$. (That is, $\Sigma'(P, c)$ is unimodular with respect to L_P .)
- (2) If the vertices of Q are ordered $v_1, v_2, \dots, v_{\dim(P)+1}$, then for all $i = 1, \dots, \dim(P)$, every face of P which contains v_i also contains v_{i+1} .

Proof. Note: This proof uses ideas and notation from the next section. We have put the proof in this section for the sake of organization.

We construct the triangulation $\Sigma'(P, c)$ of cP as follows. For each face F of P , let O_F be the barycenter of F . If F has dimension k and F' is a face of F , then we note that

$$\phi(F', F) := (c - k - 1)F' + (k + 1)O_F$$

is a lattice polytope contained in cF .

Let $d = \dim(P)$. Let F_r, F_{r+1}, \dots, F_d be a sequence of nonempty faces of P with $F_r < \dots < F_d$ and $\dim(F_i) = i$ for all $r \leq i \leq d$. We define

$$(cP)_{F_r, \dots, F_d} := \text{conv} \bigcup_{i=r}^d \phi(F_i, F_i).$$

Then the collection of all such $(cP)_{F_r, \dots, F_d}$ are the full-dimensional cells of a subdivision Σ of cP .

The final step is to refine Σ to a triangulation. When viewed as lattice polytopes in L_P , each $(cP)_{F_r, \dots, F_d}$ is lattice equivalent to the Cayley polytope

$$\mathcal{C}((c - r - 1)F_r, (c - r - 2)F_r, \dots, (c - d - 1)F_r).$$

Thus, by Lemma 3.7, there are canonical triangulations of each $(cP)_{F_r, \dots, F_d}$ which are unimodular with respect to L_P . These extend to a triangulation of Σ which is unimodular in L_P . The fact that this triangulation satisfies property (2) is easy to check, as is canonicity. \square

3.2. Cayley polytopes. Let $(P_1, \mathbf{Z}^d), (P_2, \mathbf{Z}^d) \dots, (P_n, \mathbf{Z}^d)$ be lattice polytopes with the same lattice \mathbf{Z}^d . To simplify some statements later, we will assume that L_{P_j} contains the origin for all j . (Note that we can always translate the P_j so that this holds.) Let P be the column array $(P_1 \dots P_n)^T$. We define the *Cayley polytope* $\mathcal{C}(P)$ to be the polytope

$$\mathcal{C}(P) := \text{conv} \left(\bigcup_{i=1}^n P_i \times e_i \right) \subset \mathbf{R}^d \times \mathbf{R}^n$$

where conv denotes convex hull and e_i is the i -th standard basis vector of \mathbf{R}^n . We will also occasionally write $\mathcal{C}(P_1, \dots, P_n)$ instead of $\mathcal{C}(P)$.

We make $\mathcal{C}(P)$ a lattice polytope by equipping it with the affine lattice $\mathbf{Z}^d \times \Lambda^{n-1}$, where $\Lambda^{n-1} := \{x \in \mathbf{Z}^n : x_1 + \dots + x_n = 1\} \cong \mathbf{Z}^{n-1}$. We have

$$L_{\mathcal{C}(P)} = L_P \times \Lambda^{n-1}$$

where $L_P := \langle L_{P_1}, \dots, L_{P_n} \rangle$ is the lattice generated by L_{P_1}, \dots, L_{P_n} . We define

$$\text{index}(P) := [\mathbf{Z}^d \cap (L_P \otimes \mathbf{R}) : L_P],$$

so that $\text{index}(\mathcal{C}(P)) = \text{index}(P)$.

If P_1, \dots, P_n are ordered polytopes, then we make $\mathcal{C}(P)$ an ordered polytope with the following ordering: First the vertices of $P_1 \times e_1$ in the order given by P_1 , then the vertices of $P_2 \times e_2$ in the order given by P_2 , and so on. (Note that these are precisely the vertices of $\mathcal{C}(P)$.)

Let A be an $m \times n$ matrix with nonnegative integer entries and let P be as above. We define

$$AP := \left(\sum_{j=1}^n A_{1j} P_j, \sum_{j=1}^n A_{2j} P_j, \dots, \sum_{j=1}^n A_{mj} P_j \right)^T,$$

with the sum being Minkowski sum. In other words, we define matrix multiplication in the expected way. If A is a row vector, then AP has one entry, and we identify AP with this entry.

A *polysimplex*, or *product of simplices*, is a polytope of the form $\sum_j P_j$, where $\{P_j\}$ is an affinely independent set of simplices. If there is an ordering on such a set $\{P_j\}$ and each P_j is an ordered polytope, then we make $\sum_j P_j$ an ordered polytope by lexicographic ordering on its vertices. Thus, if A is a matrix as above and $P = (P_1 \dots P_n)^T$ with $\{P_j\}_{j=1}^n$ an affinely independent set of ordered simplices, then each entry of AP is an ordered polysimplex, and $\mathcal{C}(AP)$ is an ordered polytope. In addition, with the assumption that L_{P_j} contains the origin for all j , the vertices of $\mathcal{C}(AP)$ are contained in $L_P \times \Lambda^{m-1}$.

We set some final notation regarding matrices. In the following, assume P is a column array with n lattice polytope entries and A is an $m \times n$ nonnegative integer matrix. Let A_i denote the i -th row of A . Let $\text{supp } A$ denote the set of column indices at which A is not a 0-column. Let $P[i, q]$ denote the column matrix obtained by replacing the i -th entry of P with q . Let P_A denote the column matrix obtained by restricting P to the entries indexed by $\text{supp } A$. If $\text{supp } A = \emptyset$, then we define P_A to be a single entry which is the origin.

3.3. Overview of the proof. In this section, we will reduce the problem to constructing certain canonical triangulations of Cayley polytopes of polysimplices.

Let \mathcal{F} be the category whose objects are tuples (P, A) satisfying the following.

- (1) $P = (P_1, \dots, P_n)^T$ where P_1, \dots, P_n are affinely independent ordered lattice simplices with the same lattice.
- (2) A is an $m \times n$ matrix with nonnegative integer entries.

The morphisms in \mathcal{F} are $(F, A') \rightarrow (P, A)$, where F_i is a face of P_i for all i , and A' is obtained from A by taking a subset of the rows of A . We have a full and faithful functor $\mathcal{C} : \mathcal{F} \rightarrow \mathcal{P}$ given by $(P, A) \mapsto \mathcal{C}(AP)$.

Lemma 3.7. *There is a projective canonical subdivision Σ of \mathcal{C} such that for all $(P, A) \in \text{Ob}(\mathcal{F})$, the triangulation $\Sigma(P, A)$ is unimodular with respect to $L_P \times \Lambda^{m-1}$.*

We postpone the proof to the next section, and proceed to find a way to lower the indices of the polytopes in a polytopal complex. For this, we will use the idea of ‘‘box points’’ (or Waterman points) as introduced by Waterman for the KMW theorem. (Note that our final construction will be different from theirs when restricted to simplices.)

