EXPERIMENTS WITH CERESA CLASSES OF CYCLIC FERMAT QUOTIENTS

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Abstract. We give two new examples of non-hyperelliptic curves whose Ceresa cycles have torsion images in the intermediate Jacobian. For one of them, the central value of the L-function of the relevant motive is non-vanishing, consistent with the conjectures of Beilinson and Bloch. We speculate on a possible explanation for the existence of these torsion Ceresa classes, based on some computations with cyclic Fermat quotients.

1. Introduction

Let C be a smooth projective curve over a field k with a point $e \in C(k)$. Let J be the Jacobian of C, and let $\iota\colon C \hookrightarrow J$ be the Abel–Jacobi map sending e to 0. The *Ceresa cycle* $\kappa_e(C)$ is the class of $\iota(C) - [-1]^*\iota(C)$ in the Chow group $\mathrm{CH}_1(J)$ of one-cycles modulo rational equivalence on J. Let $\kappa(C)$ be its image in the Griffiths group $\mathrm{Gr}_1(J)$ of null-homologous one-cycles modulo algebraic equivalence, which does not depend on the choice of e. When C is hyperelliptic, we have $\kappa(C) = 0$. On the other hand, Ceresa showed that $\kappa(C)$ has infinite order for the general curve of genus $g \geq 3$ [9]. It was speculated that this may be the case for all non-hyperelliptic curves, but recently two counterexamples were discovered.

Bisogno, Li, Litt, and Srinivasan showed that for the genus 7 Fricke–Macbeath curve, the image of $\kappa_e(C)$ under the étale Abel–Jacobi map is torsion [4]. Gross then showed that the appropriate L-function has non-vanishing central value, giving more evidence that $\kappa(C)$ is torsion in the Griffiths group [14], as we will explain. Zhang and Qiu have recently shown that the corresponding Gross–Kudla–Schoen cycle vanishes in the Chow group [20]. In particular, this implies that $\kappa(C)$ is torsion [13, Rem. 3.4] for the Fricke–Macbeath curve.

The second example is due to Beauville in [2], who shows that for the genus 3 plane curve $D^9\colon y^3 = x^4 + x$, the complex Abel–Jacobi image of $\kappa_e(C)$ is torsion, for appropriate choice of e. Assuming conjectures of Beilinson and Bloch, this implies that $\kappa_e(C)$ and $\kappa(C)$ are torsion. Beauville and Schoen then gave an unconditional proof that $\kappa(C)$ is torsion in [3].

1.1. **Results.** Beauville's proof in [2] is elegant and simple, making use of an automorphism of order 9 on the curve. The aim of this note is to find more non-hyperelliptic examples of Ceresa cycles which are Abel–Jacobi trivial via Beauville's method. The curves we consider are the cyclic quotients of the Fermat curves $F^m : X^m + Y^m + Z^m = 0$. Beauville's curve D^9 is a quotient of F^9 (see Proposition 3.1), so this is a natural family to consider. Ceresa cycles of cyclic Fermat quotients have been studied quite a bit [12, 15, 19, 22], but mostly when m is prime or when the Jacobian has many simple factors, which perhaps explains why these examples have been missed.

We have turned Beauville's method into an algorithm and implemented it in SageMath [24]. We ran the algorithm on all cyclic Fermat quotients of F^m up to $m = 50$. On the positive side, we found two new examples of Ceresa cycles whose Abel–Jacobi images are torsion: the trigonal curves

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with affine equations

$$D^{12}\colon y^3 = x^4 + 1 \qquad \text{and} \qquad D^{15}\colon y^3 = x^5 + 1,$$

of genus 3 and 4, respectively. On the negative side, these were the only examples we found, and enlarging m seems unlikely to discover more by Beauville's method (see Section 3.3).

Our experiments uncovered a phenomenon that points to a possible explanation of the three torsion Ceresa examples D^9, D^{12}, and D^{15}. In all three cases, there is an algebraic correspondence between D^m and a hyperelliptic quotient of F^m. We use work of Aoki to show that the minimal cyclic Fermat quotients with this property and with $m \not\equiv 2 \pmod 4$ are our curves D^9, D^{12}, D^{15}, and a genus 9 curve which is a quotient of F^{21}. This suggests the possibility that the Ceresa cycle is torsion because some multiple of it arises via correspondence from a hyperelliptic Ceresa class.

