A MODEL OF CUMMINGS AND FOREMAN REVISITED

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Abstract. This paper concerns the model of Cummings and Foreman where from ω supercompact cardinals they obtain the tree property at each $\aleph_n$ for $2 \leq n < \omega$. We prove some structural facts about this model. We show that the combinatorics at $\aleph_{\omega+1}$ in this model depend strongly on the properties of ω_1 in the ground model. From different ground models for the Cummings-Foreman iteration we can obtain either $\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$ and every stationary subset of $\aleph_{\omega+1}$ reflects or there are a bad scale at $\aleph_\omega$ and a non-reflecting stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\omega_1)$. We also prove that regardless of the ground model a strong generalization of the tree property holds at each $\aleph_n$ for $n \geq 2$.

1. Introduction

In this paper we prove some structural facts about the Cummings and Foreman [2] model for the tree property at each $\aleph_n$ for $n \geq 2$. We were led to the current results by asking about the combinatorics of weak square principles and stationary reflection at $\aleph_{\omega+1}$ in the extension by the Cummings-Foreman iteration. It turns out that these combinatorics depend in a strong way on the properties of ω_1 in the ground model for the Cummings-Foreman construction.

Before outlining the theorems of the paper, we recall some definitions and theorems which put our results in context. We begin with the notion of weak square introduced by Jensen.

Definition 1.1. *Let ν be a cardinal. A $\square^*_\nu$-sequence is a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \nu^+ \rangle$ such that*

1. *for all $\alpha < \nu^+$, $1 \leq |\mathcal{C}_\alpha| \leq \nu$,*
2. *for all $\alpha < \nu^+$ and for all $C \in \mathcal{C}_\alpha$, C is club in α and $\mathrm{otp}(C) \leq \nu$ and*
3. *for all $\alpha < \nu^+$, $C \in \mathcal{C}_\alpha$ and $\beta \in \lim(C)$, $C \cap \beta \in \mathcal{C}_\beta$.*

Definition 1.2. *$\square^*_\nu$ holds if and only if there is a $\square^*_\nu$-sequence.*

Jensen [6] proved that for a cardinal ν, $\square^*_\nu$ holds if and only if there is a special ν^+-Aronszajn tree. So in particular the failure of weak square can be seen as a weakening of the tree property at ν^+. Two related notions developed by Shelah (see [12, 14]) are *approachability* and *good scales*. For a survey of these topics we recommend Cummings paper [1] on singular cardinal combinatorics.

Date: 8/18/2013.
2010 *Mathematics Subject Classification.* 03E55,03E35.
Key words and phrases. tree property, weak squares, stationary reflection, large cardinals.
The results in this paper appear in the author's PHD thesis written under the supervision of James Cummings, to whom the author is greatly indebted.

Definition 1.3. *Let μ be a regular cardinal and $\langle x_\alpha \mid \alpha < \mu\rangle$ be a sequence of bounded subsets of μ. An ordinal $\gamma < \mu$ is* approachable with respect to $\vec{x}$ *if there is $A \subseteq \gamma$ cofinal with* $\mathrm{otp}(A) = \mathrm{cf}(\gamma)$ *such that for all $\beta < \gamma$, there is $\delta < \gamma$ such that $A \cap \beta = x_\delta$.*

Definition 1.4. *Let μ be a cardinal. A set $S \subseteq \mu$ is in the collection of approachable subsets $I[\mu]$ if and only if there are a club $C \subseteq \mu$ and a sequence $\vec{x}$ such that for all $\gamma \in S \cap C$, γ is approachable with respect to $\vec{x}$.*

The most basic fact about $I[\mu]$ is that it is an ideal. Moreover the statement $\mu \in I[\mu]$ is a kind of weak square principle. The notion of a scale comes from Shelah's PCF Theory and provides yet another weak square principle. The general setting is a singular cardinal ν with an increasing and cofinal sequence $\langle \nu_i \mid i < \mathrm{cf}(\nu)\rangle$ of regular cardinals less than ν. Given members $f, g \in \prod_i \nu_i$ we say that $f <^* g$ if and only if there is a $j < \mathrm{cf}(\nu)$ such that for all $i \geq j$, $f(i) < g(i)$. A sequence of functions $\langle f_\alpha \mid \alpha < \nu^+\rangle$ is a *scale of length ν^+ in $\prod_i \nu_i$* if it is increasing and cofinal in $\prod_i \nu_i$ under the ordering $<^*$.

A point $\gamma < \nu^+$ with $\mathrm{cf}(\gamma) > \mathrm{cf}(\nu)$ is *good* for a scale $\vec{f}$ of length ν^+ if there are $A \subseteq \gamma$ cofinal and $j < \mathrm{cf}(\nu)$ such that for all $i \geq j$ the sequence $\langle f_\alpha(i) \mid \alpha \in A\rangle$ is strictly increasing. A scale $\vec{f}$ is *good* if there is a club $C \subseteq \nu^+$ such that all γ in C of cofinality greater than $\mathrm{cf}(\nu)$ are good for $\vec{f}$. A scale $\vec{f}$ is *bad* if it is not good. We say there is a bad scale at a singular cardinal ν if there is a bad scale of length ν^+ in some product $\prod \nu_i$.

The theorem which relates the above principles is due to Shelah.

Theorem 1.5. *Suppose ν is a singular cardinal. $\square^*_\nu$ implies $\nu^+ \in I[\nu^+]$ implies that all scales of length ν^+ are good.*

So the existence of a bad scale implies that $\nu^+ \notin I[\nu^+]$ which in turn implies the failure of $\square^*_\nu$.

Recall that a stationary subset S of ν^+ reflects at point of cofinality μ if there is an $\alpha < \nu^+$ with $\mathrm{cf}(\alpha) = \mu$ such that $S \cap \alpha$ is stationary. For more on the interaction between squares, scales and stationary reflection we refer the interested reader to a paper of Cummings, Foreman and Magidor [3].

With the above definitions in mind we can now outline the results in this paper. In the first two theorems we show that the combinatorics of $\aleph_{\omega+1}$ can change dramatically based on the choice of ground model for the Cummings-Foreman iteration. In Theorem 5.1 we show that if the ω_1 of the ground model is formerly supercompact, then in the extension we have $\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$ and for every n, every stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\aleph_n)$ reflects at a point of cofinality $\aleph_{n+1}$. In Theorem 6.1 we show that if the ω_1 of the ground model is formerly a carefully chosen successor of a singular cardinal, then in the extension we have a bad scale of length $\aleph_{\omega+1}$ in some product of the $\aleph_n$'s and a non-reflecting stationary subset of $\aleph_{\omega+1}$. This theorem should be compared to a recent theorem of Neeman [11] where he defines a forcing similar to the Cummings Foreman iteration and obtains the tree property at every regular cardinal on the interval $[\aleph_2, \aleph_{\omega+1}]$. Neeman's result also requires a careful choice of ω_1.

To show that every stationary subset of $\aleph_{\omega+1}$ reflects in Theorem 5.1, we lift an elementary embedding to the entire iteration. Having done so we prove in Theorem 7.5 that regardless of the choice of ω_1 in the ground model a strengthening of the tree property due to Weiss [16] holds at each $\aleph_n$ for $n \geq 2$. This last result was discovered independently by Laura Fontanella [5].

The paper is organized as follows

- In Section 2 we define the Cummings-Foreman iteration and state some of its properties. Throughout this section we reference lemmas from the original paper [2]. A careful reader should probably have a copy on hand.
- In Section 3 we show that it is possible to lift an elementary embedding witnessing a fixed large degree of supercompactness to the extension by the full iteration.
- In Section 4 we carefully analyze the forcing which adds the embedding with a view to the work in Sections 5, 6 and 7.
- In Section 5 we give the proof of Theorem 5.1.
- In Section 6 we give the proof of Theorem 6.1.
- In Section 7 we give the proof of Theorem 7.5.

One remark is due on the notation used below. If $\mathbb{P}$ is a poset, then we use $V[\mathbb{P}]$ to denote a typical generic extension by $\mathbb{P}$ or an extension by $\mathbb{P}$ where the generic object is implicit.

2. Preliminaries on the iteration

We begin by giving an abstract definition of a poset $\mathbb{R}$ with many parameters which will form the iterates in the Cummings-Foreman iteration. The definitions below are found in Section 3 and the beginning of Section 4 in [2] and our notation is the same.

Definition 2.1. *Let $V \subseteq W$ be models of set theory. Suppose that there are cardinals τ, κ such that $W \models \tau = \mathrm{cf}(\tau)$ and κ is inaccessible. Let $\mathbb{P} =_{def} \mathrm{Add}(\tau, \kappa)_V$ and assume that $W \models \mathbb{P}$ is τ^+-cc and $< \tau$-distributive. Let $F \in W$ be a function from $\kappa \to (V_\kappa)_W$. We define $\mathbb{R} = \mathbb{R}(\tau, \kappa, V, W, F)$ in W by recursion on its restrictions to $\beta \leq \kappa$ and by setting $\mathbb{R} = \mathbb{R} \restriction \kappa$.*

Let $\mathbb{R} \restriction 0$ be the trivial forcing. Assume that we have defined $\mathbb{R} \restriction \alpha$ for all $\alpha < \beta$. Let $(p, q, f) \in \mathbb{R} \restriction \beta$ if and only if all of the following hold,

1. *$p \in \mathbb{P} \restriction \beta =_{def} \mathrm{Add}(\tau, \beta)_V$,*
2. *q is a function with $\mathrm{dom}(q) \subseteq \beta$, $|q| \leq \tau$, and if $\alpha \in \mathrm{dom}(q)$, then α is a successor ordinal, $q(\alpha) \in W^{\mathbb{P}\restriction\alpha}$ and $\Vdash^W_{\mathbb{P}\restriction\alpha} q(\alpha) \in \mathrm{Add}(\tau^+, 1)_{W[\mathbb{P}\restriction\alpha]}$ and*
3. *f is a function with $\mathrm{dom}(f) \subseteq \beta$, $|f| \leq \tau$, and for all $\alpha \in \mathrm{dom}(f)$,*
   - (a) *α is a limit ordinal,*
   - (b) *$\Vdash^W_{\mathbb{R}\restriction\alpha} F(\alpha)$ is a canonically τ^+-directed closed forcing and*
   - (c) *$f(\alpha) \in W^{\mathbb{R}\restriction\alpha}$ and $\Vdash^W_{\mathbb{R}\restriction\alpha} f(\alpha) \in F(\alpha)$.*

Assuming that $(p_1, q_1, f_1), (p_2, q_2, f_2) \in \mathbb{R} \restriction \beta$, we define $(p_1, q_1, f_1) \leq (p_2, q_2, f_2)$ if and only if

1. *$p_1 \leq p_2$ in $\mathbb{P} \restriction \beta$,*
2. *$\mathrm{dom}(q_2) \subseteq \mathrm{dom}(q_1)$,*
3. *for all $\alpha \in \mathrm{dom}(q_2)$, $p_1 \restriction \alpha \Vdash^W_{\mathbb{P}\restriction\alpha} q_1(\alpha) \leq q_2(\alpha)$,*
4. *$\mathrm{dom}(f_2) \subseteq \mathrm{dom}(f_1)$ and*
5. *for all $\alpha \in \mathrm{dom}(f_2)$, $(p_1, q_1, f_1) \restriction \alpha \Vdash^W_{\mathbb{R}\restriction\alpha} f_1(\alpha) \leq f_2(\alpha)$.*

The definition of $\mathbb{R}$ in particular and the paper in general make use of a wide range of closure properties of posets, which we record for completeness.