Let $P = (P_1 \dots P_n)^T$ where $\{P_j\}$ is an affinely independent set of lattice simplices in \mathbf{Z}^d . For simplicity, assume that L_{P_j} contains the origin for all j . Recall that L_P is the lattice generated by L_{P_1}, \dots, L_{P_n} . A *box point* of P is a nonzero element of $G_P := (\mathbf{Z}^d \cap (L_P \otimes \mathbf{R}))/L_P$. If $F = (F_1 \dots F_n)^T$ is such that F_j is a face of P_j for all j , then there is a natural inclusion $G_F \hookrightarrow G_P$, and so any box point of F can be regarded as a box point of P . Moreover, if we have two arrays P, Q with $L_P = L_Q$, then we identify the box points of P with the box points of Q .

We make the following crucial observation: If \mathbf{m} is a box point of P , then there is a unique minimal (in the product order) N -tuple (c_1, \dots, c_n) of nonnegative integers such that the polytope $\sum_{j=1}^n c_j P_j$ contains a representative of \mathbf{m} . This tuple satisfies $0 \leq c_j \leq \dim P_j$ for all j , and the representative is unique. We denote the row vector $(c_1 \dots c_n)$ by $c(P, \mathbf{m})$. If \mathbf{m} is not a box point of P , then we set $c(P, \mathbf{m})$ to be the 0-vector. Note that if F is as above and \mathbf{m} is also a box point of F , then $c(F, \mathbf{m}) = c(P, \mathbf{m})$.

Let \mathbf{m} be a box point of P_0 for some $(P_0, A_0) \in \text{Ob}(\mathcal{F})$. We define $\mathcal{F}^{\mathbf{m}}$ to be the full subcategory of \mathcal{F} whose objects are $(P, A) \in \text{Ob}(\mathcal{F})$ satisfying the following:

- (1) Let $c = c(P, \mathbf{m})$. Then for all $i \in [m]$ and $j \in [n]$, we have

$$A_{ij} = 0 \text{ or } A_{ij} \geq c_j.$$

- (2) We have

$$\text{supp } A_1 \cap \text{supp } c \supseteq \text{supp } A_2 \cap \text{supp } c \supseteq \dots \supseteq \text{supp } A_m \cap \text{supp } c.$$

Let $\mathcal{C}^{\mathbf{m}}$ be the restriction of \mathcal{C} to $\mathcal{F}^{\mathbf{m}}$.

We will prove the following in Section 3.5

Lemma 3.8. *Let \mathbf{m} be as above. Then there is a projective canonical triangulation $\Sigma^{\mathbf{m}}$ of $\mathcal{C}^{\mathbf{m}}$ such that for all $(P, A) \in \text{Ob}(\mathcal{F}^{\mathbf{m}})$ and all full-dimensional simplices Q in $\Sigma^{\mathbf{m}}(P, A)$, we have the following.*

- If \mathbf{m} is a box point of P_A , then

$$\text{index}(Q) < \text{index}(P_A).$$

- If \mathbf{m} is not a box point of P_A , then $L_Q = L_{P_A} \times \Lambda^{m-1}$.

Assuming this lemma, we can now prove Theorem 3.3.

Proof of Theorem 3.3. By Proposition 3.1, we may assume $f : X \rightarrow Y$ is a good map.

Step 1: Reducing the base.

As discussed before, the first step is to alter B so that it is a unimodular triangulation. By the KMW theorem [KKMS73], there is a positive integer c so that we have a projective unimodular triangulation $B' \rightarrow cB$. Now, by Proposition 3.2, there is a positive integer c' and an alteration $B' \rightarrow c'cB$ such that this alteration induces with respect to $c'cf$ an alteration $X_1 \rightarrow c'cX$ and map $X_1 \rightarrow B'$. Hence, we may assume B is a unimodular triangulation. Choosing any triangulation, for example, the barycentric one, we may assume that X is triangulated.

Step 2: Lowering the index.

By Lemma 3.6, there exists c such that for each $Q \in B$ we have a unimodular projective triangulation

$$\Sigma'(Q, c) \rightarrow cQ.$$

Since these triangulations are canonical, this gives a unimodular triangulation B_1 of cB .

By Proposition 3.2, for some c' , the alteration $B_1 \rightarrow c'B_1 \rightarrow c'cB$ induces an alteration $X_1 \rightarrow c'cX$ and a map $f_1 : X_1 \rightarrow B_1$. We may assume $c' > \dim X$.

Suppose $Q \in B$ and Δ is a full-dimensional simplex of the complex $f^{-1}(Q)$. Let v_1, \dots, v_n be the vertices of Q , and let

$$P_i := (f^{-1}(v_i) \cap \Delta, N_{f^{-1}(v_i)})$$

for all i . Let $P = (P_1 \dots P_n)^T$. Note that the P_i are affinely independent simplices and $\Delta = \mathcal{C}(P)$.

Let v be a vertex of B_1 contained in cQ . Define

$$P(v) := f_1^{-1}(v) \cap c'\mathcal{C}\Delta.$$

Let (a_1, \dots, a_n) be the barycentric coordinates of v with respect to the vertices cv_1, \dots, cv_n of cQ . Since Q is unimodular, ca_1, \dots, ca_n are nonnegative integers. From the definition of X_1 , we have

$$\begin{aligned} P(v) &= a_1P(cv_1) + a_2P(cv_2) + \dots + a_nP(cv_n) \\ &= c'(a_1P_1 + a_2P_2 + \dots + a_nP_n). \end{aligned}$$

Thus, if $Q_1 \in B_1$ has vertices u_1, \dots, u_m , we have

$$\begin{aligned} f_1^{-1}(Q_1) \cap c'\mathcal{C}\Delta &= \mathcal{C}(P(u_1), \dots, P(u_m)) \\ &= \mathcal{C}(AP) \end{aligned}$$

where A is an $m \times n$ matrix of nonnegative integers divisible by c' . By Lemma 3.6, we also have

$$\text{supp } A_1 \supseteq \text{supp } A_2 \supseteq \dots \supseteq \text{supp } A_m.$$

This argument shows that every $R \in X_1$ is of the form $\mathcal{C}(A_R P_R)$ with (P_R, A_R) satisfying the above conditions and where $\Delta_R := \mathcal{C}(P_R)$ is an element of X .

Let \mathbf{m} be a box point of P . By the above conditions on A_R , we have that $(P_R, A_R) \in \text{Ob}(\mathcal{F}^{\mathbf{m}})$ for all $R \in X_1$. Thus, by Lemma 3.8, we have a projective triangulation $Y \rightarrow X_1$ where each $R \in X_1$ is triangulated into $\Sigma^{\mathbf{m}}(P_R, A_R)$. If \mathbf{m} is a box point of P_R , then for every full-dimensional simplex Q of $Y|_R$ we have $\text{index}(Q) < \text{index}(P_R) = \text{index}(\Delta_R)$, and otherwise $L_Q = L_{\Delta_R}$.

Now repeat the process of Step 2 with Y instead of X . Each time we do this procedure, we lower the indices of some of the lattices spanned by elements of X while keeping the other lattices the same. Eventually all lattices will be unimodular, completing the proof. \square

3.4. Canonical triangulations of Cayley polytopes. To prove Lemma 3.8, we will first prove Lemma 3.7. We will give two distinct triangulations satisfying this lemma. Both are generalizations of the Haase et al. construction [HPPS14], and both are the same when restricted to polysimplices. The first triangulation is described in the authors' previous work [ALPT18] with Pak. We will use this triangulation mainly to help define the second triangulation, which is what we will need later. The definitions given here are entirely recursive; for a more explicit description of the first triangulation, see [ALPT18]. The advantage of the recursive approach is that it will make very abstract statements easier to prove.