1.2. **L-functions.** The kernel of the Abel–Jacobi map (restricted to cycles defined over $\bar{\mathbb{Q}}$) is conjectured to be torsion. This would imply that the Ceresa cycle of each of the curves D^9, D^{12}, and D^{15} (with respect to a certain base point) is torsion modulo rational equivalence, and hence modulo algebraic equivalence as well. For D^9, the latter statement is the main result of [3].

The existence of non-torsion algebraic cycles in the Chow group is the main theme of the conjecture of Beilinson and Bloch [5]. Given a smooth projective variety X over a number field K, the conjecture predicts that the rank of the group $\mathrm{CH}^r(X)_0$ of null-homologous cycles of codimension r is equal to the order of vanishing of the L-function of the motive $H^{2r-1}(X)$ in degree $2r-1$ at $s = r$. Similarly, refinements of this conjecture predict that in certain situations, L-functions of submotives of $H^{2r-1}(X)$ detect properties of cycles modulo algebraic equivalence (see Section 4.1).

Both curves D^9 and D^{12} have genus 3. Let J^9 and J^{12} denote their Jacobians. By the theory of complex multiplication, there are submotives M^9 and M^{12} of type $(3,0)+(0,3)$ inside $H^3(J^9)$ and $H^3(J^{12})$ respectively. They are defined over the totally real fields $\mathbb{Q}(\zeta_9)^+$ and $\mathbb{Q}(\zeta_{12})^+$ respectively, and they are predicted by Bloch to account for the algebraic equivalence classes of the Ceresa cycles of D^9 and D^{12}. Using Magma [7], we find that

$$\mathrm{ord}_{s=2}\, L(M^{12}/\mathbb{Q}(\zeta_{12})^+, s) = 0.$$

On the other hand, the argument of [3] applies to the curve D^{12} as well, showing that $\kappa(D^{12})$ is torsion. Putting these two facts together, we obtain new evidence for the Beilinson–Bloch conjectures. If we assume the Beilinson–Bloch conjectures, then this gives an example of a non-hyperelliptic Jacobian over $\mathbb{Q}$ whose Griffiths group $\mathrm{Gr}_1(J)$ has rank 0; see Section 4.3.

For Beauville's original example D^9, we found that

$$\mathrm{ord}_{s=2}\, L(M^9/\mathbb{Q}(\zeta_9)^+, s) = 1$$

despite the fact that $\kappa(D^9)$ is torsion. For D^{15} we were not able to compute the special value of the corresponding L-function in Magma.

1.3. **Outline.** Section 2 provides background on Beauville's method and cyclic Fermat quotients. In Section 3, we apply Beauville's method to a genus 4 curve as a way of illustrating the algorithmic aspect of the method, and then present the results of our implementation in SageMath. In Section 3.3, we compare and contrast the results of this note with recent work of Eskandari and Murty. After some background on the Beilinson–Bloch conjecture and Jacobi sums, we compute the orders of vanishing of the relevant L-functions in Section 4. Finally, in Section 5 we speculate on some connections with hyperelliptic cyclic Fermat quotients.

2. Preliminaries

Let C be a smooth, projective, geometrically integral curve over a field k of genus $g \geq 3$, as in the introduction. We sometimes consider affine and possibly singular models of C, but the Ceresa cycles will always be for the smooth projective model. Let $\kappa_e = \kappa_e(C)$ and $\kappa = \kappa(C)$ be the Ceresa cycles and classes defined in the introduction. Let J be the Jacobian of C.

2.1. **Beauville's method.** Assume now that $k = \mathbb{C}$. In [2] Beauville considers the image of κ_e under the complex Abel–Jacobi map

$$\mathrm{AJ}_C : \mathrm{CH}^{g-1}(J)_0 \longrightarrow J^{g-1}(J) := \frac{\mathrm{Fil}^2 H^3_{\mathrm{dR}}(J)^\vee}{\mathrm{Im}\, H_3(J(\mathbb{C}), \mathbb{Z})}. \tag{1}$$

This map is defined by the integration formula

$$\mathrm{AJ}_C(Z)(\alpha) := \int_{\partial^{-1}(Z)} \alpha, \qquad \text{for all } \alpha \in \mathrm{Fil}^2 H^3_{\mathrm{dR}}(J),$$

where $\partial^{-1}(Z)$ denotes any continuous 3-chain in $J(\mathbb{C})$ whose boundary is Z.