Definition 2.2. *Let κ be a regular cardinal and $\mathbb{P}$ be a poset.*

- *$\mathbb{P}$ is $<\kappa$-distributive if forcing with $\mathbb{P}$ adds no sequences of length less than κ.*
- *$\mathbb{P}$ is κ-closed if every decreasing sequence of elements of $\mathbb{P}$ of length less than κ has a lower bound.*
- *$\mathbb{P}$ is κ-directed closed if every directed subset of $\mathbb{P}$ of size less than κ has a lower bound.*
- *$\mathbb{P}$ is canonically κ-directed closed if every directed subset of $\mathbb{P}$ of size less than κ has a greatest lower bound.*

We repeat a few facts about $\mathbb{R}$. In each case we reference the appropriate lemma from [2].

Lemma 2.3. *Let $\mathbb{R}$ be as in Definition 2.1.*

(1) *(Lemma 3.2) $|\mathbb{R}| = \kappa$ and $\mathbb{R}$ has the κ-Knaster property.*
(2) *(Lemma 3.3) There are projections from $\mathbb{R}$ to $\mathbb{P}$, $\mathbb{P} \upharpoonright \alpha * \mathrm{Add}(\tau^+, 1)_{W[\mathbb{P}\upharpoonright\alpha]}$, and $\mathbb{R} \upharpoonright \alpha * F(\alpha)$.*
(3) *(Lemma 3.6) If $\eta \leq \tau$ and $\mathbb{P}$ is η-closed in W, then $\mathbb{R}$ is η-closed in W.*
(4) *(Lemmas from Section 3.3) The poset $\mathbb{U}$ defined to be the set $\{(0, q, f) \mid (0, q, f) \in \mathbb{R}\}$ ordered as a suborder of $\mathbb{R}$ is canonically τ^+-directed closed, κ-cc forcing and there is a projection from $\mathbb{P} \times \mathbb{U}$ to $\mathbb{R}$.*
(5) *(Corollaries from Section 3.3) Let G be $\mathbb{R}$-generic.*
    (a) *If g is the $\mathbb{P}$-generic induced by G, then every τ-sequence of ordinals from $W[G]$ is in $W[g]$.*
    (b) *In $W[G]$, τ^+ is preserved and $2^\tau = \kappa = \tau^{++}$.*
    (c) *Every $<\tau$-sequence of ordinals from $W[G]$ is in W.*
    (d) *Every set of ordinals of size τ in $W[G]$ is covered by a set of size τ in W.*
(6) *(Corollary 3.17) If u is $\mathbb{U}$-generic, then in $W[u]$ $\kappa = \tau^{++}$.*

We will actually need a strengthening of clause (3) of the above lemma. The proof is an easy adaptation of Lemma 3.6 of [2].

Lemma 2.4. *If $\eta \leq \tau$ and $\mathbb{P}$ is canonically η-directed closed in W, then $\mathbb{R}$ is canonically η-directed closed in W.*

$W[\mathbb{P} \times \mathbb{U}]$ can be viewed as a generic extension of $W[\mathbb{R}]$. Let $\mathbb{S}$ be the quotient forcing defined in $W[\mathbb{R}]$. We restate Lemma 3.20 of [2].

Lemma 2.5. *In $W[\mathbb{R}]$, $\mathbb{S}$ is τ-closed, $<\tau^+$-distributive, and κ-cc.*

To complete the analysis of $\mathbb{R}$ we need to factor a tail of $\mathbb{R}$ as we did $\mathbb{R}$. Notice that there is a projection from $\mathbb{R}$ to $\mathbb{R} \upharpoonright \beta$ for all β, which is given by the restriction of each coordinate to β. Combining lemmas from Section 3.5 of [2], we have the following.

Lemma 2.6. *Working in $W[\mathbb{R} \upharpoonright \beta]$ there are forcings $\mathbb{R}^*, \mathbb{P}^*, \mathbb{U}^*$ such that forcing with $\mathbb{R}^*$ brings us to an extension by $\mathbb{R}$, $\mathbb{P}^*$ is τ^+-cc and $\mathbb{U}^*$ is τ^+-closed and $\mathbb{R}^*$ is the projection of $\mathbb{P}^* \times \mathbb{U}^*$.*

Remark 2.7. *The actual choice of $\mathbb{P}$ in each iterate of the Cummings-Foreman iteration ensures that $\mathbb{P}^*$ has a stronger form of chain condition.*

This completes the facts that we need about $\mathbb{R}$. We move on to the definition of the iteration $\mathbb{R}_\omega$. For this we fix our increasing sequence of supercompact cardinals $\langle \kappa_n \mid n < \omega \rangle$. Let $\langle F_n \mid n < \omega \rangle$ be a sequence of Laver functions [7] for the κ_ns.

Definition 2.8. *Let $\mathbb{R}_\omega$ be the inverse limit of the sequence $\langle \mathbb{R}_n \mid n < \omega \rangle$ where $\mathbb{R}_n =_{def} \mathbb{Q}_0 * \mathbb{Q}_1 * \cdots * \mathbb{Q}_{n-1}$. We define the iterates $\mathbb{Q}_n$.*

1. *Let $\mathbb{Q}_0 =_{def} \mathbb{R}(\aleph_0, \kappa_0, V, V, F_0)$.*
2. *In $V[\mathbb{Q}_0]$ define $F_1^* : \kappa_1 \to V_{\kappa_1}$ by $F_1^*(\alpha)$ is the interpretation of $F_1(\alpha)$ if it is a $\mathbb{Q}_0$-name and 0 otherwise. Let $\mathbb{Q}_1 =_{def} \mathbb{R}(\aleph_1, \kappa_1, V, V[\mathbb{Q}_0], F_1^*)$.*
3. *Assuming that we've defined $\mathbb{Q}_k$ for all $k < n$. We work in $V[\mathbb{R}_n]$ to define $\mathbb{Q}_n$. Let F_n^* be defined in a similar fashion to F_1^* where we interpret $F_n(\alpha)$ if it is an $\mathbb{R}_n$-name. Let $\mathbb{Q}_n =_{def} \mathbb{R}(\kappa_{n-2}, \kappa_n, V[\mathbb{R}_{n-1}], V[\mathbb{R}_n], F_n^*)$.*

Throughout the proof our notation for the forcings derived from the iteration as well generic objects for these forcings will be the same as [2]. In particular in the extension by $\mathbb{R}_n$ we have the n^{th} iterate $\mathbb{Q}_n$ which is a version of the main poset $\mathbb{R}$ defined above. We write $\mathbb{P}_n, \mathbb{U}_n$ and $\mathbb{S}_n$ for the $\mathbb{P}, \mathbb{U}$ and $\mathbb{S}$ posets associated to this version of $\mathbb{R}$.

We refer the reader to Lemmas 4.2 and 4.3 of [2] to see that the definition of $\mathbb{R}_\omega$ is valid and for a catalog of its properties. For completeness we recall the cardinal structure of the final model.

Theorem 2.9. *In the generic extension by $\mathbb{R}_\omega$, ω_1 is preserved and for all $n < \omega$, $2^{\aleph_n} = \aleph_{n+2} = \kappa_n$.*

Before proceeding with our analysis of elementary embeddings, we need some general notation and facts about term forcing. We give the definitions in terms of a two step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ where $\mathbb{P}$ and $\dot{\mathbb{Q}}$ have no relation to the iteration we have just defined.

Definition 2.10. *Suppose that $\mathbb{P} * \dot{\mathbb{Q}}$ is a two step iteration. The $\mathbb{P}$-term forcing for $\dot{\mathbb{Q}}$ is the set $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$ of canonical $\mathbb{P}$-terms for elements of $\dot{\mathbb{Q}}$ together with the relation $\dot{q}_1 \leq_{term} \dot{q}_2$ if and only if $\Vdash_{\mathbb{P}} \dot{q}_1 \leq_{\dot{\mathbb{Q}}} \dot{q}_2$.*

It is easy to see that the identity map from $\mathbb{P} \times \mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$ to $\mathbb{P} * \dot{\mathbb{Q}}$ is a projection. Thus we can define the quotient forcing in $V[\mathbb{P} * \dot{\mathbb{Q}}]$. Let $\mathcal{S}(\mathbb{P}, \dot{\mathbb{Q}})$ be the forcing with underlying set the $\mathbb{P} * \dot{\mathbb{Q}}$-generic object, ordered by the ordering on $\mathbb{P} \times \mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$. We call this forcing $\mathcal{S}$, since it resembles the forcing $\mathbb{S}$ defined above. It is worth noting that by the general theory of projections $\mathbb{P} \times \mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$ is isomorphic to a dense subset of $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathcal{S}}(\mathbb{P}, \dot{\mathbb{Q}})$.

We will prove that under general circumstances $\mathcal{S}$ is equivalent to closed forcing.

Proposition 2.11. *If $\mathbb{Q}$ is τ-closed, H is $\mathbb{Q}$-generic and $\langle q_\alpha \mid \alpha < \gamma \rangle$ for some $\gamma < \tau$ is a decreasing sequence of elements of H, then there is $q \in H$ with $q \leq q_\alpha$ for all $\alpha < \gamma$.*

Proof. Let $\langle \dot{q}_\alpha \mid \alpha < \gamma \rangle$ for some $\gamma < \tau$ be a name for a decreasing sequence of elements of $\dot{G}_{\mathbb{Q}}$. We show that the set of conditions forcing a lower bound for the sequence is dense. Let $q \in \mathbb{Q}$. Build a decreasing sequence of elements of $\langle q'_\alpha \mid \alpha < \gamma \rangle$ such that $q = q'_0$, $q'_{\alpha+1}$ decides the value of $\dot{q}_\alpha$ to be q_α and $q'_{\alpha+1} \leq q_\alpha$. Let $q' \leq q'_\alpha$ for all $\alpha < \gamma$. It follows that $q' \leq q_\alpha$ for all $\alpha < \gamma$, which finishes the proposition. $\square$

Lemma 2.12. *If $\mathbb{P}$ is $< \tau$-distributive and it is forced by $\mathbb{P}$ that $\dot{\mathbb{Q}}$ is τ-closed, then in $V[\mathbb{P} * \dot{\mathbb{Q}}]$, $\mathcal{S}(\mathbb{P}, \dot{\mathbb{Q}})$ is equivalent to a forcing which is τ-closed.*

Proof. Work in V. For simplicity we write $\mathcal{A}$ for $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$ and $\mathcal{S}$ for $\mathcal{S}(\mathbb{P}, \dot{\mathbb{Q}})$. Let $G * H$ be $\mathbb{P} * \dot{\mathbb{Q}}$-generic. Let $\mathcal{S}' =_{def} \{\dot{q} \mid \exists p \ (p, \dot{q}) \in G * H\}$ ordered as a suborder of $\mathcal{A}$.