3.4.1. The first triangulation. Let $(P, A) \in \mathcal{F}$. Given indices i, j , we say that A is (i, j) -reducible if $A_{ij} > 0$. Suppose A is (i, j) -reducible, and let $\phi : \mathcal{C}(AP) \rightarrow Q$ be any affine map. We will define a set $f_{ij}(P, A, \phi)$ as follows.

Let A' be the matrix obtained from A by subtracting 1 from the A_{ij} entry. If $\dim P_j = 0$, then we set

$$f_{ij}(P, A, \phi) = \{(P, A', \phi)\}.$$

Note that in this case $\mathcal{C}(AP) = \mathcal{C}(A'P)$, since by the assumptions in the previous section P_j is the origin. We therefore have a map $\phi : \mathcal{C}(A'P) \rightarrow Q$.

Now assume $\dim P_j > 0$. First assume $\sum_{i'} A_{i'j} > 1$. Let v be the first vertex of P_j and let F_j be the facet of P_j opposite v . Let F be the array obtained from P by replacing P_j with F_j . Then $\mathcal{C}(AP)$ has a subdivision into the two full-dimensional polytopes, one of which is

$$Q_1 := \mathcal{C}((AP)[i, v + A'_i P])$$

and the other of which is

$$Q_2 := \mathcal{C}((AF)[i, \text{conv}((v + A'_i F) \cup (A_i F))]).$$

We have a lattice polytope isomorphism $\phi_1 : \mathcal{C}(A'P) \rightarrow Q_1$. In addition, we have an affine isomorphism $\phi_2 : \mathcal{C}(A''F) \rightarrow Q_2$, where A'' is obtained from A by inserting the row A'_i above the i -th row of A . Note that Q_1 and Q_2 have vertices in $L_P \times \Lambda^{m-1}$. If we view Q_2 as having the ambient lattice $L_P \times \Lambda^{m-1}$, then ϕ_2 is a lattice polytope isomorphism.

In the above situation, we define

$$f_{ij}(P, A, \phi) := \{(P, A', \phi \circ \phi_1), (F, A'', \phi \circ \phi_2)\}$$

We order $f_{ij}(P, A, \phi)$ so that $(P, A', \phi \circ \phi_1)$ is the first element and $(F, A'', \phi \circ \phi_2)$ is the second. Note that we have maps $\phi \circ \phi_1 : \mathcal{C}(A'P) \rightarrow Q$ and $\phi \circ \phi_2 : \mathcal{C}(A''F) \rightarrow Q$.

Finally, assume that $\sum_{i'} A_{i'j} = 1$. Let P^\square be the array obtained from P by replacing P_j with 0, and let A^\square be the matrix obtained from A by replacing the i -th row of A with $\dim P_j + 1$ copies of A'_i . Then we have an affine isomorphism $\phi^\square : \mathcal{C}(A^\square P^\square) \rightarrow \mathcal{C}(AP)$. This is a lattice polytope isomorphism if $\mathcal{C}(AP)$ is viewed in the lattice $L_P \times \Lambda^{m-1}$. We set

$$f_{ij}(P, A, \phi) := \{(P^\square, A^\square, \phi \circ \phi^\square)\}.$$

Again, we have a map $\phi \circ \phi^\square : \mathcal{C}(A^\square P^\square) \rightarrow Q$.

Now, for $(P, A) \in \mathcal{F}$, we say that A is j -reducible if $A_{ij} > 0$ for some i . If A is j -reducible, we define

$$f_j(P, A, \phi) := f_{i_0 j}(P, A, \phi),$$

where

$$i_0 := \min\{i : A_{ij} = \max_{i'} A_{i'j}\}.$$

This is well-defined because by definition, $A_{i_0 j} > 0$.

We now define a process as follows. Fix $(P_0, A_0) \in \mathcal{F}$. We keep track of a set \mathcal{S} , which is initially $\{(P_0, A_0, \text{id})\}$, where id is the identity map. A *move* consists of the following. First, we choose an element $\alpha = (P, A, \phi)$ of \mathcal{S} and a j such that A is j -reducible. We then modify \mathcal{S} by replacing α with the elements of $f_j(\alpha)$. We will call this a “ j -move applied to α ”. We apply such moves one-by-one until there are no moves available.

At each step of the process, we have that

$$\Phi(\mathcal{S}) := \{\phi(\mathcal{C}(AP)) : (P, A, \phi) \in \mathcal{S}\}$$

is a polytopal dissection of $\mathcal{C}(P_0 A_0)$; i.e., it is a set of full-dimensional polytopes with pairwise disjoint interiors and whose union is $\mathcal{C}(P_0 A_0)$. In fact, we have the following.

Lemma 3.9. *The above process always terminates and the final result $\mathcal{S}_{\text{final}}$ is independent of the moves chosen. Moreover, the map*

$$(P_0, A_0) \mapsto \Phi(\mathcal{S}_{\text{final}})$$

is a projective canonical triangulation Σ of \mathcal{C} such that $\Sigma(P_0, A_0)$ is unimodular with respect to $L_{P_0} \times \Lambda^{m-1}$.

Proof. For $(P, A, \phi) \in \mathcal{S}$, a consideration of how the dimensions of the entries of P and the volume of $\mathcal{C}(AP)$ change after each move easily proves termination of the process.

We now prove uniqueness of $\mathcal{S}_{\text{final}}$. By the diamond lemma, it suffices to prove the following: Suppose \mathcal{S} is an intermediate configuration and \mathcal{S}_1 and \mathcal{S}_2 are the results of applying moves M_1 and M_2 , respectively, to \mathcal{S} . Then there is a sequence of moves starting from \mathcal{S}_1 and a sequence of moves starting from \mathcal{S}_2 which both end in the same configuration.

If M_1 and M_2 apply to different elements of \mathcal{S} , then they clearly commute, and we are done. Assume they apply to the same element $\alpha = (P, A, \phi) \in \mathcal{S}$. We can assume M_1 and M_2 replace α with $f_{i_1 j_1}(\alpha)$ and $f_{i_2 j_2}(\alpha)$, respectively, with $j_1 \neq j_2$. First assume $i_1 \neq i_2$. Then applying a j_2 -move to all elements of $f_{i_1 j_1}(\alpha)$ yields the same result as applying a j_1 -move to all elements of $f_{i_2 j_2}(\alpha)$, as desired. (Note that these moves are legal due to the existence of M_1 and M_2 .)

Now assume $i_1 = i_2 = i$. First suppose that $\sum_{i'} A_{i' j_1} > 1$ and $\sum_{i'} A_{i' j_2} > 1$. Apply a j_2 -move to all elements of $f_{i j_1}(\alpha)$. Then, if $f_{i j_1}(\alpha)$ has two elements, apply a j_2 -move to $f_{j_2}(f_{i j_1}(\alpha)_2)_1$, where S_i denotes the i -th element of an ordered set S . We can check that this process gives the same result if we replace the roles of j_1 and j_2 , as desired.