Beauville's idea to prove that $\mathrm{AJ}_C(\kappa_e)$ is torsion in certain instances is simple and elegant. Suppose that C has an automorphism σ that fixes a point $e \in C(k)$. The push-forward σ_* then fixes the Ceresa cycle κ_e. By functoriality of Abel–Jacobi maps, $\mathrm{AJ}_C(\kappa_e)$ is then fixed by $(\sigma^*)^\vee$ in $J^{g-1}(J)$. If the complex torus $J^{g-1}(J)$ has only finitely many fixed points under $(\sigma^*)^\vee$, or equivalently, if 1 is not an eigenvalue for the action of σ on the tangent space $T^{g-1}(C)$ of $J^{g-1}(J)$ at 0, then $\mathrm{AJ}_C(\kappa_e)$ must be torsion. The tangent space $T^{g-1}(C)$ is given by

$$H^{g-2,g-1}(J) \oplus H^{g-3,g}(J) = \left(\bigwedge^{g-2} V \otimes \bigwedge^{g-1} V^*\right) \oplus \left(\bigwedge^{g-3} V \otimes \bigwedge^{g} V^*\right), \tag{2}$$

where $V := H^{1,0}(J) = H^0(C, \Omega^1_C)$ and V^* denotes its $\mathbb{C}$-dual.

Beauville carries out this idea for the curve $y^3 = x^4 + x$ using the order 9 automorphism sending $(x, y) \mapsto (\zeta_9^3 x, \zeta_9 y)$. Here ζ_9 denotes a fixed choice of primitive 9-th root of unity.

2.2. **Cyclic Fermat quotients.** We collect some results on cyclic Fermat quotients, from [10] and the references therein. Recall that m denotes a positive integer. For any two integers a and b satisfying $0 < a, b < m$ and $\gcd(m, a, b, a+b) = 1$, let $C^m_{a,b}$ be the smooth projective curve birational to

$$F^m_{a,b} : y^m = (-1)^{a+b} x^a (1-x)^b. \tag{3}$$

The map $F^m \longrightarrow F^m_{a,b}$ sending $(x\colon y\colon 1) \mapsto (-x^m, x^a y^b)$ induces a map $f^m_{a,b}\colon F^m \longrightarrow C^m_{a,b}$. The genus of $C^m_{a,b}$ is

$$m - (\gcd(m, a) + \gcd(m, b) + \gcd(m, a+b)) + 2. \tag{4}$$

If two other integers a' and b' satisfy the relation

$$\{a, b, m-a-b\} \equiv \{ta', tb', t(m-a'-b')\} \pmod{m}, \quad \text{for some } t \in (\mathbb{Z}/m\mathbb{Z})^\times, \tag{5}$$

henceforth written $(a, b) \sim_m (a', b')$, then there is an isomorphism $C^m_{a,b} \simeq C^m_{a',b'}$. We will consequently restrict our attention to equivalence classes $[a, b]$ of pairs with respect to $\sim_m$.

Proposition 2.1. *The cyclic Fermat quotient $C^m_{a,b}$ is hyperelliptic if and only if*

$$(a, b) \sim_m (1, 1) \qquad \textit{or} \qquad m = 2n \textit{ and } (a, b) \sim_m (1, n).$$

Proof. This is [10, §IV Prop. 8 & Cor. 8.1]. □

Denote by μ_m the group of m-th roots of unity and let ζ_m be a fixed choice of primitive m-th root of unity. Note that μ_m acts on $C = C^m_{a,b}$ by scaling the y-coordinate in (3). One can see the action of μ_m on the space $V = H^0(C, \Omega^1_C)$ by pulling back the differential forms in V to the Fermat curve F^m via the map $f = f^m_{a,b}$. A basis for f^*V is

$$L^m_{a,b} := \{w_{r,s} \,:\, 0 < r, s, r+s < m \;\text{ and }\; br \equiv as \pmod{m}\}, \tag{6}$$

where $w_{r,s} := w^m_{r,s} := x^{r-m} y^{s-1} dy$ with functions $x = X/Z$ and $y = Y/Z$ on F^m.

For integers $1 \leq i, j \leq m$, the automorphisms $\sigma_{i,j} : (x, y) \mapsto (\zeta_m^i x, \zeta_m^j y)$ of F^m descend to the curve C. Note that some of them will be trivial, as $\mathrm{Aut}(F^m)$ contains μ_m^2 and $\mathrm{Aut}(C)$ only contains μ_m in general. The automorphisms of F^m act on the eigenbasis (6) by the following rule:

$$\sigma_{i,j}^*(w_{r,s}) = \zeta_m^{ir+js} w_{r,s}. \tag{7}$$

It follows that $\sigma_{i,j}$ acts on the dual space V^* with eigenvalues

$$\{\zeta_m^{-(ir+js)} : 0 < r, s, r+s < m \text{ and } br \equiv as \pmod{m}\}. \tag{8}$$

The eigenvalues of $\sigma_{i,j}$ acting on the tangent space (2) are computed from those on V and V^* using standard properties of wedge products.

