# FORCING EXERCISES
# DAY 8

**Problem 1.** *Do the exercise from class about showing that some basic concepts are absolute.*

Let $X$ be a countable elementary substructure of some $H_\theta$ where $\theta$ is regular and uncountable. Note that $X$ is not transitive!

**Problem 2.** *Suppose that $A \in X$ and $H_\theta \models$ "A is countable". Show that $X \models$ "A is countable" and $A \subseteq X$.*

**Problem 3.** *Show that $X \cap \omega_1 \in \omega_1$.*

**Problem 4.** *Define a countable set of ordinals that is not a member of $X$. Hint: It's easy.*

**Problem 5.** *Show that $\omega_1 \in X$. Hint: It's easy.*

**Problem 6.** *Define a subset of $\omega$ which is not a member of $X$.*

**Problem 7.** *Let $\langle X_\alpha \mid \alpha < \omega_1 \rangle$ be a sequence of elementary substructures of $H_\theta$ for some regular uncountable $\theta$ such that $X_\alpha \in X_{\alpha+1}$ for all $\alpha < \omega_1$ and $X_\gamma \bigcup_{\alpha<\gamma} X_\alpha$ for all limit $\gamma$. Show the following:*

(1) *For all $\alpha < \beta < \omega_1$, $X_\alpha \prec X_\beta$.*
(2) *$\{X_\alpha \cap \omega_1 \mid \alpha < \omega_1\}$ is club in $\omega_1$.*
(3) *There is a club $C$ in $\omega_1$ such that for all $\gamma \in C$, $X_\gamma \cap \omega_1 = \gamma$.*

**Problem 8.** *Show that $(X, \in)$ is a well-founded extensional model.*

**Definition 1.** *Let $M, N$ be transistive models of set theory. Let $\pi : M \to N$ be an elementary embedding. Define the critical point of $\pi$ to be the least ordinal $\alpha$ such that $\pi(\alpha) > \alpha$.*

Let $M$ be the Mostowski Collapse of $X$. Let $\pi : M \to X$ be the inverse of the Mostowski collapse map. Note that viewed as a map from $M$ to $H_\kappa$, $\pi$ is an elementary embedding.

**Problem 9.** *Show that the critical point of $\pi$ is a countable ordinal (which one?) and that $\pi$ applied to the critical point is $\omega_1$.*

**Problem 10.** *Let $M$ be a transitive model of set theory. Let $\kappa$ be a regular cardinal with $U$ an ultrafilter on $\kappa$. Define a structure $(\mathrm{Ult}_U(M), E)$ as follows. $\mathrm{Ult}_U(M)$ is the collection equivalence classes of functions from $\kappa$ to $M$ under the following equivalence relation. If $f, g \in \mathrm{Ult}_U(M)$, then $f =_U g$ if and only if there is a measure one set $A \in U$ such that for all $\alpha \in A$, $f(\alpha) = g(\alpha)$. We define $[f]\, E\, [G]$ if and only if there is a measure one set $A \in U$ such that for all $\alpha \in A$, $f(\alpha) \in g(\alpha)$. Prove the following facts:*

(1) *$=_U$ is an equivalence relation.*
(2) *The $E$ relation is well-defined.*

(3) *For every formula* $\phi$ *and functions* $f_1, \dots f_n$, $(\mathrm{Ult}_U(M), E) \vDash \phi([f_1], \dots [f_n])$ *if and only if the set* $\{\alpha < \kappa \mid M \vDash \phi(f_1(\alpha), \dots f_n(\alpha)\} \in U$.

(4) *Show that the function* $j : M \to \mathrm{Ult}_U(M)$ *given by* $j(x) = [c_x]$ *where* $c_x$ *is the constantly* $x$ *function is an elementary embedding.*

*Hint: Use induction on formulas for (3).*