A BRIEF ACCOUNT OF SOLOVAY'S MODEL

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We give a brief account of Solovay's model modulo well-known facts about the Levy Collapse and HOD (or more generally definability). There is also a brief appearance of the Boolean algebra view of forcing.

Theorem 1 (Solovay [1]). *From a model V of ZFC with an inaccessible, we can build a model of ZF, DC and all sets of reals are Lebesgue measurable and have the Baire and perfect set properties.*

First we describe the model. Let κ be inaccessible and let $\mathbb{P} = \mathrm{Coll}(\omega, <\kappa)$. Let G be $\mathbb{P}$-generic over V. The desired model is $\mathrm{HOD}(S)$ as computed in $V[G]$ where S is the class of countable sequences of ordinals. We could take other models here like $L(\mathbb{R})$ or $\mathrm{HOD}(\mathbb{R})$, but DC is more difficult to verify in these models. The proofs for the Lebesgue measurability, Baire property and perfect set property work with only minor revisions.

The key fact that we need about $\mathrm{HOD}(S)$ is that if $U \in \mathrm{HOD}(S)$, then there are a formula φ, a sequence of ordinals $\vec{\alpha}$ so that $x \in U$ if and only if $\varphi(x, \vec{\alpha})$ holds.

We start with the proofs of the regularity properties. The first move is to notice that the properties we want to establish are absolute, so for a set of reals $U \in \mathrm{HOD}(S)$ it is enough to show that U has the desired regularity property in $V[G]$.

We can treat Lebesgue measurability and the Baire property uniformly, since each can be described the collection of Borel sets modulo some σ-ideal.

Claim 1. *Let $M \subseteq N$ be transitive models of set theory such that $(2^{\aleph_0})^M$ is countable in M. In N the set of reals which are M-generic for the random algebra is conull.*

This is an easy argument, since there are just countably many null sets in M. The analogous claim is true where random algebra is replaced by Cohen algebra and conull is replaced by comeager.

Claim 2. *In $V[G]$ every set of reals in $\mathrm{HOD}(S)$ is Lebesgue measurable.*

Proof. Let φ and $\vec{\alpha}$ be the formula and parameter defining a set U of reals. By the κ-cc of the Levy collapse, there is a real t so that $\vec{\alpha} \in V[t]$. Let $V' = V[t]$ and x be a real of $V[G]$. By properties of the Levy collapse $V[G] = V'[x][H]$ where H is generic for $\mathrm{Coll}(\omega, <\kappa)$ over $V'[x]$. By the homogeneity of the Levy collapse, $\varphi(x, \vec{\alpha})$ holds in $V[G]$ if and only if it holds in $V'[x]$.

It is enough to show that there is a Borel set B (coded by a real in V') such that $x \in U$ if and only if $x \in B$ and x is generic for the random algebra over V'. In V', let B be the boolean value of $\varphi(\dot{x}, \vec{\alpha})$ where $\dot{x}$ is the canonical name for the generic real. It is straightforward to see that B works. $\square$

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The same proof using the Cohen algebra shows that every set of reals is Baire measurable. We turn our attention to the perfect set property.

Claim 3. *In* $V[G]$ *every set of reals in* $\mathrm{HOD}(S)$ *has the perfect set property.*

Proof. Let φ and $\vec{\alpha}$ be the formula and parameter defining a set of reals U. We assume that U is uncountable in $V[G]$ and show that it has a perfect subset.

Again we choose a real t so that $\vec{\alpha} \in V[t]$ and let $V' = V[t]$. Since the reals of V' are countable in $V[G]$, there is a real $x \in U$, which is not in V'. By the κ-cc of the Levy collapse there is an ordinal $\xi < \kappa$ such that x is in $V'[G_\xi]$ where G_ξ is a $\mathrm{Coll}(\omega, \xi)$-generic induced by G.

So in V' we have a $\mathrm{Coll}(\omega, \xi)$-name for a real $\dot{x}$ which is not a member of V' such that $\varphi(x, \vec{\alpha})$ holds. This uses the homogeneity of the Levy collapse and the fact that $V[G] = V'[G_\xi][H]$ for some suitable H. Working in $V[G]$ it is easy to build a perfect tree of generics for $\mathrm{Coll}(\omega, \xi)$ so that any two force some disagreement about $\dot{x}$. The 2^ω interpretations of $\dot{x}$ via these generics form a perfect subset of U. □

Finally we show that DC holds in $\mathrm{HOD}(S)$. To do so we need the following fact about $\mathrm{OD}(X)$ for some class X.

Claim 4. *There is an* $\mathrm{OD}(X)$ *surjection from* $\mathrm{On} \times X$ *to* $\mathrm{HOD}(X)$*.*

This is straight forward. Define $F(\alpha, x)$ to be y if and only if α codes a formula and a tuple ordinals which (together with x) define y.

Claim 5. *In* $V[G]$ *if* f *is a function on* ω *with values in* $\mathrm{HOD}(S)$*, then* $f \in \mathrm{HOD}(S)$*.*

Proof. Let F be as above. Let β_n and $\vec{\alpha}_n$ be such that $F(\beta_n, \vec{\alpha}_n) = f(n)$. Code the countably many ordinal present as a single sequence $\vec{\alpha}$. It is easy to see that f is definable from $\vec{\alpha}$. □

It is easy to see that DC holds in $\mathrm{HOD}(S)$ since the witness is a function f as above, which exists using choice in $V[G]$. We will now verify that DC holds in $\mathrm{HOD}(\mathbb{R})$, since we could not find the argument in print.

Claim 6. $\mathrm{HOD}(\mathbb{R})$ *satisfies* DC.

Proof. Let $R \in \mathrm{HOD}(\mathbb{R})$ be a relation on a set A such that for every $a \in A$ there is a $b \in A$ with $(a, b) \in R$. Working in the real world we define $\langle x_n \mid n < \omega \rangle$, $\langle \alpha_n \mid n < \omega \rangle$ and $f : \omega \to \mathrm{HOD}(\mathbb{R})$.

Choose $f(0) \in A$ and let α_0 be the least ordinal α such that there is an x with $F(\alpha, x) = f(0)$. Let x_0 be such that $F(\alpha_0, x_0) = f(0)$. Now inductively choose α_{n+1} to be the least α such that there is an x with $(f(n), F(\alpha, x)) \in R$ and let x_{n+1} be such an x. We define $f(n+1) = F(\alpha_{n+1}, x_{n+1})$.

We code $\langle x_n \mid n < \omega \rangle$ as a single real x and notice that f is definable from x and the parameters needed to define $f(0)$. Notice that the sequence of ordinals can be recovered in the inductive definition of f and so we don't need it as a parameter. □

References

1. Robert M Solovay, *A model of set-theory in which every set of reals is lebesgue measurable*, Annals of Mathematics (1970), 1–56.