Next suppose $\sum_{i'} A_{i' j_1} = 1$ and $\sum_{i'} A_{i' j_2} > 1$. We first define a sequence of moves starting from \mathcal{S}_1 . First, apply a j_2 -move to the one element of $f_{i j_1}(\alpha)$. Then apply a j_2 -move to $f_{j_2}(f_{i j_1}(\alpha)_1)_1$. Then apply a j_2 -move $f_{j_2}(f_{j_2}(f_{i j_1}(\alpha)_1)_1)_1$, and so on, until we have applied a total of $\dim P_i + 1$ moves. Now we define a sequence

of moves starting from \mathcal{S}_2 . Starting from the set $\{f_{i_{j_2}}(\alpha)\}$, apply j_1 -moves to elements until there are no j_1 -moves left. We again check that these two processes have the same result, as desired.

Finally, suppose that $\sum_{i'} A_{i'j_1} = 1$ and $\sum_{i'} A_{i'j_2} = 1$. Apply j_2 -moves to $\{f_{i_{j_1}}(\alpha)\}$ until there are no j_2 -moves available and apply j_1 -moves to $\{f_{i_{j_2}}(\alpha)\}$ until there are no j_1 -moves available. It is a straightforward proof by induction that these two processes have the same result. This completes the proof that $\mathcal{S}_{\text{final}}$ is unique, and thus Σ is well-defined.

We next show that every element of $\Sigma(P_0, A_0)$ is simplex which is unimodular in $L_{P_0} \times \Lambda^{m-1}$. It is easy to see that if $(P, A, \phi) \in \mathcal{S}_{\text{final}}$, then $\mathcal{C}(AP)$ must be a unimodular simplex. Since each ϕ is a lattice polytope isomorphism to a polytope with ambient lattice $L_{P_0} \times \Lambda^{m-1}$, the result holds.

To show that Σ is a triangulation, we note that f_j is canonical in the following sense: Let $(F, A') \rightarrow (P, A)$ be a morphism in \mathcal{F} . Assume (P, A) is j -reducible. Then either $\mathcal{C}(A'F)$ is a face of one of the elements of $\Phi(f_j(P, A, \text{id}))$, or (F, A') is j -reducible and

$$\Phi(f_j(F, A', \text{id})) = \Phi(f_j(P, A, \text{id}))|_{\mathcal{C}(A'F)}.$$

where $|_{\mathcal{C}(A'F)}$ denotes the induced dissection on $\mathcal{C}(A'F)$. This implies that $\Sigma(F, A') = \Sigma(P, A)|_F$.

We can now prove $\Sigma(P, A)$ is a triangulation by induction on the volume of $\mathcal{C}(AP)$. If $\mathcal{C}(AP)$ is not already a unimodular simplex, then there is a sequence of moves which subdivides $\mathcal{C}(AP)$ into two full-dimensional polytopes; call their common facet F . By induction, Σ gives triangulations of these two polytopes, and the aforementioned canonicity implies that these triangulations agree on F . Thus Σ gives a triangulation of $\mathcal{C}(AP)$. This argument also shows that the triangulation is canonical. Projectivity is standard to check. \square

3.4.2. The second triangulation. This triangulation is similar to the first one but differs in the order we apply the operators f_{ij} . To describe the triangulation, we need some additional notation.

A *stratified matrix* is a matrix A of nonnegative integers with rows indexed by a set I_A and columns indexed by $[n]$ for some n , along with an ordered partition $\Pi_A = (I_1, \dots, I_p)$ of I_A , where each I_k is an ordered set. For an ordinary matrix A with $I_A = [m]$, we can view A as a stratified matrix with the *standard stratification* $\Pi_A = (\{1\}, \{2\}, \dots, \{m\})$. Analogously to the definition of \mathcal{F} , we define $\tilde{\mathcal{F}}$ to be the category whose objects are tuples (P, A) , except that A is a stratified matrix. The morphisms are $(F, A') \rightarrow (P, A)$ where F_i is a face of P_i for all i , A' is obtained from A by restricting to a subset of rows of A , and if $\Pi_A = (I_1, \dots, I_p)$, then $\Pi_{A'} = (I_1 \cap I_{A'}, \dots, I_p \cap I_{A'})$. Note that the standard stratification makes \mathcal{F} a full subcategory of $\tilde{\mathcal{F}}$. We also have a full and faithful functor $\tilde{\mathcal{C}} : \tilde{\mathcal{F}} \rightarrow \mathcal{P}$ given by $(P, A) \mapsto \mathcal{C}(AP)$ which restricts to \mathcal{C} on \mathcal{F} .

Let $(P, A) \in \tilde{\mathcal{F}}$ with $\Pi_A = (I_1, \dots, I_p)$. Let A_{I_k} be the ordinary matrix obtained by restricting A to the rows indexed by I_k . We say that A is (I_k, j) -reducible if A_{I_k} is j -reducible. If A is (I_k, j) -reducible, then we define $f_{I_k, j}(P, A, \phi)$ to be the set obtained by applying f_j locally to (P, A_{I_k}) ; in other words,

$$f_{I_k, j}(P, A, \phi) := \{(P', \tilde{A}', \phi \circ \tilde{\phi}') : (P', A', \phi') \in f_j(P, A_{I_k}, \text{id})\}$$

where \tilde{A}' is obtained from A by replacing A_{I_k} with A' (and I_k with $I_{A'}$ in Π_A), and $\tilde{\phi}' : \mathcal{C}(\tilde{A}'P') \rightarrow \mathcal{C}(AP')$ is the affine map restricting to ϕ' on $\mathcal{C}(A'P')$ and the identity on $\mathcal{C}(A_{I_l}P')$ for $l \neq k$. We say that A is I_k -reducible if it is (I_k, j) -reducible for some j .

Fix $(P_0, A_0) \in \tilde{F}$, and define a process as follows. We keep track of a set \mathcal{S} , which is initially $\{(P_0, A_0, \text{id})\}$. A *move* consists of the following. First, choose an $\alpha = (P, A, \phi) \in \mathcal{S}$. Let $\Pi_A = (I_1, \dots, I_p)$, and let k be the smallest index for which A is I_k -reducible. Choose a j such that A is (I_k, j) -reducible, and modify \mathcal{S} by replacing α with the elements of $f_{I_k, j}(\alpha)$. Apply such moves until there are no moves available.

Defining $\Phi(\mathcal{S})$ as before, we have the following.

Lemma 3.10. *The above process always terminates and the final result $\mathcal{S}_{\text{final}}$ is independent of the moves chosen. Moreover, the map*

$$(P_0, A_0) \mapsto \Phi(\mathcal{S}_{\text{final}})$$

is a projective canonical triangulation Σ of $\tilde{\mathcal{C}}$ such that $\Sigma(P_0, A_0)$ is unimodular with respect to $L_{P_0} \times \Lambda^{|I_A|-1}$.

The proof is identical to the proof of Lemma 3.9. Also, note that the triangulation of Lemma 3.9 is obtained from this Lemma by viewing A_0 as a stratified matrix with $\Pi_{A_0} = ([m])$. So Lemma 3.10 can be seen as a generalization of Lemma 3.9 to \tilde{F} . However, we obtain a different triangulation than Lemma 3.9 if we use the standard stratification of A_0 .