3. Algorithm

3.1. **A genus 4 example.** Take $m = 15$, $a = 3$, $b = 5$, and $\sigma = \sigma_{2,1}$. Write $\zeta = \zeta_{15}$. By (4), the genus of $C = C_{3,5}^{15}$ is 4. It follows from (2) that

$$T^3(C) = \left(\bigwedge^2 V \otimes \bigwedge^3 V^*\right) \oplus \left(V \otimes \bigwedge^4 V^*\right).$$

The eigenbasis of V is $L = L_{3,5}^{15} = \{w_{3,5}, w_{3,10}, w_{6,5}, w_{9,5}\}$. Using (7) and (8), the eigenvalues of σ acting on V and V^* are respectively

$$\{\zeta^{11}, \zeta, \zeta^2, \zeta^8\} \quad \text{and} \quad \{\zeta^4, \zeta^{14}, \zeta^{13}, \zeta^7\}.$$

From properties of wedge products, we deduce that the eigenvalues of σ acting on

- $\bigwedge^2 V$ are $\{\zeta^{12}, \zeta^{13}, \zeta^4, \zeta^3, \zeta^9, \zeta^{10}\}$,
- $\bigwedge^3 V^*$ are $\{\zeta, \zeta^9, \zeta^{10}, \zeta^4\}$,
- $\bigwedge^4 V^*$ is ζ^8.

We obtain that the eigenvalues of σ acting on $\bigwedge^2 V \otimes \bigwedge^3 V^*$ and $V \otimes \bigwedge^4 V^*$ are respectively

$$\{\zeta^{13}, \zeta^6, \zeta^7, \zeta, \zeta^{14}, \zeta^8, \zeta^2, \zeta^5, \zeta^4, \zeta^{12}, \zeta^{10}, \zeta^3, \zeta^{11}\} \quad \text{and} \quad \{\zeta^4, \zeta^9, \zeta^{10}, \zeta\}.$$

We conclude that 1 is not an eigenvalue of σ acting on $T^3(C)$. By the arguments of Section 2.1, $\mathrm{AJ}_C(\kappa_e)$ is torsion for any point $e \in C(\bar{\mathbb{Q}})$ fixed by σ.

3.2. **Algorithm and results.** We have implemented Beauville's method in SageMath [24]; see [17]. The algorithm takes as input a positive integer m and outputs the list of all cyclic Fermat quotients of F^m in the format $(m, a, b, m-a-b)$ together with their genus. If a cyclic Fermat quotient has an automorphism $\sigma_{i,j}$ for which 1 is not an eigenvalue for the action on the tangent space (2), then the algorithm also outputs (m, a, b, i, j). Only the first such automorphism is listed, as the existence of an automorphism is what matters for our purpose.

We ran our algorithm for all positive integers m up to 50. For each even $m = 2n$, the cyclic Fermat quotient $C_{1,n-1}^m$ has an automorphism for which 1 is not an eigenvalue, hence the arguments of Section 2.1 apply. However $(1, n-1) \sim_m (1, n)$ and thus by Proposition 2.1 the curve $C_{1,n-1}^m$ is hyperelliptic. The only other examples of torsion Ceresa class we found were for the curves

$$C_{1,2}^9,\ C_{1,3}^{12},\ \text{and}\ C_{1,5}^{15}. \tag{9}$$

The first two have genus 3 and the third has genus 4. By Proposition 2.1, these are non-hyperelliptic.

Proposition 3.1. *The curves $C_{1,2}^9, C_{1,3}^{12}$, and $C_{1,5}^{15}$ are respectively isomorphic over $\mathbb{Q}$ to the smooth projective plane curves with affine models*

$$\begin{aligned} D^9 &: y^3 = x^4 + x \\ D^{12} &: y^3 = x^4 + 1 \\ D^{15} &: y^3 = x^5 + 1. \end{aligned}$$

Proof. Magma [7]. □

Remark 3.1. For each of the three curves D^m above, it is known that D^m is the unique curve over $\mathbb{C}$ of genus $g(D^m)$ with an automorphism of order m; for the genus 3 curves, see [18, pg. 2].