Claim. *Forcing with $\mathcal{S}$ is equivalent to forcing with $\mathcal{S}'$.*

Proof of Claim. We will show that the map $(p, \dot{q}) \mapsto \dot{q}$ is a dense embedding from $\mathcal{S}$ to $\mathcal{S}'$. It is clear that the map is order preserving and has a dense range. It remains to show that for all $(p_1, \dot{q}_1)$ and $(p_2, \dot{q}_2)$, $(p_1, \dot{q}_1)$ and $(p_2, \dot{q}_2)$ are compatible in $\mathcal{S}$ if and only if $\dot{q}_1$ and $\dot{q}_2$ are compatible in $\mathcal{S}'$. This is obvious since p_1 and p_2 are taken from the filter G and the ordering of the $\mathbb{P}$-terms is the same in $\mathcal{S}$ and $\mathcal{S}'$. $\square$

Next we show that $\mathcal{S}'$ is τ-closed. Suppose that $\langle \dot{q}_\alpha \mid \alpha < \gamma \rangle$ for some $\gamma < \tau$ is a decreasing sequence in $\mathcal{S}'$. Working in $V[G]$ we can choose a condition $q \in H$ which decides the value of and is a lower bound for $\langle q_\alpha \mid \alpha < \gamma \rangle$. We choose a name $\dot{q}$ for q and a $p \in G$ which forces that $\dot{q} \leq \dot{q}_\alpha$ for all $\alpha < \gamma$. Since $\mathbb{P}$ is $< \tau$-distributive, $\langle \dot{q}_\alpha \mid \alpha < \gamma \rangle$ is in V. Moreover it is a decreasing sequence in $\mathcal{A}$. Let $\dot{q}'$ be a name such that $\Vdash_{\mathbb{P}} \dot{q}' \leq \dot{q}_\alpha$ for all $\alpha < \gamma$. Construct a name $\dot{q}^*$ such that $p \Vdash \dot{q}^* = \dot{q}$ and if $p' \perp p$, then $p' \Vdash \dot{q}^* = \dot{q}'$. It follows that $\dot{q}^* \in \mathcal{S}'$ and $\dot{q}^*$ is a lower bound for $\langle \dot{q}_\alpha \mid \alpha < \gamma \rangle$ in $\mathcal{S}'$. $\square$

We end this section with a proposition about term forcing whose proof is straightforward.

Proposition 2.13. *If it is forced by $\mathbb{P}$ that $\dot{\mathbb{Q}}$ is canonically κ-directed closed, then $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$ is canonically κ-directed closed.*

3. Generic embeddings

In this section we work to lift an elementary embedding with critical point κ_n for a given $n \in \omega$. Let F be the Laver function that we used at stage n of the construction. Let θ be large and regular and $j : V \to M$ witness that κ_n is θ-supercompact in V. Moreover we assume that $j(F)(\kappa_n)$ returns an $\mathbb{R}_n$-name for a $\mathbb{Q}_n$-name for the forcing

$$\mathbb{U}_{n+1} \times \mathbb{P}_{n+2} \times \mathbb{U}^T_{n+2} \times \mathbb{A}$$

where $\mathbb{U}_{n+1}, \mathbb{P}_{n+2}$, are as in [2],

$$\mathbb{U}^T_{n+2} = \mathcal{A}(\mathbb{Q}_{n+1}, \dot{\mathbb{U}}_{n+2}) \text{ and}$$
$$\mathbb{A} = \mathcal{A}(\mathbb{Q}_{n+1} * \mathbb{Q}_{n+2}, \mathbb{R}_\omega / \mathbb{R}_{n+3}).$$

This forcing is in the scope of the definition of the third coordinate of $\mathbb{R}$ in Definition 2.1, since each of its components is canonically κ_n-directed closed in $V[\mathbb{R}_n * \mathbb{Q}_n]$. This follows from Lemma 2.3 clause (4), Lemma 2.4, Proposition 2.13 and general facts about iterated forcing. We are now ready to lift the embedding j. For ease of notation we denote all extensions of j by j.

Let $G_0 * \dots G_{n-1}$ be generic for $\mathbb{R}_n$. As $\mathbb{R}_n$ has size less than κ_n we can lift to $j : V[G_0 * \dots G_{n-1}] \to M[G_0 * \dots * G_{n-1}]$. Denote these models V_{n-1} and M_{n-1}. Note that if $n = 0$, then there is no forcing to do at this stage and we set $V_{-1} =_{def} V$ and $M_{-1} =_{def} M$. Let G_n be $\mathbb{Q}_n$-generic over V_{n-1} and $g_{n+2} \times$

$u_{n+1} \times u^T_{n+2} \times G^T_\infty$ be $i_{G_n}(j(F)(\kappa_n))$-generic over $V_{n-1}[G_n]$. It follows that we can view $M_{n-1}[G_n][g_{n+2} \times u_{n+1} \times u^T_{n+2} \times G^T_\infty]$ as a generic extension of M_{n-1} by $j(\mathbb{Q}_n) \restriction \kappa_n + 1$. We force to prolong this extension to one by $j(\mathbb{Q}_n)$ and denote the resulting generic by H_n. Since $\mathbb{Q}_n$ is κ_n-cc and for all $r \in \mathbb{Q}_n$, $j(r) = r$, we can lift j again to $j : V_n \to M_{n-1}[H_n]$.

Note that by the agreement between V and M,

$$j(\mathbb{P}_{n+1}) = \mathrm{Add}(\aleph_{n+1}, j(\kappa_{n+1}))_{V_{n-1}}.$$

By Lemma 2.6 of [2], $j(\mathbb{P}_{n+1})$ is κ_n-Knaster in V_n. We force with it to obtain a generic object h_{n+1}, which by the κ_n-cc induces a generic object g_{n+1} for $\mathbb{P}_{n+1}$, so that $j``g_{n+1} \subseteq h_{n+1}$.

Next we note that $\mathbb{U}_{n+1}$ is κ_n-directed closed in V_n and hence $j(\mathbb{U}_{n+1})$ is $j(\kappa_n)$-directed closed in $M_{n-1}[H_n]$. In $M_{n-1}[H_n]$, $j``u_{n+1}$ is a directed set of size $\kappa_{n+1}$ and hence we find a condition $t \leq j``u_{n+1}$ and force below it to obtain a generic x_{n+1} with $j``u_{n+1} \subseteq x_{n+1}$.

As in [2] one can argue that if G_{n+1} is $\mathbb{Q}_{n+1}$-generic filter generated by $g_{n+1} \times u_{n+1}$ and H_{n+1} is the $j(\mathbb{Q}_{n+1})$-generic filter generated by $h_{n+1} \times x_{n+1}$, then $j``G_{n+1} \subseteq H_{n+1}$. This allows us to lift to $j : V_{n+1} \to M_{n-1}[H_n][H_{n+1}]$.

Again following [2] we can find a master condition for $j``g_{n+2}$ and force to obtain a generic object $h_{n+2}$ with $j``g_{n+2} \subseteq h_{n+2}$. If we only lifted to this extension then we would essentially have done the work need to lift the embedding in [2]. However we are interested in larger objects so need to lift to the entire iteration. This is where we will use the extra posets selected by $j(F)(\kappa_n)$.

In $M_{n-1}[H_n][H_{n+1}]$, we have u^T_{n+2} a generic for $\mathbb{U}^T_{n+2}$. u^T_{n+2} has size κ_{n+2}, which has been collapsed by the addition of H_n to have size $\aleph_{n+1}$ and $j(\mathbb{U}^T_{n+2})$ is $j(\kappa_{n+1})$-directed closed in $M_{n-1}[H_n]$. Thus we can find a master condition $v \leq j``u^T_{n+2}$. We force with $j(\mathbb{U}^T_{n+2})$ below $v$ to obtain a generic object $x^T_{n+2}$ with $j``u^T_{n+2} \subseteq x^T_{n+2}$. We now interpret our term generic objects. Let $u_{n+2} = \{i_{G_{n+1}}(\sigma) \mid \sigma \in u^T_{n+2}\}$ and $x_{n+2} = \{i_{H_{n+1}}(\sigma) \mid \sigma \in x^T_{n+2}\}$. It is straight forward to see that $j``u_{n+2} \subseteq x_{n+2}$. Using a similar argument to the one we used to lift $\mathbb{Q}_{n+1}$, we can generate $\mathbb{Q}_{n+2}$ and $j(\mathbb{Q}_{n+2})$ generic filters and lift to obtain $j : V_{n+2} \to M_{n-1}[H_n][H_{n+1}][H_{n+2}]$.

Lastly we need to lift to the tail of the iteration. By GCH in V, $j``G^T_\infty$ has size $(\sup \kappa_n)^+$. We assumed enough supercompactness to obtain $j(\kappa_n) > (\sup \kappa_n)^+$. It follows that $(\sup \kappa_n)^+$ is collapsed by the addition of H_n to have size $(\aleph_{n+1})_{M_{n-1}[H_n]}$. In $M_{n-1}[H_n]$, $j(\mathbb{A})$ is $j(\kappa_n) = (\aleph_{n+2})_{M_{n-1}[H_n]}$-directed closed. We force below a master condition to obtain a $j(\mathbb{A})$-generic object H^T_∞ with $j``G^T_\infty \subseteq H^T_\infty$. We interpret each term generic to obtain generics $G_\infty$ and $H_\infty$ for $\mathbb{R}_\omega/(G_0 * \cdots * G_{n+2})$ and $j(\mathbb{R}_\omega)/(G_0 * \cdots * G_{n-1} * H_n * H_{n+1} * H_{n+2})$. It is clear that $j``G_\infty \subseteq H_\infty$ and hence we can lift to $j : V_{n+2}[G_\infty] \to M_{n-1}[H_n][H_{n+1}][H_{n+2}][H_\infty]$.

4. The forcing which adds the embedding

In this section we analyze the forcing which adds the elementary embedding with critical point κ_n from the previous section.

Lemma 4.1. *If $j : V_{n+2}[G_\infty] \to M_{n-1}[H_n][H_{n+1}][H_{n+2}][H_\infty]$ is as in the previous section with critical point κ_n which witnesses that κ_n is λ-supercompact for λ large enough, then every λ-sequence from $M_{n-1}[H_n][H_{n+1}][H_{n+2}]$ $[H_\infty]$ is in $M_{n-1}[H_n][h_{n+1}]$.*

Proof. Working in $M_{n-1}[H_n][h_{n+1}]$ we see that $j(\mathbb{Q}_{n+1})/j(h_{n+1}) * j(\mathbb{R}_\omega)/j(\mathbb{R}_{n+2})$ is $< j(\kappa_n)$-distributive. Since $j(\kappa_n) > \lambda$ the result follows. $\square$

For the arguments below we need a careful analysis of the forcing which takes us from $M[G_\omega]$ up to $M_{n-1}[H_n][h_{n+1}]$. To do this analysis we need to isolate some instances of the $\mathbb{S}$ forcings. Let $\mathbb{S}_{n+1}$ be the $\mathbb{S}$ forcing for $\mathbb{Q}_{n+1}$ and $\mathbb{S}_{n+2}$ be the same but for $\mathbb{Q}_{n+2}$. Let $\mathbb{S}^{\mathbb{U}}_{n+2}$ be $\mathcal{S}(\mathbb{Q}_{n+1}, \dot{\mathbb{U}}_{n+2})$, which is defined in the model $V_{n+1}[u_{n+2}]$. Let $\mathbb{S}_\infty$ be $\mathcal{S}(\mathbb{Q}_{n+1} * \mathbb{Q}_{n+2}, \mathbb{R}_\omega/\mathbb{R}_{n+3})$, which is defined in the model $V[G_\omega]$.