3.5. Proof of Lemma 3.8. We are now ready to complete the proof of Theorem 3.3 by proving Lemma 3.8. Recall the definition of $\tilde{\mathcal{F}}$ from Section 3.4.2. We will focus on the following subcategory of $\tilde{\mathcal{F}}$.

Definition 3.11. *Let \mathbf{m} be a box point of P_0 for some $(P_0, A_0) \in \text{Ob}(\tilde{\mathcal{F}})$. We define $\tilde{\mathcal{F}}^{\mathbf{m}}$ to be the full subcategory of $\tilde{\mathcal{F}}$ whose objects are $(P, A) \in \text{Ob}(\tilde{\mathcal{F}})$ satisfying the following:*

(1) *Let $c = c(P, \mathbf{m})$. Then for all $i \in I_A$ and $j \in [n]$, we have*

$$A_{ij} = 0 \text{ or } A_{ij} \geq c_j.$$

(2) *Let $\Pi_A = (I_1, \dots, I_p)$. Then for all $k \in [p]$ and all $i, i' \in I_k$, we have*

$$\text{supp } A_i \cap \text{supp } c = \text{supp } A_{i'} \cap \text{supp } c := J_k.$$

Moreover, we have

$$J_1 \supseteq J_2 \supseteq \dots \supseteq J_p.$$

Let $\tilde{\mathcal{C}}^{\mathbf{m}}$ be the restriction of $\tilde{\mathcal{C}}$ to $\tilde{\mathcal{F}}^{\mathbf{m}}$.

Clearly $\mathcal{F}^{\mathbf{m}}$ is a subcategory of $\tilde{\mathcal{F}}^{\mathbf{m}}$ via the standard stratification, and $\mathcal{C}^{\mathbf{m}}$ is the restriction of $\tilde{\mathcal{C}}^{\mathbf{m}}$ to this subcategory. It thus suffices to prove the following refinement of Lemma 3.8.

Lemma 3.12. *Let \mathbf{m} be as above. Then there is a projective canonical triangulation $\Sigma^{\mathbf{m}}$ of $\tilde{\mathcal{C}}^{\mathbf{m}}$ such that for all $(P, A) \in \text{Ob}(\tilde{\mathcal{F}}^{\mathbf{m}})$ and all full-dimensional simplices Q in $\Sigma^{\mathbf{m}}(P, A)$, we have the following.*

- *If \mathbf{m} is a box point of P_A , then*

$$\text{index}(Q) < \text{index}(P_A).$$

- If \mathbf{m} is not a box point of P_A , then $L_Q = L_{P_A} \times \Lambda^{|I_A|-1}$.

Proof. Let $(P, A) \in \text{Ob}(\mathcal{F}^{\mathbf{m}})$. If \mathbf{m} is not a box point of P_A , then we set $\Sigma^{\mathbf{m}}(P, A)$ to be the triangulation from Lemma 3.10. This satisfies the desired properties.

Now assume \mathbf{m} is a box point of P_A . We proceed by induction on the size of the set of configurations reachable from $\{(P, A, \text{id})\}$ through moves, as defined in of Section 3.4. If no moves are available, then all entries of A are 0. In this case $P_A = \{0\}$ so \mathbf{m} is not a box point of P_A , which we already considered. So we may assume there are moves available and A is not a 0-matrix. Note that deleting a 0-column of A and the corresponding entry of P does not change P_A or $\mathcal{C}(AP)$. So we may assume A has no 0-columns and $P_A = P$.

Let $c := c(P, \mathbf{m})$. Since \mathbf{m} is a box point of P , we have $\text{supp } c \neq \emptyset$. Let $\Pi_A = (I_1, \dots, I_p)$, and for each $k \in [p]$, define J_k as in Definition 3.11. Since A has no 0-columns, and $J_1 \supseteq \dots \supseteq J_p$, we must have $J_1 = \text{supp } c$. By the definition of $\tilde{\mathcal{F}}$, this implies $A_{ij} \geq c_j$ for all $i \in I_1$ and $j \in [n]$.

We separate the argument into two cases.

Case 1: $A_{ij} > c_j$ for some $i \in I_1$ and $j \in [n]$.

Let $j \in [n]$ be the smallest index for which $A_{ij} > c_j$ for some $i \in I_1$. Then A is (I_1, j) -reducible. Let $i \in I_1$ be the unique index such that $f_j(P, A_{I_1}, \text{id}) = f_{ij}(P, A_{I_1}, \text{id})$. By our assumption and the definition of i , $A_{ij} > c_j$.

Now, recall that $\Phi(f_{I_1, j}(P, A, \text{id}))$ subdivides $\mathcal{C}(AP)$ into one or two polytopes, and comes with an ordering on these polytopes. Call the first of these polytopes Q_1 . Then we have an affine isomorphism $\phi_1 : \mathcal{C}(A'P) \rightarrow Q_1$, where either

- A' is obtained from A by subtracting 1 from the A_{ij} entry, or
- $A' = A^\square$ as defined in Section 3.4.1.

In either case, since $A_{ij} > c_j$, it is easy to check that $(P, A') \in \mathcal{F}^{\mathbf{m}}$ and \mathbf{m} is a box point of $P_{A'}$. Thus, by induction, we have a triangulation Σ_1 of $\mathcal{C}(A'P)$ into full-dimensional simplices each with index less than $\text{index}(P_{A'})$.

We now wish to show that $\text{index}(\phi_1(Q)) < \text{index}(P)$ for all full-dimensional $Q \in \Sigma_1$. If we have case (a), then ϕ_1 is a lattice polytope isomorphism and $P_{A'} = P_A = P$, so the claim follows. Suppose we have case (b). Let $Q \in \Sigma_1$ be full dimensional. Then

$$\text{index}(\phi_1(Q)) = \text{index}(P_j) \text{index}(Q).$$

Also, $\text{index}(P) = \text{index}(P_j) \text{index}(P_{A'})$. Thus $\text{index}(\phi_1(Q)) < \text{index}(P)$, as desired.

If $f_j(P, A, \text{id})$ has one element, we are done. Otherwise, let Q_2 be the second element of $\Phi(f_j(P, A, \text{id}))$. We need to give a triangulation of Q_2 . Recall the definitions of A'' , v , and F from Section 3.4.1. Then we have an affine isomorphism $\phi_2 : \mathcal{C}(A''F) \rightarrow Q_2$.

First suppose \mathbf{m} is a box point of F . Since $c(F, \mathbf{m}) = c$ and $A_{ij} > c_j$, we have $(F, A'') \in \mathcal{F}^{\mathbf{m}}$. We thus have a triangulation Σ_2 of $\mathcal{C}(A''F)$ using the inductive hypothesis. Let $Q \in \Sigma_2$ be full-dimensional; we wish to show that $\text{index}(\phi_2(Q)) < \text{index}(P)$. Since \mathbf{m} is a box point of F , we have by the inductive hypothesis $\text{index}(Q) < \text{index}(F_{A''}) = \text{index}(F)$. Now, let h be the lattice distance between the parallel hyperplanes $\text{AffSpan}(v + A'_i F)$ and $\text{AffSpan}(A'_i F)$. Then

$$\text{index}(\phi_2(Q)) = h \text{index}(Q).$$

Also, h is the lattice distance between v and F_j , so $\text{index}(P) = h \text{index}(F)$. We conclude that $\text{index}(\phi_2(Q)) < \text{index}(P)$, as desired.