Proposition 3.2. *Let C be either $C^9_{1,2}, C^{12}_{1,3}$, or $C^{15}_{1,5}$, and let $e = (0,0,1)$. Then $\mathrm{AJ}_C(\kappa_e)$ is torsion.*

Proof. In each of the three cases we have $a = 1$, and $(0,0)$ is a non-singular point of $F^m_{1,b}$ fixed by the μ_m-action. The result then follows from the arguments of Section 2.1 and the output of our algorithm (9). □

Remark 3.2. Note that

$$(-2)\cdot\{3,5,7\} \equiv \{9,5,1\} \equiv \{1,5,9\} \pmod{15},$$

which implies that $(3,5) \sim_{15} (1,5)$, and thus $C^{15}_{1,5} \simeq C^{15}_{3,5}$. Proposition 3.2 therefore recovers the example calculated by hand in Section 3.1.

Remark 3.3. Propositions 3.1 & 3.2 generalize Beauville's result [2] for the curve D^9.

3.3. **Discussion and related work.** The genus of $C^m_{a,b}$ is of size $O(m)$ as $m \to \infty$. The dimension of the tangent space (2) therefore grows as $O(m^3)$. Since there are $O(m)$ choices for eigenvalues, it becomes very unlikely that 1 is not an eigenvalue for the action of $\sigma_{i,j}$ as $m \to \infty$. It therefore seems unlikely that Beauville's method alone will provide more examples of non-hyperelliptic curves with torsion Ceresa cycle under the complex Abel–Jacobi map. The output of our algorithm for large m is consistent with this heuristic.

Our results can be compared with recent work of Eskandari and Murty [12]. They show that for every choice of base point, the image of the Ceresa cycle of F^m under the complex Abel–Jacobi map is non-torsion, for any integer m divisible by a prime greater than 7. For an excellent survey of results concerning the non-triviality of Ceresa cycles of Fermat curves and their quotients see [11]. In Remark 2 on page 17 of [11], Eskandari and Murty speculate whether their method can be extended to prove that for every choice of base point, the image of the Ceresa cycle of the cyclic Fermat quotient $C^p_{1,b}$ under the complex Abel–Jacobi map is non-torsion for all prime numbers $p > 7$. Indeed, we ran our algorithm for quotients of Fermat curves of prime degree up to 71 and found that Beauville's method applies to none of them. It is interesting that for composite values of m, we do find curves $C^m_{1,b}$ with torsion Ceresa cycle, but only when the prime factors $p \mid m$ all satisfy $p < 7$.

4. L-functions

We make explicit the Beilinson–Bloch conjectures for the Jacobians $J^9_{1,2}$ and $J^{12}_{1,3}$, which predicts the rank of the Griffiths group of one-cycles in terms of certain L-functions. For $J^{12}_{1,3}$, we give new evidence for this conjecture.

4.1. **The Beilinson–Bloch conjecture.** Let C be as in Section 2, defined over a number field K. The conjecture of Beilinson and Bloch [5] predicts the equality

$$\operatorname{rank}_{\mathbb{Z}} \mathrm{CH}^{g-1}(J)_0 = \operatorname{ord}_{s=g-1} L(H^{2g-3}(J)/K, s), \tag{10}$$

where $L(H^{2g-3}(J)/K, s)$ denotes the L-function of the compatible system of ℓ-adic Galois representations coming from the cohomology of the Jacobian J in degree $2g-3$. Both the finite generation of $\mathrm{CH}^{g-1}(J)_0$ and the analytic continuation of the L-function are not known in general, hence are implicit in the conjecture.

Suppose there is a motivic decomposition $H^{2g-3}(J) = I \oplus M$, where M is pure of type $(g, g-3)+(g-3, g)$ and I has type $(g-1, g-2)+(g-2, g-1)$. Then the subgroup of algebraically trivial cycles $\mathrm{CH}^{g-1}(J)_{\mathrm{alg}}$ maps trivially under the étale Abel–Jacobi map to $H^1(\mathrm{Gal}(\bar{K}/K), M_{\mathrm{et}}(g-1))$ [5]. Hence, it is natural to expect the equality

$$\operatorname{rank}_{\mathbb{Z}} \mathrm{Gr}^{g-1}(J) = \operatorname{ord}_{s=g-1} L(M/K, s), \tag{11}$$

where $\mathrm{Gr}^{g-1}(J) := \mathrm{CH}^{g-1}(J)_0/\mathrm{CH}^{g-1}(J)_{\mathrm{alg}}$ is the Griffiths group. The equality (11) is Bloch's "Son of Recurring Fantasy". Even if $H^{2g-3}(J)$ does not have such a decomposition, Bloch defines a coniveau filtration $F^i \subset H^{2g-3}(J)$ and conjectures that the rank of $\mathrm{Gr}^{g-1}(J)$ is equal to the order of vanishing of $L(F^0/F^1, s)$ at its central point. See [6] for more details.