Lemma 4.2. *The forcing poset to go from $M[G_\omega]$ to $M_{n-1}[H_n][h_{n+1}]$ is*

$$\mathbb{S}_\infty \times (\mathbb{S}_{n+2} * \mathbb{S}^{\mathbb{U}}_{n+2}) \times \mathbb{S}_{n+1} \times j(\mathbb{P}_{n+1})/j``\mathbb{P}_{n+1} \times j(\mathbb{Q}_n)/(j(\mathbb{Q}_n) \upharpoonright \kappa_n + 1)$$

Remark 4.3. *In the proof below we use $\mathbb{P} \simeq \mathbb{Q}$ to mean that one of the posets is isomorphic to a dense subset of the other.*

Proof of Lemma 4.2. To avoid confusion we argue in stages. First we force with all of the $\mathbb{S}$ forcings to obtain a generic $s_\infty \times (s_{n+2} * s^{\mathbb{U}}_{n+2}) \times s_{n+1}$ and show that we have come to a reasonable model. After giving the list of models we give a list justifying each step in our reduction.

$$\begin{aligned}
&M_{n+2}[G_{n+3,\omega}][s_\infty \times (s_{n+2} * s^{\mathbb{U}}_{n+2}) \times s_{n+1}] && (1)\\
&= M_{n+2}[G^T_\infty \times (s_{n+2} * s^{\mathbb{U}}_{n+2}) \times s_{n+1}] && (2)\\
&= M_{n+1}[G_{n+2}][(s_{n+2} * s^{\mathbb{U}}_{n+2}) \times s_{n+1} \times G^T_\infty] && (3)\\
&= M_{n+1}[g_{n+2} \times u_{n+2}][s^{\mathbb{U}}_{n+2} \times s_{n+1} \times G^T_\infty] && (4)\\
&= M_{n+1}[g_{n+2} \times u^T_{n+2} \times s_{n+1} \times G^T_\infty] && (5)\\
&= M_n[G_{n+1}][s_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty] && (6)\\
&= M_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty] && (7)\\
&= M_n[u_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty \times g_{n+1}] && (8)
\end{aligned}$$

In the following list, each numbered entry justifies the associated line in our reduction. The claims about the $\simeq$ relation follow from the discussion of term forcing preceding Proposition 2.11.

(1) This line is the starting model $M[G_\omega]$ extended by a generic for the product $\mathbb{S}_\infty \times (\mathbb{S}_{n+2} * \mathbb{S}^{\mathbb{U}}_{n+2}) \times \mathbb{S}_{n+1}$.
(2) In M_{n+2}, $((\mathbb{R}_\omega/\mathbb{R}_{n+3}) * \mathbb{S}_\infty) \simeq \mathbb{A}$.
(3) This line is an application of mutual genericity.
(4) In M_{n+1}, $\mathbb{Q}_{n+2} * \mathbb{S}_{n+2} \simeq \mathbb{P}_{n+2} \times \mathbb{U}_{n+2}$.
(5) In M_{n+1} and hence in $M_{n+1}[g_{n+1}]$, $\mathbb{U}_{n+2} \times \mathbb{S}^{\mathbb{U}}_{n+2} \simeq \mathbb{U}^T_{n+2}$.
(6) This line is an application of mutual genericity.
(7) In M_n, $\mathbb{Q}_{n+1} * \mathbb{S}_{n+1} \simeq \mathbb{P}_{n+1} \times \mathbb{U}_{n+1}$.
(8) This line is an application of mutual genericity

Next we force with $\mathbb{P}^*_{n+1} = j(\mathbb{P}_{n+1})/j``\mathbb{P}_{n+1} = \mathrm{Add}(\kappa_{n-1}, j(\kappa_{n+1}) \setminus j``\kappa_{n+1})_{M_{n-1}}$ and apply the fact that $j(\mathbb{P}_{n+1}) \simeq \mathbb{P}_{n+1} \times \mathbb{P}^*_{n+1}$ to come to the model

$$M_n[u_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty \times h_{n+1}].$$

Next we force with $\mathbb{Q}_n^* = j(\mathbb{Q}_n)/(j(\mathbb{Q}_n) \restriction \kappa_n + 1)$ to obtain a generic G_n^* and provide another sequence of reductions. Note the slight departure from the notation of [2]. Our $\mathbb{Q}_n^*$ is called $\mathbb{R}_n^*$ in [2].

$$M_n[u_{n+1} \times g_{n+2} \times u_{n+2}^T \times G_\infty^T \times h_{n+1}][G_n^*] \tag{9}$$
$$= M_{n-1}[G_n][u_{n+1} \times g_{n+2} \times u_{n+2}^T \times G_\infty^T][G_n^* \times h_{n+1}] \tag{10}$$
$$= M_{n-1}[H_n][h_{n+1}] \tag{11}$$

Again the numbered entries below correspond the associated line above.

(9) The model after forcing with $\mathbb{Q}_n^*$.
(10) We use the fact that $M_n = M_{n-1}[G_n]$ and an application of mutual genericity.
(11) Here by the choice of j we can view $G_n * (u_{n+1} \times g_{n+2} \times u_{n+2}^T \times G_\infty^T)$ as generic for $j(\mathbb{Q}_n) \restriction \kappa_n + 1$ and apply the fact that $j(\mathbb{Q}_n) \simeq \mathbb{Q}_n \restriction \kappa_n + 1 * j(\mathbb{Q}_n)/(j(\mathbb{Q}_n) \restriction \kappa_n + 1)$.

This finishes the proof of the lemma. □

We now give a step by step analysis of the properties of the forcing which adds the embedding.

Lemma 4.4. *In $M[G_\omega]$, $\mathbb{S}_{n+1} \times (\mathbb{S}_{n+2} * \mathbb{S}_{n+2}^{\mathbb{U}}) \times \mathbb{S}_\infty$ is equivalent to forcing which is $\aleph_{n+1}$-closed and adds no $\aleph_{n+1}$-sequences.*

Before proving the lemma we need a few propositions about the $\mathbb{S}$-forcings. The following two propositions are easy applications of Lemma 2.12.

Proposition 4.5. *$\mathbb{S}_{n+2}^{\mathbb{U}}$ is equivalent to a forcing which is $\aleph_{n+1}$-closed in $V_{n+1}[u_{n+2} \times g_{n+2}]$.*

Proposition 4.6. *$\mathbb{S}_\infty$ is equivalent to a forcing which is $\aleph_{n+1}$-closed in $V[G_\omega]$.*

We are now ready to prove Lemma 4.4.

Proof. By the agreement between V and M all of the $\mathbb{S}$ forcings are the same whether computed in V or M. We claim that their product is equivalent to one which is $\aleph_{n+1}$-closed in $M[G_\omega]$. It is enough to establish the closure for each part of the product. By Lemma 2.5 $\mathbb{S}_{n+1}$ is $\aleph_{n+1}$-closed in M_{n+1} and the tail of the iteration adds no $< \aleph_{n+1}$-sequences, hence $\mathbb{S}_{n+1}$ is still $\aleph_{n+1}$-closed in $M[G_\omega]$. Lemmas 2.5 and 4.5 show that the two step iteration $\mathbb{S}_{n+2} * \mathbb{S}_{n+2}^{\mathbb{U}}$ is equivalent to $\aleph_{n+1}$-closed forcing in M_{n+2}. A similar argument as for $\mathbb{S}_{n+1}$ shows that $\mathbb{S}_{n+2} * \mathbb{S}_{n+2}^{\mathbb{U}}$ is still $\aleph_{n+1}$-closed in $M[G_\omega]$. Lastly we have that $\mathbb{S}_\infty$ is equivalent to $\aleph_{n+1}$-closed forcing in $M[G_\omega]$ by Lemma 4.6. It remains to show that the forcing adds no $\aleph_{n+1}$-sequences. By the proof of Lemma 4.2, we have come to the model

$$M_n[u_{n+1} \times g_{n+2} \times u_{n+2}^T \times G_\infty^T \times g_{n+1}].$$

Over M_n, $u_{n+1} \times g_{n+2} \times u_{n+2}^T \times G_\infty^T$ is generic for $\kappa_n = \aleph_{n+2}$-directed closed forcing and g_{n+1} is generic for $\aleph_{n+2}$-cc forcing. So all $\aleph_{n+1}$-sequences from the model in question are actually in $M_n[g_{n+1}] \subseteq M[G_\omega]$. This finishes the proof. □

Lemma 4.7. *In the model $M_n[u_{n+1} \times g_{n+2} \times u_{n+2}^T \times G_\infty^T \times g_{n+1}]$ the forcing $\mathbb{P}_{n+1}^*$ is $\aleph_n$-closed, $< \aleph_{n+1}$-distributive and its square is $\kappa_n = \aleph_{n+2}$-cc.*

Proof. Recall that $\mathbb{P}^*_{n+1} = \mathit{Add}(\aleph_{n+1}, j(\kappa_n) \setminus j``\kappa_n)_{M_{n-1}}$ and $\mathbb{Q}_n$ is the projection of $\mathbb{P}_n \times \mathbb{U}_n$. $\mathbb{P}_n$ is $< \aleph_n$-distributive and $\aleph_{n+1}$-cc and $\mathbb{U}_n$ is $\aleph_{n+1}$-closed in M_{n-1}. Closure is now immediate from the fact that $u_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty \times g_{n+1}$ adds no $\aleph_{n-1}$-sequences over M_n.

It is not hard to see that $\mathbb{P}_{n+1} \times (\mathbb{P}^*_{n+1})^2$ is κ_n-cc in M_n, hence by Easton's lemma it is still κ_n-cc in

$$M_n[u_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty].$$

It follows that $(\mathbb{P}^*_{n+1})^2$ is κ_n-cc in the desired model. It remains to show that forcing with $\mathbb{P}^*_{n+1}$ does not add any $\aleph_n$-sequences of ordinals over the model in question. By the proof of Lemma 4.2, this forcing brings us to the model

$$M_n[u_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty \times h_{n+1}].$$

By Easton's lemma every $< \aleph_{n+2}$-sequence from this model is in $M_n[h_{n+1}]$. $j(\mathbb{P}_{n+1})$ is $< \aleph_{n+1}$-distributive in M_n, since $\mathbb{P}_{n+1}$ is $< \aleph_{n+1}$-distributive in V_n by Lemma 2.3 clause (5c). This completes the proof. □

Lemma 4.8. *In the model $M_n[u_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty \times h_{n+1}]$ the forcing $\mathbb{Q}^*_n$ is the projection of $\mathbb{P}^*_n \times \mathbb{U}^*_n$ where $(\mathbb{P}^*_n)^2$ is $\aleph_{n+1}$-cc and $\mathbb{U}^*_n$ is $\aleph_{n+1}$-closed.*

Proof. This follows from Lemma 2.6 and the choice of $\mathbb{P}_n = \mathrm{Add}(\aleph_n, \kappa_n)_{V_{n-2}}$[1]. □

Lemma 4.9. *For $n \geq 1$, $\mathbb{P}^*_n$ is $\aleph_{n-1}$-closed in $M_n[u_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty \times h_{n+1}]$.*

Proof. $\mathbb{P}^*_n$ is $\mathrm{Add}(\aleph_n, j(\kappa_n) \setminus (\kappa_n + 1))_{M_{n-1}}$ and every $< \aleph_{n-1}$-sequence from the desired model is in M_{n-1}. So the result follows. □

It follows from the above sequence of lemmas that

Corollary 4.10. *In $V[G_\omega]$ for all $n \geq 1$ and all sufficiently large λ, there are a supercompactness embedding $j : V \to M$ witnessing λ-supercompactness of κ_n in V and an $\aleph_{n-1}$-closed forcing in $V[G_\omega]$ so that in the extension by the closed forcing j lifts to an elementary embedding with domain $V[G_\omega]$.*

To add such an embedding we force with the large product of posets from Lemma 4.2 and then with the posets to add a tail end of the generics for $j(\mathbb{R}_\omega)$ (in particular H_{n+2} and H_∞). It is straightforward to see that the forcing to add the generic for a tail end of $j(\mathbb{R}_\omega)$ is $\aleph_{n-1}$-closed.