Now assume \mathbf{m} is not a box point of F . In particular, this implies $j \in \text{supp } c$. Let m be the unique representative of \mathbf{m} in cP . Consider the polytope

$$m + (A_i - c)F$$

which is well-defined since $A_{ik} - c_k \geq 0$ for all k . The affine span of this polytope is parallel to $\text{AffSpan}(v + A'_i F)$ and $\text{AffSpan}(A_i F)$, and since \mathbf{m} is not a box point of F , it lies strictly between these two hyperplanes. Moreover, this polytope is contained in $\text{conv}((v + A'_i F) \cup (A_i F))$.

We use this to construct a subdivision of Q_2 as follows. First, we have the polytopes

$$R_1 := \mathcal{C}((AF)[i, \text{conv}((m + (A_i - c)F) \cup (A_i F))])$$

$$R_2 := \mathcal{C}((AF)[i, \text{conv}((m + (A_i - c)F) \cup (v + A'_i F))]).$$

In addition, let \mathfrak{F} denote the set of all $F' = (F'_1 \dots F'_n)^T$ such that F'_k is a face of P_k for all k , $\sum_k F'_k$ is a facet of $\sum_k P_k$, and \mathbf{m} is not a box point of F' . Let \mathfrak{G} denote the set of all $G = (G_1 \dots G_n)^T$ such that G_k is a face of F_k for all k , $\sum_k G_k$ is a facet of $\sum_k F_k$, and such that there exists $F' \in \mathfrak{F}$ with

$$G_k = F_k \cap F'_k$$

for all k . For each $G \in \mathfrak{G}$, we define the polytope

$$R_G := \mathcal{C}((AG)[i, \text{conv}((m + (A_i - c)G) \cup (v + A'_i G) \cup (A_i G))]).$$

Then $\{R_1, R_2\} \cup \{R_G\}_{G \in \mathfrak{G}}$ forms a subdivision of Q_2 .

Let $B, B',$ and B'' be the matrices obtained by inserting $A_i - c$ above the i -th row of $A, A',$ and A'' , respectively (this operation is done locally in I_1). Then we have affine isomorphisms

$$R_1 \cong \mathcal{C}(F, B) \quad R_2 \cong \mathcal{C}(F, B') \quad R_G \cong \mathcal{C}(G, B'')$$

Using Lemma 3.10, we obtain triangulations $\Xi_1, \Xi_2,$ and Ξ_G of $R_1, R_2,$ and R_G . We prove that these triangulations give the desired triangulation of Q_2 . Suppose Q is a full-dimensional simplex in Ξ_1 . Then

$$\text{index}(Q) = h' \text{index}(F)$$

where h' is the lattice distance between $\text{AffSpan}(m + (A_i - c)F)$ and $\text{AffSpan}(A_i F)$. Since $h' < h$, where h is as defined previously, we have $\text{index}(Q) < \text{index}(P)$, as desired. Similarly, $\text{index}(Q) < \text{index}(P)$ for all full-dimensional $Q \in \Xi_2$.

Finally, suppose $Q \in \Xi_G$ is full-dimensional. Then

$$\text{index}(Q) = h'' \text{index}(F')$$

where $F' \in \mathfrak{F}$ is such that $G_k = F_k \cap F'_k$ for all k , and h'' is the lattice distance between m and $\sum_k c_k F'_k$. By replacing F with F' in the previous arguments, we have that $\text{index}(Q) < \text{index}(P)$, as desired.

All that remains is to show canonicity. Let $\Sigma^{\mathbf{m}}$ denote the above triangulation. Let $(F, A') \rightarrow (P, A)$ be a morphism in $\tilde{\mathcal{F}}$. We wish to show that $\Sigma^{\mathbf{m}}(F, A') = \Sigma(P, A)|_{\mathcal{C}(A'F)}$. If $\mathcal{C}(A'F)$ is a face of one of Q_1 and Q_2 , we are done by induction. Assume otherwise. First suppose \mathbf{m} is a box point of F . Then it is straightforward to check that if we repeat the above process with (P, A) replaced by (F, A') , we get $\Sigma^{\mathbf{m}}(P, A)|_{\mathcal{C}(A'F)}$. The crucial detail is that since $c(F, \mathbf{m}) = c(P, \mathbf{m})$, we end up choosing the same values for i and j .

Now assume \mathbf{m} is not a box point of (F, A') . One can check that $\Sigma^{\mathbf{m}}(P, A)|_{\mathcal{C}(A'F)}$ is the triangulation obtained by triangulating each element of $\Phi(f_{I_1, j}(F, A', \text{id}))$ by the inductive hypothesis, i.e. by the triangulation Σ from Lemma 3.10. Then Lemma 3.10 implies that this triangulation is $\Sigma(F, A')$. Hence $\Sigma(P, A)|_{\mathcal{C}(A'F)} = \Sigma(F, A') = \Sigma^{\mathbf{m}}(F, A')$, as desired.

Case 2: $A_i = c$ for all $i \in I_1$.

Define m and \mathfrak{F} as in the previous case. Let i be the first element of I_1 . Then $\mathcal{C}(AP)$ has a subdivision into the full-dimensional polytopes

$$R_F := \mathcal{C}((AF)[i, \text{conv}(\{m\} \cup cF)])$$

where F ranges over all elements of \mathfrak{F} , and

$$R_A := \mathcal{C}((AP)[i, \{m\}])$$

if R_A is full-dimensional. Let A' be the stratified matrix obtained by deleting the i -th row of A . Then we have affine isomorphisms $\phi_F : \text{Cone}(\mathcal{C}(AF)) \rightarrow R_F$ and $\phi_A : \text{Cone}(\mathcal{C}(A'P)) \rightarrow R_A$, where

$$\text{Cone}(Q) := \mathcal{C}(\{0\}, Q).$$

By Lemma 3.10, we obtain a triangulation of $\mathcal{C}(AF)$, which gives a triangulation of R_F by coning and ϕ_F . Similar arguments to the previous section imply all full-dimensional simplices of this triangulation have index less than $\text{index}(P)$, as desired. Finally, if R_A is full-dimensional, then $P_{A'} = P_A = P$ and $(P, A') \in \tilde{\mathcal{F}}$, so by induction we have a triangulation of $\mathcal{C}(A'P)$ whose full-dimensional simplices have index less than $\text{index}(P)$. Coning this triangulation gives the desired triangulation of R_A . Canonicity is easy to check; we note that in this case, if $(F, A') \rightarrow (P, A)$ is a morphism and \mathbf{m} is not a box point of F , then $\mathcal{C}(A'F)$ is a face of some R_F . \square

4. MORE SEMISTABLE REDUCTION THEOREMS

4.1. The main theorem for log schemes. Our goal now is to lift the canonical alteration of fans constructed in Theorem 2.7 to log schemes.

4.1.1. Toric stacks. It is well known that subdivisions of fans naturally lift to modifications of log schemes (in the old language this was worked out in [KKMS73]). Lifting more general morphisms, including the case of alterations that we need, involves toric stacks that we briefly recall now.