Next, we recall an explicit description of the relevant L-functions in our examples.

4.2. **Jacobi sums.** Fix m, a, and b as in Section 2.2, and let $C = C^m_{a,b}$ and $J = J^m_{a,b}$. For any n dividing m, there is a map $F^m \longrightarrow F^{m/n}$ given by $(x, y) \mapsto (x^n, y^n)$. Together with $f = f^m_{a,b}$, it induces a map of Jacobians $\mathrm{Jac}(F^{m/n}) \longrightarrow J$. Define J^{old} to be the subvariety of J generated by the images of the above maps for all proper divisors n of m. Define the *new part* $A = A^m_{a,b} := J/J^{\mathrm{old}}$, a quotient of $J^m_{a,b}$. Let

$$H^m_{a,b} := \{h \in (\mathbb{Z}/m\mathbb{Z})^\times : \langle ha\rangle + \langle hb\rangle < m\},$$

where $\langle\cdot\rangle$ denotes the unique representative in $\{1, \dots, m-1\}$ of the residue class modulo m, and define

$$W^m_{a,b} := \{h \in (\mathbb{Z}/m\mathbb{Z})^\times : hH^m_{a,b} = H^m_{a,b}\}.$$

The abelian variety A has CM by $\mathbb{Z}[\zeta_m]$ with CM type equal to $H^m_{a,b} \subset \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. Moreover, A is simple if and only if $W^m_{a,b} = \{1\}$. In general, A is isogenous over $\mathbb{C}$ to $|W^m_{a,b}|$ isomorphic simple factors each having CM by the subfield $K^m_{a,b}$ of $\mathbb{Q}(\zeta_m)$ fixed by $W^m_{a,b}$ and CM type given by $H^m_{a,b}/W^m_{a,b} \subset \mathrm{Gal}(K^m_{a,b}/\mathbb{Q})$.

For each prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_m]$, let $\mathbb{F}_\mathfrak{p}$ denote the residue field $\mathbb{Z}[\zeta_m]/\mathfrak{p}$. If $\mathfrak{p} \nmid m$, then the m-th power residue symbol $\chi_\mathfrak{p} : \mathbb{F}_\mathfrak{p}^\times \longrightarrow \mu_m \subset \mathbb{Q}(\zeta_m)$ is uniquely determined by the congruence $\chi_\mathfrak{p}(z) = z^{\frac{N(\mathfrak{p})-1}{m}} \pmod{\mathfrak{p}}$. Define the Hecke character (defined on ideals of $\mathbb{Z}[\zeta_m]$ prime to m)

$$\tau^m_{a,b} = \tau_{a,b} : I_{\mathbb{Q}(\zeta_m)}(m) \longrightarrow \mathbb{Q}(\zeta_m)^\times$$

whose value on prime ideals $\mathfrak{p}$ is the Jacobi sum

$$\tau_{a,b}(\mathfrak{p}) := -\sum_{z \in \mathbb{F}_\mathfrak{p} \setminus \{0,1\}} \chi^a_\mathfrak{p}(z)\chi^b_\mathfrak{p}(1-z).$$

The algebraic integer $\tau_{a,b}(\mathfrak{p})$ has absolute value $N(\mathfrak{p})^{1/2}$ in every complex embedding [25].

Fix a primitive character $\chi : \mu_m \longrightarrow \mathbb{C}^\times$ and define $\tau^{m,\chi}_{a,b} := \tau^\chi_{a,b} := \chi \circ \tau_{a,b} : I_{\mathbb{Q}(\zeta_m)}(m) \longrightarrow \mathbb{C}^\times$. This is a Grossencharacter for $\mathbb{Q}(\zeta_m)$ whose infinity type is given by the CM type of A. The complex L-function $L(A/\mathbb{Q}, s)$, attached to the cohomology of A in degree 1, has degree $\varphi(m)$ and we have the equality

$$L(A/\mathbb{Q}, s) = L(\tau^\chi_{a,b}/\mathbb{Q}(\zeta_m), s),$$

where the latter is the Hecke L-function attached to the Grossencharacter $\tau^\chi_{a,b}$ [25]. In particular, it has analytic continuation and satisfies a functional equation [23].