5. Stationary reflection and approachability

As mentioned in the introduction we need a formerly supercompact ω_1 in the ground model to obtain stationary reflection and approachability in the extension.

Theorem 5.1. *Suppose that W is a model of* GCH *with a sequence of supercompact cardinals $\langle \kappa_n \mid n \geq -1 \rangle$ and that C is $\mathrm{Coll}(\omega, < \kappa_{-1})$-generic over W. If we set $V = W[C]$, then in $V[\mathbb{R}_\omega]$*

(1) *$\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$ and*
(2) *for every $n < \omega$, every stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\aleph_n)$ reflects at a point of cofinality $\aleph_{n+1}$.*

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[1]This choice of notation is fine if we set $V_{-2} = V_{-1} = V$.

Using this notation for the κ_n's we can apply the work from the previous sections without risk of confusion. We work towards showing that approachability holds. To do so we use an equivalent formulation of approachability in terms of expansions of the structure $\langle H_\theta, \in, <_\theta\rangle$. Here $<_\theta$ is a fixed well ordering of H_θ and θ is a large regular cardinal.

Definition 5.2. *Let μ be a regular cardinal and $\mathcal{A}$ be an expansion of the structure $\langle H_\theta, \in, <_\theta\rangle$ by countably many constants, functions and relations. We say that $\gamma < \mu$ is* approachable with respect to $\mathcal{A}$ *if and only if there is an unbounded $A \subseteq \gamma$ with $\mathrm{otp}(A) = \mathrm{cf}(\gamma)$ and for all $\beta < \gamma$, $A \cap \beta \in \mathrm{Hull}^{\mathcal{A}}(\gamma)$.*

It is not hard to show that the notion of approachability defined in the introduction and the notion obtained from the previous definition are the same.

Our proof follows work of Magidor [8]. The key difficulty is to give the same proof with a more complex iteration. To proceed with the proof we choose a nice collection of filtrations of ordinals less than ν^+ where $\nu = \sup_{n<\omega} \kappa_n$. For $\beta < \nu^+$, let $\langle x^n_\beta \mid n < \omega\rangle$ be a sequence of subsets of β such that

(1) for all n, $|x^n_\beta| = \kappa_n$,
(2) for all m, n, if $n < m$, then $x^n_\beta \subseteq x^m_\beta$,
(3) $\bigcup_{n<\omega} x^n_\beta = \beta$ and
(4) if $\alpha \in x^n_\beta$, then $x^n_\alpha \subseteq x^n_\beta$

To choose such sequences, we choose any sequences with the first three properties and then an easy recursive construction improves our sequences to satisfy the fourth condition. For $\alpha < \beta < \nu^+$, we define $d(\alpha, \beta) = n$ if and only if n is least such that $\alpha \in x^n_\beta$. We have the following lemma, which is analogous to Lemma 7 of [8].

Lemma 5.3. *In $V[\mathbb{R}_\omega]$, for all $\lambda < \nu^+$ with $\mathrm{cf}\,\lambda > \omega$ there is $A \subseteq \lambda$ unbounded such that $\mathrm{otp}(A) = \mathrm{cf}\,\lambda$ and $d \restriction [A]^2$ is constant.*

Proof. In V let $\mathrm{cf}(\lambda) = \rho$. There is an n such that $\kappa_n \leq \rho < \kappa_{n+1}$. We claim that in $V[\mathbb{R}_\omega]$, $\mathrm{cf}(\lambda) = \kappa_n$. It is enough to ask about the cofinality of ρ in the extension. If $n = -1$, then ρ is collapsed over $V[\mathrm{Coll}(\omega, <\kappa_{-1})]$ by $\mathbb{Q}_0$. $\mathbb{Q}_0$ projects on to $\mathrm{Add}(\omega, \rho+1) * \mathrm{Add}(\omega_1, 1)$. Since ρ is regular, it follows that ρ is collapsed to have cofinality κ_{-1} by $\mathbb{Q}_0$. A similar argument works for $n > -1$.

Using the strong compactness of κ_n in V, there is a κ_n-complete ultrafilter U on λ so that for all $\alpha < \lambda$, $[\alpha, \lambda) \in U$. For all $\alpha < \lambda$, there are a $B_\alpha \in U$ and $i_\alpha < \omega$ such that for all $\beta \in B_\alpha$, $d(\alpha, \beta) = i_\alpha$.

Claim. *For every sequence of fewer than κ_n U-measure one sets in $V[\mathbb{R}_\omega]$, there is a measure one set contained in their intersection.*

Note that for every n we can write $\mathbb{R}_\omega$ as a two step iteration where the first iterate is κ_n-cc and the second is $<\kappa_n$ distributive. In particular we can write $\mathbb{R}_\omega \simeq (\mathbb{Q}_{-1} * \cdots * \mathbb{Q}_n * \mathbb{P}_{n+1}) * (\mathbb{Q}_{n+1}/\mathbb{P}_{n+1} * \mathbb{Q}_{n+2} * \dots)$. Any $<\kappa_n$-sequence is in the extension by the first iterate. Since the first iterate is κ_n-cc, we can cover a set of fewer than κ_n-many measure one sets with a set from V of size less than κ_n. The intersection of this set in V is as required.

We return to the construction on the set A. In $V[\mathbb{R}_\omega]$, let $\langle \alpha_\beta \mid \beta < \kappa_n\rangle$ be increasing and cofinal in λ. We construct a sequence $\langle \gamma_\beta \mid \beta < \kappa_n\rangle$ as follows. Let $\gamma_0 = \alpha_0$. Suppose that we have constructed γ_δ for $\delta < \beta$. We choose γ_β to be the least member of $\bigcap_{\delta<\beta} B_{\gamma_\delta}$ greater than α_β. Such a γ_β exists using the above claim.

Now we take $A \subseteq \{\gamma_\beta \mid \beta < \kappa_n\}$ of size κ_n such that the map $i \mapsto i_{\gamma_\beta}$ is constant on A with value i. We claim that $d \restriction [A]^2$ is constant with value i. Let $\beta < \beta' < \kappa_n$ so that $\gamma_\beta, \gamma_{\beta'} \in A$. Since $\gamma_{\beta'} \in B_{\gamma_\beta}$, $d(\gamma_\beta, \gamma_{\beta'}) = i_{\gamma_\beta}$. Now we have $\gamma_\beta \in A$, so $i_{\gamma_\beta} = i$. □

We can now prove that every stationary subset of $\aleph_{\omega+1}$ is approachable in $V[\mathbb{R}_\omega]$.

Lemma 5.4. *In $V[\mathbb{R}_\omega]$, $\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$*

Proof. Fix a large regular θ and consider the structure

$$\mathcal{A} = \langle H_\theta, \in, <_\theta, d, \langle \vec{x}_\beta \mid \beta < \aleph_{\omega+1}\rangle\rangle.$$

There is a club of $\lambda < \aleph_{\omega+1}$ such that $\mathrm{Hull}^{\mathcal{A}}(\lambda)$ is of the form

$$N = \langle X, \in, <_\theta \restriction [X]^2, d \restriction [\lambda]^2, \langle \vec{x}_\beta \mid \beta < \lambda\rangle\rangle.$$

where $\lambda = X \cap \aleph_{\omega+1}$ and $\mathcal{P}(\kappa_i) \subseteq X$ for all $i \geq -1$.

Claim. *λ as above is approachable with respect to $\mathcal{A}$ as witnessed by N.*

We may assume that $\mathrm{cf}(\lambda) > \omega$, since points of cofinality ω are always approachable. By the previous lemma there is $A \subseteq \lambda$ unbounded such that $\mathrm{otp}(A) = \mathrm{cf}(\lambda)$ and A is homogeneous for d with color i. We claim that $A \cap \alpha \in N$ for all $\alpha < \lambda$. We fix $\beta \in A \cap X \setminus \alpha$. We have

$$A \cap \alpha \subseteq \{\delta < \alpha \mid d(\delta, \beta) = i\} \subseteq \{\delta < \alpha \mid \delta \in x^i_\beta\}.$$

The first is since $\beta \in A$ and the second uses the definition of d. Since $i, \beta \in X$ we have $\kappa_i \subseteq X$ and $x^i_\beta \in X$. Let $f \in X$ be a bijection from κ_i to x^i_β. $f^{-1}(A \cap \alpha)$ is a subset of κ_i, which is in X since $\mathcal{P}(\kappa_i) \subseteq X$. It follows that $A \cap \alpha \in X$ as required. □

We move on to the proof of stationary reflection. Our argument will go in two stages using the following fact, which was observed in a paper of Cummings and Shelah [4].