For a scheme X and a lattice Λ let $T_{X, \Lambda} = X \times \text{Spec}(\mathbf{Z}[\Lambda])$ denote the corresponding X -torus. Given a fine monoid P let $F_P = \text{Spec}(P)$, $Z_P = \text{Spec}(\mathbf{Z}[P])$ and $\mathcal{Z}_P = [Z_P/T_{P^{\text{gp}}}]$ denote the corresponding affine fan, toric scheme and toric stack, respectively. To a log scheme X with a global chart $X \rightarrow Z_P$ and a homomorphism of fine monoids $\phi: P \rightarrow Q$ we associate the relative toric scheme $X_P[Q] = X \times_{Z_P} Z_Q$ and the relative toric stack

$$\mathcal{X}_P[Q] = X \times_{Z_P} \mathcal{Z}_Q = [X_P[Q]/T_{X, Q^{\text{gp}}/P^{\text{gp}}}]$$

Remark 4.1. (i) The stack $\mathcal{X}_P[Q]$ plays an important role in logarithmic geometry, and it only depends on the fan $\bar{P} \rightarrow \bar{M}_X$ and the sharpening $\bar{\phi}: \bar{P} \rightarrow \bar{Q}$, see [Ols03, Proposition 5.17] or [MT19, Lemma 3.2.4]. Thus, $\mathcal{X}_P[Q]$ can be viewed as the base change of the map of fans $F_Q \rightarrow F_P$ and the dependence on Q is only through \bar{Q} .

(ii) Unlike $\mathcal{X}_P[Q]$, the scheme $X_P[Q]$ essentially depends on the chart $P \rightarrow M_X$ and the homomorphism ϕ . So it is more correct to view it as a base change of a

morphism of monoschemes. This technical point was essential in [IT14a, Section 3] and [ALPT18] but not in this paper since we will only work with $\mathcal{X}_P[Q]$.

(iii) If $P^{\text{gp}} = Q^{\text{gp}}$, then $X_P[Q] = \mathcal{X}_P[Q]$.

4.1.2. *Globalization.* Assume that $F_R \hookrightarrow F_Q$ is an open immersion, that is, $\overline{Q} \rightarrow \overline{R}$ is a sharpened localization. Then it is easy to see that $\mathcal{X}_P[R] \rightarrow \mathcal{X}_Q[R]$ is an open immersion too. It follows that the construction globalizes to fans: to any fan F over F_P one associates a toric stack $\mathcal{X}_P[F]$ with the natural log structure.

4.1.3. *Lifting fan functors.* Let \mathcal{L} be a construction that associates to any fan F a morphism of fans $\mathcal{L}(F) \rightarrow F$. We say that \mathcal{L} is *quasi-local* if for any surjective local isomorphism $f: F' \rightarrow F$ we have that $\mathcal{L}(F') = \mathcal{L}(F) \times_F F'$. The following theorem is proved in [MT19]. We briefly recall the argument, because a similar reasoning will be used later.

Theorem 4.2. *Let \mathcal{L} be a quasi-local construction on fans. Then there exists a unique construction \mathcal{L}^{log} associating to any log scheme X a morphism $\mathcal{L}^{\text{log}}(X) \rightarrow X$ and such that the following two conditions are satisfied:*

(i) *If $Y \rightarrow X$ is a strict surjective morphism, then $\mathcal{L}^{\text{log}}(Y) = \mathcal{L}^{\text{log}}(X) \times_Y X$.*

(ii) *If $X = \coprod_i X_i$ and $X_i \rightarrow F_{P_i}$ are global fans, then $\mathcal{L}^{\text{log}}(X) = \coprod (\mathcal{X}_i)_{P_i}[F_i]$, where F_i is the preimage of F_{P_i} in $\mathcal{L}(\coprod_i F_{P_i})$.*

Proof. If $X = \coprod_i X_i$ and each X_i possesses a global affine fan, then (ii) dictates the definition of $\mathcal{L}^{\text{log}}(X)$, and the independence of choices follows from Remark 4.1(i). In the general case, find a strict étale covering $\coprod_i X_i \rightarrow X$, such that each X_i possesses a global chart, and use (i) and étale descent. \square

Example 4.3. (i) If $\mathcal{L}(X) \rightarrow X$ is a subdivision, then it follows from Remark 4.1(iii) that $\mathcal{L}^{\text{log}}(X)$ is a scheme, which is étale locally of the form $X_P[Q]$, and $\mathcal{L}^{\text{log}}(X) \rightarrow X$ is a log étale modification. For example, in this way one can define the barycentric subdivision X_{bar} of the monoidal structure of X . Similarly, there are various resolution of fans $F \mapsto F_{\text{res}}$ constructed in the literature, and some of them are local constructions. Lifting such a construction one obtains a monoidal resolution $X \mapsto X_{\text{res}}$ and a log étale modification $X_{\text{res}} \rightarrow X$. The monoids $\overline{M}_{X_{\text{res}},x}$ are free, in particular, if X is log regular then X_{res} is log regular and the underlying scheme is regular. Since [KKMS73] such constructions are used to resolve toroidal singularities, see [IT14a, Section 3.3] for a modern treatment.

(ii) Fix $c > 0$. For an fs monoid P the group \overline{P}^{gp} is torsion free and we let $c^{-1}\overline{P}$ denote the saturation of \overline{P} in $c^{-1}\overline{P}^{\text{gp}}$. Setting $cF_P = F_{c^{-1}\overline{P}}$ we obtain a functor compatible with localizations and hence extending to a local functor $F \mapsto cF$ on the category of saturated fans. The induced functor on fs log schemes is nothing else but the root stack introduced in [BV12] and denoted there X_c . Note that $X_c \rightarrow X$ is a proper log étale morphism, which is non-representable whenever $c > 1$ and the log structure is non-trivial.

(iii) In the same manner, if $\mathcal{L}(X) \rightarrow X$ is an alteration, then $\mathcal{L}^{\text{log}}(X) \rightarrow X$ is a proper log étale morphism, which is an isomorphism over the triviality locus of the log structure. We call such a morphism *monoidal alteration*.

4.1.4. *Monoidal semistable reduction.* A morphism of log schemes $f: Y \rightarrow X$ is called *monoidally semistable* if Y and X are monoidally regular and $f: Y \rightarrow X$ is saturated. By a (*projective*) *monoidal resolution* of a morphism $Y \rightarrow X$ of

log schemes we mean a (projective) monoidal alteration $X' \rightarrow X$ and a (projective) monoidal subdivision $Y' \rightarrow Y \times_X X'$ such that $f': Y' \rightarrow X'$ is monoidally semistable.

Theorem 4.4. *There exists a construction associating to each morphism of fine log schemes $f: X \rightarrow B$ a projective monoidal resolution $f': X' \rightarrow B'$ and compatible with surjective strict morphisms: if $g: Y \rightarrow C$ is another morphism and $\alpha: Y \rightarrow X$, $\beta: C \rightarrow B$ are strict surjective and such that $\beta \circ g = f \circ \alpha$, then the monoidal resolution g' of g is the base change of f' in the sense that $C' = C \times_B B'$ and $Y' = Y \times_X X'$.*

Proof. This is done similarly to the proof of Theorem 4.2. The only difference is that this time we lift the quasi-local construction that associates a morphism of fans to a morphism of fans.

