FORCING EXERCISES
DAY 13

Definition 1. *A map* $\pi : \mathbb{P} \to \mathbb{Q}$ *is a* projection *if*

(1) $\pi(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{Q}}$,
(2) *for all* $p_1, p_2 \in \mathbb{P}$, $p_1 \leq p_2$ *implies that* $\pi(p_1) \leq \pi(p_2)$ *and*
(3) *for all* $p \in \mathbb{P}$ *and all* $q \leq \pi(p)$, *there is* $p' \in \mathbb{P}$ *such that* $p' \leq p$ *and* $\pi(p') \leq q$.

Let $\pi : \mathbb{P} \to \mathbb{Q}$ be a projection.

Problem 2. *If* G *is* $\mathbb{P}$*-generic, then the upwards closure of* $\{\pi(p) \mid p \in G\}$ *is* $\mathbb{Q}$*-generic.*

Definition 3. *Let* κ *be a regular cardinal.* $\mathbb{P}$ *has the* κ *chain condition if every antichain of* $\mathbb{P}$ *has size less than* κ.

Problem 4. *If* $\mathbb{P}$ *is* κ*-cc, then* $\mathbb{Q}$ *is* κ*-cc.*

Problem 5. *Let* H *be* $\mathbb{Q}$*-generic. In* $M[H]$ *define a poset* $\mathbb{P}/H$ *whose underlying set is* $\{p \in \mathbb{P} \mid \pi(p) \in H\}$ *and is ordered as a suborder of* $\mathbb{P}$. *(Note that* $M[H]$ *is a perfectly good transitive model of ZFC, so we can force over it.) Let* G *be* $\mathbb{P}/H$*-generic over* $M[H]$. *Show that* G *is a* $\mathbb{P}$*-generic filter over* M.

Problem 6. *Let* $\mathbb{C}$ *be Cohen forcing that is functions whose domain is a natural number and range is contained in* ω. *Define* $\pi : \mathbb{C} \to \mathbb{C}$ *by* $\pi(p)(n) = p(2n)$ *whenever* $2n$ *is in the domain of* p. *Show that* π *is a projection. (Notice that this solves Problem 3 part 2 from Day 11.)*

Problem 7. *Let* G *be* $\mathbb{C}$*-generic. In* $M[G]$ *define* $\mathbb{C}/G$ *as above using* π *as in the previous problem. What is this poset? Can you give a concrete characterization?*

Problem 8. *An automorphism* $i : \mathbb{P} \to \mathbb{P}$ *is a bijection such that* $p \leq q$ *if and only if* $i(p) \leq i(q)$. *A poset is* almost homogeneous *if for any* $p, q \in \mathbb{P}$ *there is an automorphism* $i : \mathbb{P} \to \mathbb{P}$ *such that* $i(p)$ *and* q *are compatible. Do the following:*

(1) *Show that if* I *is an infinite set and* J *is any set then* $Fn(I, J) = \{p \mid dom(p) \subseteq I$ *is finite and* $ran(p) \subseteq J\}$ *ordered by reverse inclusion is almost homogeneous. (Note that the forcing to make* $2^\omega = \omega_2$ *is* $Fn(\omega_2, 2)$.*)*
(2) *Given an automorphism* $i : \mathbb{P} \to \mathbb{P}$ *with* $i \in M$ *and a* $\mathbb{P}$*-name* τ *we define recursively the* $\mathbb{P}$*-name* $i(\tau)$ *by letting* $i(\tau)$ *be all the pairs* $\langle i(\sigma), i(p)\rangle$ *where* $\langle \sigma, p\rangle \in \tau$. *Show that for any* $x \in M$, $i(\check{x}) = \check{x}$.
(3) *Let* $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$. *Show that* $p \Vdash \phi(\tau_1, \ldots, \tau_n)$ *if and only if* $i(p) \Vdash \phi(i(\tau_1), \ldots, i(\tau_n))$.
(4) *Suppose that* $\mathbb{P}$ *is almost homogeneous, and let* G *be* $\mathbb{P}$*-generic. Let* $x_1, \ldots x_n \in M$. *Show that if* $M[G] \vDash \phi(x_1, \ldots, x_n)$ *then in fact* $\mathbb{1} \vDash \phi(\check{x_1}, \ldots, \check{x_n})$.
(5) *Conclude that if* $\mathbb{P}$ *is almost homogeneous, then for any* $\mathbb{P}$*-generic filters* G *and* H, $M[G]$ *and* $M[H]$ *are elementarily equivalent (with respect to first order logic).*

Problem 9. *Show that if* $\mathbb{P}$ *is* κ^+*-cc where* κ *is regular, then forcing with* $\mathbb{P}$ *adds a subset to* κ^+*. Hint: Use a previous exercise.*

Problem 10. *Remove the assumption that* κ *is regular in the previous problem.*

Problem 11. *Let* $\mathbb{P}$ *be* ω_1*-Knaster. Show that if* T *is a tree of height* ω_1 *with no cofinal branch, then for all* M*-generic* G*,* T *has no cofinal branch in* $M[G]$*.*

Problem 12 (*)**.** *Let* T *be a tree of height* ω_1 *and* $\mathbb{P}$ *be a poset such that* $\mathbb{P} \times \mathbb{P}$ *is ccc. Let* G *be* M*-generic over* $\mathbb{P}$*. Show that if* $b \in M[G]$ *is a cofinal branch through* T*, then* $b \in M$*.*