EXTRA SET THEORY PROBLEMS

Here are some extra problems. Don't let them distract you from the real homework. I will occasionally add to this file as I think of more problems that I wanted to add.

Problem 1. *Assume that $\kappa^{<\kappa} = \kappa$. Construct a linear order L of size κ such that for all $A, B \subseteq L$ with $|A|, |B| < \kappa$ and for all $a \in A$ and $b \in B$ $a < b$, there is $c \in L$ such that $a < c < b$ for all $a \in A$ and $b \in B$.*

Problem 2. *Assume $\kappa^{<\kappa} = \kappa$. Use the linear order L from the previous problem to construct a κ^+-tree T with no cofinal branch. The tree you construct should be* special *in the sense that there is a function $f : T \to \kappa$ such that $s < t$ implies $f(s) \neq f(t)$.*

Problem 3. *Show that* $\mathrm{MA}(\aleph_1)$ *implies all ccc posets are ω_1-Knaster through the following sequence of claims about a ccc poset $\mathbb{P}$:*

(1) *Let $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ be a sequence of elements of $\mathbb{P}$. Show that there is an $\alpha < \omega_1$ such that p_α is compatible with uncountably many p_β. To do this suppose otherwise and let F be a function such that $F(\alpha)$ is the least ordinal γ such that for all β if p_α, p_β are compatible, then $\beta < \gamma$. Show that the collection of $\gamma < \omega_1$ such that $\{F(\alpha) \mid \alpha < \gamma\} \subseteq \gamma$ is a club $C \subseteq \omega_1$. Show that $\{p_\alpha \mid \alpha \in C\}$ is an antichain, a contradiction.*
(2) *Fix an α^* as in the first part. Next show that the set $D_\alpha = \{p \mid p \leq p_\gamma$ for some $\gamma > \alpha\}$ is dense in the ccc poset $\mathbb{P} \restriction p_{\alpha^*} = \{p \in \mathbb{P} \mid p \leq p_{\alpha^*}\}$.*
(3) *Apply* $\mathrm{MA}(\aleph_1)$.

Problem 4. *Assume CH. Show that $\mathbb{R} \times \mathbb{R}$ can be written as the union of countably many functions some of which have domain the x-axis and some of which have domain the y-axis.*