FORCING EXERCISES
DAY 5

In addition to the problems below you should look over the problems from the week that you didn't get. If you can't figure out the old problems have someone explain them to you.

Problem 1. *Show that a filter $\mathcal{F}$ is an ultrafilter if and only if it is maximal.*

Problem 2. *Show that every filter can be extended to an ultrafilter.*

Problem 3. *Let $\mathcal{F}$ be a filter on ω and let $\langle a_n \mid n < \omega\rangle$ be a sequence of real numbers. Define $\lim^{\mathcal{F}} a_n = a$ if for every ϵ there is a set $A \in \mathcal{F}$ such that $|a - a_n| < \epsilon$ for all $n \in A$.*

1. *For which filter $\mathcal{F}$ does this notion of convergence coincide with the usual one?*
2. *Suppose that $\mathcal{U}$ is a nonprincipal ultrafilter on ω and that $\langle a_n \mid n < \omega\rangle$ is a bounded sequence of real numbers. Show that there is a unique $a \in \mathbb{R}$ such that $\lim^{\mathcal{U}} a_n = a$.*

For the remainder of the problems and definitions κ is a regular cardinal.

Definition 1. *A set $C \subseteq \kappa$ is* closed *if for all $\alpha < \kappa$, if $C \cap \alpha$ is unbounded in α, then $\alpha \in C$. A set $C \subseteq \kappa$ is* club *if it is closed and unbounded in κ.*

Problem 4. *Show that the collection of club subsets of κ form a filter on κ. We call this filter the club filter on κ.*

Definition 2. *Let λ be a regular cardinal. We say that a filter $\mathcal{F}$ is λ-complete if it is closed under intersections of size less than λ.*

Problem 5. *Show that the club filter on κ is κ-complete.*

Definition 3. *$\mathcal{F}$ be a filter on κ. We say that $\mathcal{F}$ is* normal *if for every sequence $\langle A_\alpha \mid \alpha < \kappa\rangle$ of elements of $\mathcal{F}$, the set $\triangle_{\alpha<\kappa} A_\alpha = \{\alpha < \kappa \mid \alpha \in \bigcap_{\beta<\alpha} A_\beta\}$ is in $\mathcal{F}$. This set is called the* diagonal intersection *of the sets A_α.*

Problem 6. *Show that the club filter on κ is normal*

Definition 4. *A set $S \subseteq \kappa$ is* stationary *if for every club C in κ, $S \cap C \neq \emptyset$.*

Problem 7. *Let S be a stationary subset of κ. Show that for every function $F : S \to \kappa$ such that for all $\alpha \in S$ $F(\alpha) < \alpha$, there is a stationary set $S' \subseteq S$ on which F is constant.*