Assume first that f possess a quasi-local chart in the following sense: $B = \coprod_i B_i$ with charts $B_i \rightarrow Z_{P_i}$, $X = \coprod_{ij} X_{ij}$ with charts $Z_{Q_{ij}}$, and f restricts to morphisms $X_{ij} \rightarrow B_i$ with charts $Z_{Q_{ij}} \rightarrow Z_{P_i}$. Consider the corresponding map of fans $g: \coprod_{ij} F_{Q_{ij}} \rightarrow \coprod_i F_{P_i}$ and let g' be its resolution constructed in Theorem 2.7. The source (resp. target) of g' is an alteration of the source (resp. target) of g , hence they have the same decomposition to connected components, say $g': \coprod_{ij} F_{ij} \rightarrow \coprod_i F_i$. Clearly, the morphism $f': X' \rightarrow B'$, where $B' = \coprod_i (\mathcal{B}_i)_{P_i} [F_i]$ and $X' = \coprod_{ij} (\mathcal{X}_{ij})_{Q_{ij}} [F_{ij}]$, is a resolution of f , and its independence of the chart follows from Remark 4.1(i) and quasi-locality of the resolution g' of g . Moreover, by the same reason this construction is compatible with surjective strict morphisms.

In general, there exist surjective strict étale morphisms $X_0 \rightarrow X$ and $B_0 \rightarrow B$ with a morphism $f_0: X_0 \rightarrow B_0$ compatible with f such that f_0 possesses a quasi-local chart. The induced morphism $f_1: X_0 \times_X X_0 \rightarrow B_0 \times_B B_0$ also possesses a quasi-local chart (for example, either pullback of the chart of f_0), hence the resolution of f_0 descends to a resolution of f by descent. \square

4.2. Applications.

4.2.1. *Semistable morphisms.* We say that a morphism of log schemes $f: X \rightarrow B$ is *semistable* if f is monoidally semistable, and in addition B is log regular and f is log smooth. This happens if and only if the following conditions hold:

- (i) X and B are regular and the log structures are given by normal crossings divisors $Z \hookrightarrow X$ and $W \hookrightarrow B$.
- (ii) Étale-locally at any $x \in X$ with $b = f(x)$ there exist regular parameters $t_1, \dots, t_n, t'_1, \dots, t'_{n'} \in \mathcal{O}_x$ and $\pi_1, \dots, \pi_l, \pi'_1, \dots, \pi'_{l'} \in \mathcal{O}_b$ such that $Z = V(t_1 \dots t_n)$ at x , $W = V(\pi_1 \dots \pi_l)$ at b , $f^\#(\pi_i) = t_{n_i+1} \dots t_{n_i+1}$ for $0 = n_1 < n_2 < \dots < n_{l+1} \leq n$,
- (iii) f is log smooth.

In characteristic zero, (iii) can be replaced by the condition that $f^\#(\pi'_j) = t'_j$ for $1 \leq j \leq l'$. (In positive characteristic, this is neither necessary, nor sufficient.)

Theorem 4.5. *Assume that $f: X \rightarrow B$ is a log smooth morphism between fine log schemes. Then there exists a monoidal alteration $b: B' \rightarrow B$ and a monoidal subdivision $a: X' \rightarrow X \times_B B'$ such that the morphism $f': X' \rightarrow B'$ is semistable.*

Proof. We simply apply Theorem 4.4 to f . Since a and b are log étale, this implies that f' is log smooth and monoidally semistable, that is, f' is semistable. \square

Remark 4.6. If X and B are schemes, then using Kawamata's trick as in [AK00], one can achieve that X' and B' are also schemes. However, in this case b can only be taken to be a usual alteration, and its choice is non-canonical. In particular, it does not have to be nice (e.g. étale) over the triviality locus of the log structure of B .

The above theorem and the toroidalization theorem of Abramovich-Karu imply the semistable conjecture. We will use a finer toroidalization proved in [ATW19] to obtain a strong version of the semistable reduction conjecture, where one also controls the modification loci, see part (ii) below. For convenience of applications we use closed subsets in the formulation, but one can easily reformulate this result in the language of log schemes.

Theorem 4.7. *Assume that $X \rightarrow B$ is a dominant morphism of finite type between integral schemes of characteristic zero and $Z \subsetneq X$ is a closed subset. Then there exists a stack-theoretic modification $b: B' \rightarrow B$, a projective modification $a: X' \rightarrow (X \times_B B')^{\text{pr}}$, and divisors $W' \hookrightarrow B'$, $Z' \hookrightarrow X'$ such that:*

(i) $a^{-1}(Z) \cup f'^{-1}(W') \subseteq Z'$ and the morphism $f': (X', Z') \rightarrow (B', W')$ is semistable. In particular, X', B' are regular and Z', W' are snc.

(ii) If a regular open $B_0 \subseteq B$ is such that $X_0 = X \times_B B_0 \rightarrow B_0$ is smooth and $Z_0 = Z \times_B B_0$ is a relative divisor over B_0 with normal crossings (in other words, $(X_0, Z_0) \rightarrow B_0$ is semistable), then a and b are isomorphisms over X_0 and B_0 , respectively.

Proof. Replacing X by the blow up along Z one does not modify X_0 . So, one can assume that Z is a divisor, and then X possesses a log structure $M_X \hookrightarrow \mathcal{O}_X$ whose triviality locus is $X \setminus Z$. Choose such an M_X and provide B with the trivial log structure. Resolving the morphism of log schemes $f: X \rightarrow B$ by the main result of [ATW19], we find modifications of log schemes $b'': B'' \rightarrow B$ and $a'': X'' \rightarrow X$ with a log smooth morphism $f'': X'' \rightarrow B''$ compatible with f . By the loc.cit., if f is log smooth over an open $B_1 \subseteq B$, then one can achieve that b'' and a'' are isomorphisms over B_1 and $X \times_B B_1$, respectively. This applies to $B_0 = B_1$ because $(X_0, Z_0) \rightarrow B_0$ is even semistable. In addition, resolving B'' we can also achieve that it is log regular. Again, this does not modify B_0 because it is regular. Now, it remains to apply Theorem 4.5 to f' . \square

In the same way, using [Tem17, Theorem 4.3.1] one obtains altered semistable reduction in all characteristics.

Theorem 4.8. *Let $X \rightarrow B$ be a dominant morphism of finite type between integral schemes and $Z \subsetneq X$ a closed subset. Assume that B is of finite type over a qc scheme of dimension at most 3. Then there exists a stack-theoretic $\text{char}(B)$ -alteration $b: B' \rightarrow B$, a projective $\text{char}(B)$ -alteration $a: X' \rightarrow (X \times_B B')^{\text{pr}}$, and divisors $W' \hookrightarrow B'$ and $Z' \hookrightarrow X'$, such that $a^{-1}(Z) \cup f'^{-1}(W') \subseteq Z'$ and the morphism $f': (X', Z') \rightarrow (B', W')$ is semistable.*

In both results one can use the Kawamata's trick to achieve that b is also projective at cost of making it an alteration of an arbitrary degree that modifies B_0 .

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