Lemma 5.5. *Suppose that $\mu < \lambda < \lambda' < \theta$ are all regular cardinals. If every stationary subset of $\theta \cap \mathrm{cof}(\mu)$ reflects at a point of cofinality λ' and every stationary subset of $\lambda' \cap \mathrm{cof}(\mu)$ reflects at a point of cofinality λ, then every stationary subset of $\theta \cap \mathrm{cof}(\mu)$ reflects at a point of cofinality λ.*

Proof. Suppose that $S \subseteq \theta \cap \mathrm{cof}(\mu)$ is stationary. By assumption there is an ordinal η with $\mathrm{cf}(\eta) = \lambda'$ such that $S \cap \eta$ is stationary. Let f be a continuous increasing and cofinal map from λ' to η. Let $T = f^{-1}(S \cap \eta)$. We claim that T is stationary in λ'. Suppose $C \subseteq \lambda'$ is club. Then $f``C$ is club in $\eta$ and so $S \cap f``C \neq \emptyset$. It follows that $T \cap C \neq \emptyset$. By assumption there is an ordinal $\gamma < \lambda'$ with $\mathrm{cf}(\gamma) = \lambda$ such that $T \cap \gamma$ is stationary. We claim that $S \cap f(\gamma)$ is stationary. Suppose that $D \subseteq f(\gamma)$ is club. We claim that $D' = \{\alpha \mid f(\alpha) \in D\}$ is club in γ. D' is closed since f is continuous and D is closed. To see that D' is unbounded fix an $\alpha < \gamma$ and construct a sequence $\langle \alpha_i \mid i < \omega\rangle$ such that $\alpha = \alpha_0$ and for all $i \geq 1$, $f(\alpha_i)$ is greater than the least member of $D \setminus f(\alpha_{i-1})$. This is possible since f is cofinal. Now $f(\sup \alpha_i) = \sup f(\alpha_i) \in D$ since D is closed. Hence D' is unbounded. So $T \cap D' \neq \emptyset$ implies that $S \cap f(\gamma) \cap D \neq \emptyset$ as required, since $\mathrm{cf}(f(\gamma)) = \lambda$. □

To apply the above lemma we will prove

Lemma 5.6. *In $V[G_\omega]$ for every $n \geq 2$, every stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\aleph_{n-2})$ reflects at a point of cofinality $\aleph_{n+1}$.*

Lemma 5.7. *In $V[G_\omega]$ for every n and every $k \leq n$, every stationary subset of $\aleph_{n+2} \cap \mathrm{cof}(\aleph_k)$ reflects at a point of cofinality $\aleph_{n+1}$.*

Part (2) of Theorem 5.1 follows from the above lemmas and two applications of Lemma 5.5. The proof of Lemma 5.6 is a straight forward adaptation of work of Magidor [8] using a lemma due to Shelah which also appears in [8].

Lemma 5.8. *Let κ, μ be cardinals with μ regular. Suppose that $S \subseteq \mu \cap \mathrm{cof}(\kappa)$ is stationary and $S \in I[\mu]$, then S is still stationary after any κ^+-closed forcing.*

We only sketch the proof of Lemma 5.6 and leave the details to the reader.

(1) Fix a stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\aleph_{n-2})$ in $V[G_\omega]$.
(2) Apply Corollary 4.10 to see that there is an elementary embedding j witnessing a large degree of supercompactness for κ_n such that we can extend j to an elementary embedding with domain $V[G_\omega]$ by forcing with $\aleph_{n-1}$-closed forcing. We call the extended elementary embedding j as well.
(3) Note that the stationarity of S is preserved by the forcing to extend j by Lemma 5.8.
(4) If $\rho = \sup j``\aleph_{\omega+1}$, then by standard arguments using the stationarity of S, $j(S) \cap \rho$ is stationary.
(5) The lemma follows from elementarity and the fact that ρ is an ordinal of cofinality $\aleph_{n+1}$ in the codomain of j.

We turn our attention to the proof of Lemma 5.7. We will need a fact about approachability at the successor of a regular cardinal.

Fact 5.9 (Shelah [13])**.** *If μ is regular, then $\mu^+ \cap \mathrm{cof}(< \mu) \in I[\mu^+]$.*

Proof of Lemma 5.7. Suppose that $S \in V[G_\omega]$ is a stationary subset of $\aleph_{n+2} \cap \mathrm{cof}(\aleph_k)$ for some $k \leq n$. Let j be the generic elementary embedding from Section 3 with critical point $\kappa_n = \aleph_{n+2}$. Recall that

$$j : V[G_\omega] \to M_{n-1}[H_n][H_{n+1}][H_{n+2}][H_\infty]$$

for some carefully chosen generic objects. By Lemma 4.1, $S \in M_{n-1}[H_n][h_{n+1}]$. It is enough to show that the stationarity of S is preserved by the forcing which takes us from $M[G_\omega]$ to $M_{n-1}[H_n][h_{n+1}]$, since $j(S) \cap \kappa_n = S$ and κ_n is collapsed to have cofinality $\aleph_{n+1}$ in $M_{n-1}[H_n]$. Note that by Fact 5.9 $S \in I[\aleph_{n+2}]$ in $M[G_\omega]$. By Lemma 4.4 and Lemma 5.8, S is still stationary after forcing with the $\mathbb{S}$ forcings and $\aleph_{n+2}$ is preserved. By Lemma 4.7 forcing with $\mathbb{P}^*_{n+1}$ is $\aleph_{n+2}$-cc and hence S is still stationary in this extension. Moreover S is still in $I[\aleph_{n+2}]$ in this extension by another application of Fact 5.9. By Lemma 4.8, forcing with $\mathbb{U}^*_n$ is $\aleph_{n+1}$-closed and by Lemma 5.8 S is still stationary after forcing with $\mathbb{U}^*_n$. In the current extension $\aleph_{n+2}$ has been collapsed to have cofinality $\aleph_{n+1}$ and $\mathbb{P}^*_n$ is $\aleph_{n+1}$-cc. It follows that S is still stationary in $M_n[H_n][h_{n+1}]$ as required. □

6. A bad scale and non-reflecting stationary set

In this section we start with a model V_0 of GCH with infinitely many supercompact cardinals κ_n for $n < \omega$. Then we let V be the extension of V_0 by the Laver preparation [7] for κ_0. Note that each κ_n is still supercompact in V. Recall that we defined $\sup_{n<\omega} \kappa_n = \nu$. For each $\mu < \kappa_0$ we define $\mathbb{R}^\mu_\omega$ to be the Cummings Foreman iteration in the extension by $\mathrm{Coll}(\omega, \mu)$. We can now state the theorem for this section.

Theorem 6.1. *There are a $\mu < \kappa_0$ and a generic object $d * G_\omega$ for $\mathrm{Coll}(\omega, \mu) * \mathbb{R}^\mu_\omega$ such that in $V[d * G_\omega]$ there are a bad scale at $\aleph_\omega$ and a non-reflecting stationary subset of $\aleph_{\omega+1}$.*

To begin we fix a scale $\vec{f}$ of length ν^+ in some product of the κ_n's in V_0. By work of Shelah (see [1] Theorem 18.1) the set S of bad points for $\vec{f}$ whose cofinality is less than κ_0 is stationary in ν^+. The Laver preparation is countably closed, so $\vec{f}$ is still a scale in V. In order to find the cardinal μ and the generic object $d * G_\omega$ as in the theorem we find an outer model W of $V[d * G_\omega]$ in which $\vec{f}$ is still a bad scale. This is enough, since if $C \in V[d * G_\omega]$ was a club of good points, then it would still be a club of good points in W. It follows that $\vec{f}$ is a bad scale in $V[d * G_\omega]$. We also show that the set of bad points for $\vec{f}$ does not reflect. Its non-reflection is witnessed by the fact that every point of cofinality greater than ω_1 in $V[d * G_\omega]$ is good for $\vec{f}$.

We begin by identifying a certain term forcing which is κ_0-directed closed in V. Define $\mathbb{Y}$ to be the disjoint sum of the posets $\mathrm{Coll}(\omega, \mu)$ where $\mu < \kappa$. That is we take the disjoint union of the underlying sets with the natural ordering and add a trivial condition which is above the top condition in each poset in the sum. So the poset $\mathbb{Y}$ chooses a poset from among $\mathrm{Coll}(\omega, \mu)$ for $\mu < \kappa_0$ and then forces with it. Let $\dot{\mu}$ be the canonical $\mathbb{Y}$-name for the ordinal chosen to be collapsed. Let $\mathbb{R}^{\dot{\mu}}_\omega$ be a $\mathbb{Y}$-name for the Cummings Foreman iteration. In the extension by $\mathbb{Y}$, $\mathbb{R}^{\dot{\mu}}_\omega$ is equivalent to $\mathbb{R}^{\dot{\mu}}_3 * \mathbb{R}^{\dot{\mu}}_\omega / \mathbb{R}^{\dot{\mu}}_3$. In the extension by $\mathbb{Y} * \mathbb{R}^{\dot{\mu}}_3$, $\mathbb{R}^{\dot{\mu}}_\omega / \mathbb{R}^{\dot{\mu}}_3$ is $\kappa_0 = \aleph_2$-directed closed. In V we define $\mathbb{X} = \mathcal{A}(\mathbb{Y} * \mathbb{R}^{\dot{\mu}}_3, \mathbb{R}^{\dot{\mu}}_\omega / \mathbb{R}^{\dot{\mu}}_3)$. It follows that $\mathbb{X}$ is κ_0-directed closed in V.

We show that forcing with $\mathbb{X}$ preserves ν and ν^+. For ease of notation we drop the superscript $\dot{\mu}$ in the arguments below. By work from Section 4, over $V[\mathbb{Y} * \mathbb{R}_3]$ the extension by $\mathbb{R}_\omega / \mathbb{R}_3$ is contained in an extension by

$$\mathbb{R}_{n+1}/\mathbb{R}_3 \times \mathcal{A}(\mathbb{R}_{n+1}/\mathbb{R}_3, \mathbb{P}_{n+1}) \times \mathcal{A}(\mathbb{R}_{n+1}/\mathbb{R}_3, j(F)(\kappa_n)).$$

By general considerations about term forcing this is the product of κ_n-cc forcing and κ_n-closed forcing. It follows by some further work with term forcing that the extension by $\mathbb{X}$ is contained in an extension by a product of κ_n-cc forcing and κ_n-closed forcing. So $\mathbb{X}$ preserves κ_n for each $n \geq 3$ and thus also preserves ν and ν^+.

We choose our outer model W to be $V[\mathbb{X} \times \mathbb{Y} * \mathbb{R}_3]$ for a suitable choice of generics. First we show that $\vec{f}$ is a scale in W regardless of the choice of generics.

Lemma 6.2. *Let $\mu < \kappa_0$. $\vec{f}$ is still a scale in W*

Proof. The size of $\mathbb{Y} * \mathbb{R}_3$ is κ_2. So each κ_n for $n \geq 3$ is preserved in W, since it is forced by $\mathbb{X}$ that $\mathbb{Y} * \mathbb{R}_3$ preserves cardinals greater than κ_2. Every ω-sequence from the extension is in $V[\mathbb{Y} * \mathbb{P}_0]$ and a standard argument using chain condition shows that $(\prod \kappa_n)^V$ is cofinal in $(\prod \kappa_n)^{V[\mathbb{Y} * \mathbb{P}_0]}$. The lemma follows. $\square$

Next we show that $\vec{f}$ has a stationary set of bad points in W for the right choice of generic objects. First we show that there is a stationary set of bad points for $\vec{f}$ in $V[\mathbb{X}]$.

Lemma 6.3. *There is a singular cardinal μ of V_0 such that the set of bad points for $\vec{f}$ of cofinality μ^+ is stationary in $V[\mathbb{X}]$.*

Proof. By the proof of Laver indestructibility there is a generic elementary embedding $j : V[\mathbb{X}] \to M$ which lifts an embedding $j_0 : V_0 \to M_0$ witnessing that κ_0 is (at least) ν^+-supercompact in V_0. As noted above $\sup j_0“\nu^+ \in j_0(S)$, hence also $\sup j“\nu^+ \in j(S)$. By standard reflection arguments the set $S \cap \{\gamma < \nu^+ \mid \gamma$ is bad for $\vec{f}\} \cap \{\gamma < \nu^+ \mid \mathrm{cf}^{V_0}(\gamma) = \mathrm{cf}^{V[\mathbb{X}]}(\gamma)$ is a successor of a singular cardinal of $V_0\}$ is stationary in $V[\mathbb{X}]$. It follows that there are a singular cardinal μ of V_0 and a stationary set of bad points for $\vec{f}$ of cofinality μ^+ in $V[\mathbb{X}]$. $\square$

The cardinal μ given by the previous lemma will be the μ required for Theorem 6.1. Let S' be the stationary set given by the previous lemma. S' is still stationary in $V[\mathbb{X} \times \mathbb{Y} * \mathbb{R}_3^{\dot{\mu}}]$ since the latter forcing has size κ_2. We work below a condition in $\mathbb{Y}$ which is in $\mathrm{Coll}(\omega, \mu)$. Next we show that good points of cofinality ω_1 in W are good in $V[\mathbb{X}]$. To show this we need to show that both countably closed forcing and ω_1-Knaster forcing cannot make points of cofinality ω_1 good.