Remark 4.1. The equality of L-functions is independent of the choice of χ. Indeed, the polynomial $\prod_{\mathfrak{p}|p}(1 - \tau_{a,b}(\mathfrak{p})T^{f_\mathfrak{p}})$ has integer coefficients.

4.3. ***L*-function calculations.** Inspired by Bloch's L-function calculations [5] for the Klein quartic $C^7_{2,1}$, we compute the order of vanishing of the L-function relevant for the algebraic equivalence class of the Ceresa cycle in the case of Beauville's curve $C^9_{1,2}$ and the curve $C^{12}_{1,3}$.

4.3.1. *The L-function for $C^9_{1,2}$.* Fix a character $\chi : \mu_9 \longrightarrow \mathbb{C}^\times$ by sending ζ_9 to $e^{\frac{2\pi i}{9}}$ and identify as usual $\mathrm{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q})$ with $(\mathbb{Z}/9\mathbb{Z})^\times = \{1,2,4,5,7,8\}$. We order the complex embeddings by pairs of complex conjugates as $\{(1,8),(2,7),(4,5)\}$. In this subsection, $C = C^9_{1,2}$.

The morphism $f : F^9 \longrightarrow C$ is given by $(x\colon y\colon 1) \mapsto (-x^9, xy^2)$. We have $V := H^{1,0}(C)$ and a basis of f^*V is (in the notation of Section 2.2)

$$L^9_{1,2} = \{w_{1,2}, w_{2,4}, w_{5,1}\}.$$

The automorphism $\sigma(x\colon y\colon 1) = (\zeta_9^5 x\colon \zeta_9^7 y\colon 1)$ of F^9 descends to an order 9 automorphism of C. We have

$$\sigma^* w_{1,2} = \zeta_9 w_{1,2}, \qquad \sigma^* w_{2,4} = \zeta_9^2 w_{2,4}, \qquad \sigma^* w_{5,1} = \zeta_9^5 w_{5,1}. \tag{12}$$

Let $J = J^9_{1,2}$. For each prime ℓ, the ℓ-adic cohomology group $H^1_{\mathrm{et}}(J_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell)$ inherits an action of $\mu_9 \subset \mathrm{Aut}(C)$ and admits an eigenspace decomposition

$$H^1_{\mathrm{et}}(J_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell) = \bigoplus_\phi H^\phi,$$

where the sum is over primitive characters $\phi : \mu_9 \longrightarrow \mathbb{C}^\times$ and H^ϕ is the corresponding eigenspace. This is a direct sum of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_9))$-representations, and the decomposition carries an action of $\mathrm{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q}) = (\mathbb{Z}/9\mathbb{Z})^\times$ by $a(H^\phi) = H^{\phi^a}$. It follows from (12) that

$$H^{1,0}(C) = H^\chi \oplus H^{\chi^2} \oplus H^{\chi^5}, \tag{13}$$

after base change to $\mathbb{C}$.

Remark 4.2. We have $J = J^9_{1,2} = A^9_{1,2}$, which is a simple abelian variety with CM by $\mathbb{Z}[\zeta_9]$. The decomposition (13) is reflected by the fact that the CM type of J is $H^9_{1,2} = \{1,2,5\}$.

Let H denote the motive $H^1(J)$. The $(3,0)+(0,3)$ type submotive of $H^3(J) = \bigwedge^3 H$ is

$$M = M^9_{1,2} := \left(H^\chi \otimes H^{\chi^2} \otimes H^{\chi^5}\right) \oplus \left(H^{\chi^8} \otimes H^{\chi^7} \otimes H^{\chi^4}\right). \tag{14}$$

The motive M is defined over the maximal totally real subfield $\mathbb{Q}(\zeta_9)^+$ of $\mathbb{Q}(\zeta_9)$. The Beilinson–Bloch conjecture (11) in this case predicts that

$$\mathrm{rank}_{\mathbb{Z}} \mathrm{Gr}^2(J_{\mathbb{Q}(\zeta_9)^+}) = \mathrm{ord}_{s=2} L(M/\mathbb{Q}(\zeta_9)^+, s). \tag{15}$$

Proposition 4.1. *The order of vanishing of $L(M/\mathbb{Q}(\zeta_9)^+, s)$ at $s = 2$ is equal to 1.*