Proposition 6.4. *Let $\vec{g}$ be a scale and γ less than the length of $\vec{g}$ have cofinality ω_1. If $\mathbb{P}$ is countably closed and $\Vdash_{\mathbb{P}} \gamma$ is good, then γ is good in V.*

Proof. Assume the hypotheses and let $\langle \dot{\gamma}_\alpha \mid \alpha < \omega_1 \rangle$ be a name for an increasing enumeration of a set witnessing the goodness of γ with natural number $\dot{n}$. Build a decreasing sequence of conditions which decides the value of all $\dot{\gamma}_\alpha$ for all α and $\dot{n}$ to be γ_α and n. Although there is no lower bound for the sequence of conditions it still follows that $\{\gamma_\alpha \mid \alpha < \omega_1\}$ and n witness that γ is good in V. $\square$

Proposition 6.5. *Let $\vec{g}$ be a scale and γ less than the length of $\vec{g}$ have cofinality ω_1. If $\mathbb{P}$ is ω_1-Knaster and $\Vdash_{\mathbb{P}} \gamma$ is good, then γ is good in V.*

Proof. Suppose as before that $\dot{\gamma}_\alpha$ and $\dot{n}$ are names which witness the goodness of γ in $V[\mathbb{P}]$. Let $p \in \mathbb{P}$ decide the value of $\dot{n}$ to be n. Next choose p_α deciding the value of $\dot{\gamma}_\alpha$ to be γ_α. Let $I \subseteq \omega_1$ be an unbounded set on which the conditions p_α are pairwise compatible. Then the set $\{\gamma_\alpha \mid \alpha \in I\}$ and n witness that γ is good in V. $\square$

Lemma 6.6. *Suppose that γ is good point for $\vec{f}$ of cofinality ω_1 in $V[\mathbb{X} \times \mathbb{Y} * \mathbb{R}_3]$, then γ is good for $\vec{f}$ in $V[\mathbb{X}]$.*

Proof. Assume that $\mathrm{cf}(\gamma) = \omega_1$ and γ is good for $\vec{f}$ in W. Every ω_1-sequence in the extension is a member of $V[\mathbb{X} \times (\mathrm{Coll}(\omega, \mu) * (\mathbb{Q}_0 \times \mathbb{P}_1))]$. Moreover this extension is a submodel of $V[\mathbb{X} \times (\mathrm{Coll}(\omega, \mu) * (\mathbb{P}_0 \times \mathbb{U}_0 \times \mathbb{P}_1))]$ for appropriate choice of generic objects. Recall that $\mathbb{P}_1$ is chosen from the ground model of the Cummings-Foreman iteration. In $V[\mathbb{X} \times (\mathrm{Coll}(\omega, \mu) * (\mathbb{U}_0 \times \mathbb{P}_1))]$, $\mathbb{P}_0$ is ω_1-Knaster, since it is just Cohen forcing. It follows that γ is good in $V[\mathbb{X} \times (\mathrm{Coll}(\omega, \mu) * (\mathbb{U}_0 \times \mathbb{P}_1))]$. In $V[\mathbb{X} \times \mathrm{Coll}(\omega, \mu)]$, $\mathbb{U}_0 \times \mathbb{P}_1$ is countably closed and hence γ is good in this model. Lastly by standard arguments if A and n witness that γ is good in $V[\mathbb{X} \times \mathrm{Coll}(\omega, \mu)]$, then there is an unbounded $B \subseteq A$ with $B \in V[\mathbb{X}]$. It follows that γ is good in $V[\mathbb{X}]$ witnessed by B and n. $\square$

Let $d * G_\omega$ be the $\mathrm{Coll}(\omega, \mu) * \mathbb{R}_\omega^{\dot{\mu}}$-generic object generated by generics for $\mathbb{X}$ and $\mathbb{Y} * \mathbb{R}_3^{\dot{\mu}}$ with μ given by Lemma 6.3. By the argument above $\vec{f}$ is a bad scale in

$V[d * G_\omega]$. It remains to show that the stationary set of bad points in $V[d * G_\omega]$ does not reflect.

First we argue that every point of cofinality greater than ω_1 is good for $\vec{f}$ in $V[d*G_\omega]$. In V by Shelah's trichotomy theorem (see Theorem 19.1 of [1]) every point of cofinality greater than $(2^\omega)^V$ has an exact upper bound. Suppose that γ is a point of cofinality greater than $\aleph_1$ in $V[d * G_\omega]$. It follows that there is $n < \omega$ such that $\kappa_n \leq \mathrm{cf}(\gamma) < \kappa_{n+1}$ in V. Let f^* be an exact upper bound for $\vec{f} \upharpoonright \gamma$. By standard arguments and since $\kappa_n \leq \mathrm{cf}(\gamma) < \kappa_{n+1}$, $\{n \mid \mathrm{cf}(f^*(n)) < \kappa_n \vee \mathrm{cf}(f^*(n)) \geq \kappa_{n+1}\}$ is finite. So in the extension all but finitely many outputs of f^* are collapsed to have cofinality $\kappa_n = \aleph_{n+2}$. It follows that f^* is an exact upper bound of uniform cofinality for $\vec{f} \upharpoonright \gamma$ in $V[d * G_\omega]$. Hence γ is good in $V[d * G_\omega]$.

Note that if γ is good then there is a club of good points below γ which is the set of limit points of the unbounded set witnessing goodness. It follows that the set of bad points in $V[d * G_\omega]$ does not reflect. This finishes the proof of Theorem 6.1.

7. A generalization of the tree property

In this section we prove that the Cummings Foreman iteration always establishes the a strong generalization of the tree property at each $\aleph_n$ for $n \geq 2$ whose definition is due to Weiss [16].

Definition 7.1. *Let κ and λ be cardinals with κ regular. A sequence $D = \langle d_x \mid x \in \mathcal{P}_\kappa(\lambda)\rangle$ is a $\mathcal{P}_\kappa(\lambda)$-*list *if and only if for each $x \in \mathcal{P}_\kappa(\lambda)$, $d_x \subseteq x$.*

Definition 7.2. *A $\mathcal{P}_\kappa(\lambda)$-list has an* ineffable branch *$d \subseteq \lambda$ if and only if there is a stationary set $S \subseteq \mathcal{P}_\kappa(\lambda)$ such that for all $x \in S$, $d_x = d \cap x$.*

For a fixed regular κ, we can rewrite a theorem of Magidor [9] as follows. κ is supercompact if and only if for every $\lambda \geq \kappa$, every $\mathcal{P}_\kappa(\lambda)$-list has an ineffable branch. This statement is hiding an interesting property of $\mathcal{P}_\kappa(\lambda)$-lists namely *thinness*.

Definition 7.3. *A $\mathcal{P}_\kappa(\lambda)$-list D is* thin *if and only if there is a club $C \subseteq \mathcal{P}_\kappa(\lambda)$ such that for all $c \in C$, $|\{d_x \cap c \mid c \subseteq x \in \mathcal{P}_\kappa(\lambda)\}| < \kappa$.*

It is easy to see that κ inaccessible implies that every $\mathcal{P}_\kappa(\lambda)$-list is thin. When κ is not inaccessible, we have a definition that gives an interesting combinatorial property.

Definition 7.4. ITP(κ, λ) *holds if and only if every thin $\mathcal{P}_\kappa(\lambda)$-list has an ineffable branch.*

We can restate Magidor's theorem as "κ is supercompact if and only if κ is inaccessible and for all $\lambda \geq \kappa$, ITP(κ, λ) holds." Furthermore, some easy coding shows that ITP(κ, κ) implies the tree property at κ. We prove the following theorem

Theorem 7.5. ITP$(\aleph_n, \lambda)$ *holds in $V[\mathbb{R}_\omega]$ for all $n \geq 2$ and all $\lambda \geq \aleph_n$.*

The key point in the proof of ITP$(\aleph_n, \lambda)$ will be to show that certain forcing could not have added a branch through a $\mathcal{P}_\kappa(\lambda)$-list. The analogous point in the proof from [2] is the use of *branch lemmas*. Branch lemmas are statements of the form 'Forcing of type X cannot add a branch through a tree of type Y'. In [2] two branch lemmas are used. We are going to need the analog of these branch lemmas in the setting of $\mathcal{P}_\kappa(\lambda)$. For this we need the notions of approximation and thin approximation.

Definition 7.6. *Let κ be an uncountable regular cardinal.*

(1) *A poset $\mathbb{P}$ has* κ-approximation *in a model of set theory W if and only if for every ordinal μ and every $\mathbb{P}$-name $\dot{d}$ for a subset of μ, if for every $z \in (\mathcal{P}_\kappa(\mu))^W$, $\Vdash_{\mathbb{P}} \dot{d} \cap z \in W$, then $\Vdash_{\mathbb{P}} \dot{d} \in W$.*
(2) *A poset $\mathbb{P}$ has* thin κ-approximation *in a model of set theory W if and only if for every ordinal μ and every $\mathbb{P}$-name $\dot{d}$ for a subset of μ, if for every $z \in (\mathcal{P}_\kappa(\mu))^W$, $\Vdash_{\mathbb{P}} \dot{d} \cap z \in W$ and $|\{x \in W \mid \text{there is } p \in \mathbb{P}, p \Vdash_{\mathbb{P}} x = \dot{d} \cap z\}| < \kappa$, then $\Vdash \dot{d} \in W$.*

We call names which satisfy the hypotheses of either approximation or thin approximation, *κ-approximated* or *thinly κ-approximated*. We begin with a lemma implicit in Mitchell's PHD thesis [10] which first appeared in its current form in [15]. Before we state and prove the lemma, we need a general proposition about approximated sets.

Proposition 7.7. *Suppose that $\mathbb{P}$ is a poset and $\dot{d}$ is a $\mathbb{P}$-name for a subset of some cardinal μ. Assume that for all $z \in \mathcal{P}_\kappa(\mu)$, $\Vdash \dot{d} \cap z \in V$, but $\Vdash_{\mathbb{P}} \dot{d} \notin V$. Then for all $p \in \mathbb{P}$ and all $y \in \mathcal{P}_\kappa(\mu)$, there are $p_1, p_2 \leq p$ and $z \supseteq y$ such that p_1, p_2 decide the value of $\dot{d} \cap z$ and they decide different values.*

Proof. Suppose that the conclusion fails. Then we have p and y so that for any two extensions of p and any $z \supseteq y$ if these extensions decide the value of $\dot{d} \cap z$, then they give the same value. It follows that p forces $\dot{d} \in V$. □

Lemma 7.8. *Let κ be a regular cardinal. Suppose that $\mathbb{P}$ is a poset so that $\mathbb{P} \times \mathbb{P}$ is κ-cc, then $\mathbb{P}$ has the κ-approximation property.*

Proof. Suppose that the lemma is false. Then we have a poset $\mathbb{P}$ and a name $\dot{d}$ which is κ-approximated, but $\Vdash_{\mathbb{P}} \dot{d} \notin V$. We work by recursion to construct an antichain of size κ in $\mathbb{P} \times \mathbb{P}$. In particular, we construct $\langle (p^0_i, p^1_i) \mid i < \kappa \rangle$ and a function $f : \kappa \to \mathcal{P}_\kappa(\mu)$. Assume that for some $j < \kappa$ we have constructed (p^0_i, p^1_i) for $i < j$ and $f \restriction j$. Let $y = \bigcup f``j$ which is in $\mathcal{P}_\kappa(\mu)$. Choose $p_j \in \mathbb{P}$ which decides the value of $\dot{d} \cap y$ to be d_j. Apply the proposition to p_j and y to obtain conditions p^0_j, p^1_j and $f(j) \in \mathcal{P}_\kappa(\mu)$. We record the values that each condition decides as d^0_j, d^1_j. This completes the construction.

We claim that $\{(p^0_i, p^1_i) \mid i < \kappa\}$ is an antichain of size κ. Suppose that we had $i < j$ such that $(p^0_i, p^1_i), (p^0_j, p^1_j)$ are compatible. Then $d^k_j \cap f(i) = d^k_i$ for $k = 0, 1$. Note that $d^0_j \cap \bigcup f``j = d^1_j \cap \bigcup f``j = d_j$ and $d^k_i \subseteq \bigcup f``j$ for $k = 0, 1$. This implies that $d^0_i = d^1_i$ a contradiction. □

We also need a lemma about closed forcing and thin approximation. In this case the usual branch lemma due to Silver generalizes easily.

Lemma 7.9. *Suppose that χ is a cardinal with $\chi < \kappa$ and $2^\chi \geq \kappa$. If $\mathbb{P}$ is χ^+-closed, then $\mathbb{P}$ has the thin κ-approximation property.*

For a careful proof we refer the reader to Proposition 2.1.12 of [16]. We also prove a lemma about the preservation of the hypotheses of thin approximation.

Lemma 7.10. *Suppose that $\dot{d}$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a subset of some ordinal μ, which is thinly κ-approximated. If $\mathbb{P}$ has κ-cc, then in $V[\mathbb{P}]$, $\dot{d}$ is still thinly approximated.*

Proof. We note that by the κ-cc of $\mathbb{P}$, $\mathcal{P}_\kappa(\mu)_V$ is cofinal in $\mathcal{P}_\kappa(\mu)_{V[\mathbb{P}]}$. Working in $V[\mathbb{P}]$, if $x \in \mathcal{P}_\kappa(\mu)$ and $x \subseteq y \in \mathcal{P}_\kappa(\mu)_V$, then $\dot{d} \cap x$ is determined by $\dot{d} \cap y$ and x. So $\Vdash_{\mathbb{Q}} \dot{d} \cap x \in V[\mathbb{P}]$. Notice that every value for $\dot{d} \cap x$ can be extended to one for $\dot{d} \cap y$. This defines an injective map from $\{a \in V[\mathbb{P}] \mid \text{there is } q \in \mathbb{Q}, q \Vdash a = \dot{d} \cap x\}$ to $\{b \in V \mid \text{there is } q \in \mathbb{Q}, q \Vdash b = \dot{d} \cap y\}$. The latter set is a subset of a set of size less than κ from V, namely the set values for $\dot{d} \cap y$ forced by some condition in $\mathbb{P} * \dot{\mathbb{Q}}$. This finishes the proof. $\square$

We are now ready to give the proof of Theorem 7.5. Work in $V[G_\omega]$ where G_ω is generic for $\mathbb{R}_\omega$. Let $n < \omega$ and $\lambda \geq \aleph_{n+2}$. We fix a thin $\mathcal{P}_{\aleph_{n+2}}(\lambda)$-list D. It will be enough to show that D has an ineffable branch in $V[G_\omega]$. We break the proof into two phases. First we show that the generic embedding constructed in Section 3 with $\theta >> \lambda$ gives an ineffable branch d through D. Second we show that the forcing to add the generic embedding could not have added d and hence $d \in V[G_\omega]$.

Phase 1. We can choose the degree of supercompactness θ large enough so that M and V agree about the definition of the iteration $\mathbb{R}_\omega$ and stationarity in $\mathcal{P}_{\aleph_{n+2}}(\lambda)$ in the extension by $\mathbb{R}_\omega$. Using this choice of θ we prove

Lemma 7.11. *Let $n < \omega$, $\lambda \geq \aleph_{n+2}$ be a cardinal and D be a thin $\mathcal{P}_{\aleph_{n+2}}(\lambda)$-list in $V[G_\omega]$. There is an ineffable branch through D in $M_{n-1}[H_n * H_{n+1} * H_{n+2}][H_\infty]$.*

Proof. Let $D = \langle d_x \mid x \in \mathcal{P}_{\aleph_{n+2}}(\lambda)\rangle$ be a thin $\mathcal{P}_{\aleph_{n+2}}(\lambda)$-list in $V[\mathbb{R}_\omega]$. Let $j : V[G_\omega] \to M_{n-1}[H_n * H_{n+1} * H_{n+2}][H_\infty]$ be the generic embedding from Section 3 with θ as above. We write $j(D) = \langle e_x \mid x \in \mathcal{P}_{j(\aleph_{n+2})}(j(\lambda))\rangle$. We claim that $d =_{def} j^{-1} e_{j``\lambda}$ is an ineffable branch through D. To see this we need to show that the set $S =_{def} \{x \in \mathcal{P}_{\aleph_{n+2}}(\lambda) \mid d_x = d \cap x\}$ meets every club in $\mathcal{P}_{\aleph_{n+2}}(\lambda)$ from $V[G_\omega]$. Using the elementary embedding it is enough to show that $j``\lambda \in j(S)$. From the definition of $d$, we have $e_{j``\lambda} = j(d) \cap j``\lambda$. $\square$

This completes Phase 1 of the proof. We move on to the more complex Phase 2.

Phase 2. We note that by the agreement between V and M, $D \in M[G_\omega]$. It suffices to show that the forcing to get from $M[G_\omega]$ to $M_{n-1}[H_n * H_{n+1} * H_{n+2}][H_\infty]$ could not have added the branch, so $d \in M[G_\omega]$. To begin we locate the branch d more precisely. Note that d is a λ-sequence and hence by Lemma 4.1, $d \in M_n[H_n][h_{n+1}]$. So we only need to show that the forcing to get from $M[G_\omega]$ to $M_n[H_n][h_{n+1}]$ could not have added a branch. For ease of reference we call this forcing $\mathbb{T}$. $\mathbb{T}$ is the forcing that we analyzed carefully in Section 4.

There are two components to showing that $\mathbb{T}$ could not have added d. First we need to show that each intermediate extension has an appropriate approximation property. Second we need to show that the hypotheses of approximation hold about a name for d in each intermediate extension. We break this phase of the proof in to a sequence of lemmas.

Let $\dot{d}, \dot{S}$ be $\mathbb{T}$-names for the branch and its associated stationary set. We may assume that it is forced that $\dot{d}$ is an ineffable branch as witnessed by $\dot{S}$. In the arguments below we refer to partial interpretations of $\dot{d}$ in outer models of $M[G_\omega]$ as $\dot{d}$.

Lemma 7.12. *In $M[G_\omega]$, $\dot{d}$ is thinly $\aleph_{n+2}$-approximated.*

Proof. We show that for every $x \in \mathcal{P}_{\aleph_{n+2}}(\lambda)_{M[G_\omega]}$, $\dot{d} \cap x \in M[G_\omega]$ and that there are not too many possibilities for $\dot{d} \cap x$. We claim that $\Vdash_{\mathbb{T}} \dot{d} \cap x \in M[G_\omega]$ for all $x \in \mathcal{P}_{\aleph_{n+2}}(\lambda)_{M[G_\omega]}$. Since $\Vdash_{\mathbb{T}} S$ is stationary in $(\mathcal{P}_{\aleph_{n+2}}(\lambda))_{M[G_\omega]}$, it is forced to be cofinal. So given $x \in \mathcal{P}_{\aleph_{n+2}}(\lambda)$, we can find a condition and a $y \supseteq x$ so that the condition forces $d_y = \dot{d} \cap y$. This condition forces that $\dot{d} \cap x = d_y \cap x \in M[G_\omega]$ and so we have the claim. For each $x \in \mathcal{P}_{\aleph_{n+2}}(\lambda)_{M[G_\omega]}$, the fact that the set of possible values for $\dot{d} \cap x$ has size less than $\aleph_{n+2}$ follows directly from the fact that D is thin. $\square$

Lemma 7.13. *In $M_n[u_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty \times g_{n+1}]$, $\dot{d}$ is thinly $\aleph_{n+2}$-approximated.*

Proof. By Lemma 4.4, we have that the definition of $\mathcal{P}_{\aleph_{n+2}(\lambda)}$ is unchanged from $M[G_\omega]$. The lemma follows. $\square$

Lemma 7.14. *In $M_n[u_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty \times h_{n+1}]$, $\dot{d}$ is thinly $\aleph_{n+2}$-approximated.*

Proof. The lemma follows directly from Lemma 7.10 and Lemma 4.7. $\square$

Lemma 7.15. *In $M_n[u_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty \times h_{n+1}][\mathbb{U}^*_n]$, $\dot{d}$ is $\aleph_{n+1}$-approximated.*

Proof. The lemma follows directly from Lemma 4.8 and the previous lemma. $\square$

Theorem 7.5 follows from an easy application of the above sequence of lemmas, the work from Section 4 and Lemmas 7.8 and 7.9. Working backwards through the extensions we have

- $d \in M_n[u_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty \times h_{n+1}][\mathbb{U}^*_n]$, since $(\mathbb{P}^*_n)^2$ is $\aleph_{n+1}$-cc and $\dot{d}$ is $\aleph_{n+1}$-approximated in this model.
- $d \in M_n[u_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty \times h_{n+1}]$, since $\mathbb{U}^*_n$ is $\aleph_{n+1}$-closed $2^{\aleph_n} = \aleph_{n+2}$ and $\dot{d}$ is thinly $\aleph_{n+2}$-approximated in this model.
- $d \in M_n[u_{n+1} \times g_{n+2} \times u^T_{n+2} \times G^T_\infty \times g_{n+1}]$, since $(\mathbb{P}^*_{n+1})^2$ is $\aleph_{n+2}$-cc and $\dot{d}$ is $\aleph_{n+2}$-approximated in this model.
- $d \in M[G_\omega]$, since the product of $\mathbb{S}$ forcings is equivalent to $\aleph_{n+1}$-closed forcing, $2^{\aleph_n} = \aleph_{n+2}$ and $\dot{d}$ is thinly $\aleph_{n+2}$-approximated in this model.

This finishes the proof of Theorem 7.5.

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