Proof. From the discussion in Section 4.2, we have $L(H^\chi/\mathbb{Q}(\zeta_9), s) = L(\tau^\chi_{1,2}, s)$, where $\tau^\chi_{1,2}$ has infinity type $[[1,0],[1,0],[0,1]]$. The conductor of this character has norm 3^4 (see [8, Table pg. 59]). We have the equalities of L-functions

$$L(M/\mathbb{Q}(\zeta_9)^+, s) = L(\tau^\chi_{1,2}\tau^{\chi^2}_{1,2}\tau^{\chi^5}_{1,2}/\mathbb{Q}(\zeta_9), s) = L(\tau^\chi_{1,2}\tau^\chi_{2,4}\tau^\chi_{5,1}/\mathbb{Q}(\zeta_9), s),$$

where the latter is defined as

$$L(\tau^\chi_{1,2}\tau^\chi_{2,4}\tau^\chi_{5,1}/\mathbb{Q}(\zeta_9), s) = \prod_{\mathfrak{p} \lhd \mathbb{Q}(\zeta_9)} (1 - \tau^\chi_{1,2}\tau^\chi_{2,4}\tau^\chi_{5,1}(\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1} = \prod_p L_p(s).$$

If $\mathfrak{p}$ is a prime of $\mathbb{Q}(\zeta_9)$ above $p \neq 3$, the factor $L_p(s)$ can be described as follows:

$p \equiv 1 \pmod 9$:

$$(1 - \tau^\chi_{1,2}\tau^\chi_{2,4}\tau^\chi_{5,1}(\mathfrak{p})p^{-s})^{-1}(1 - \tau^\chi_{2,4}\tau^\chi_{4,8}\tau^\chi_{1,2}(\mathfrak{p})p^{-s})^{-1}(1 - \tau^\chi_{4,8}\tau^\chi_{8,7}\tau^\chi_{2,4}(\mathfrak{p})p^{-s})^{-1}$$
$$(1 - \tau^\chi_{5,1}\tau^\chi_{1,2}\tau^\chi_{7,5}(\mathfrak{p})p^{-s})^{-1}(1 - \tau^\chi_{7,5}\tau^\chi_{5,1}\tau^\chi_{8,7}(\mathfrak{p})p^{-s})^{-1}(1 - \tau^\chi_{8,7}\tau^\chi_{7,5}\tau^\chi_{4,8}(\mathfrak{p})p^{-s})^{-1}$$

$p \equiv 2, 5 \pmod 9$: $(1 + p^9 p^{-6s})^{-1}$

$p \equiv 8 \pmod 9 : (1 - \tau^\chi_{1,2}\tau^\chi_{2,4}\tau^\chi_{5,1}(\mathfrak{p})p^{-2s})^{-1}(1 - \tau^\chi_{4,8}\tau^\chi_{8,7}\tau^\chi_{2,4}(\mathfrak{p})p^{-2s})^{-1}(1 - \tau^\chi_{7,5}\tau^\chi_{5,1}\tau^\chi_{8,7}(\mathfrak{p})p^{-2s})^{-1}$
$p \equiv 4, 7 \pmod 9 : (1 - \tau^\chi_{1,2}(\tau^\chi_{2,4})^2(\mathfrak{p})p^{-3s})^{-1}(1 - (\tau^\chi_{1,2})^2\tau^\chi_{2,4}(\mathfrak{p})p^{-3s})^{-1}.$

The infinity type of

$$\begin{cases} \tau^\chi_{1,2} & \text{is } [[1,0],[1,0],[0,1]] \\ \tau^\chi_{2,4} & \text{is } [[1,0],[0,1],[0,1]] \\ \tau^\chi_{5,1} & \text{is } [[1,0],[1,0],[1,0]]. \end{cases}$$

Thus, the infinity type of $\tau^\chi_{1,2}\tau^\chi_{2,4}\tau^\chi_{5,1}$ is $[[3,0],[2,1],[1,2]]$.

We can calculate the local factors of $L(M/\mathbb{Q}(\zeta_9)^+, s)$ in SageMath. By comparing local L-factors at all primes above $p \in \{7, 11, 13, 17, 19, 37\}$, we can identify the Grossencharacter $\tau^\chi_{1,2}\tau^\chi_{2,4}\tau^\chi_{5,1}$ with one of the finitely many Hecke characters in Magma with appropriate infinity type and conductor:

```
K:=CyclotomicField(9);
I:=Factorization(3*IntegerRing(K))[1][1]^4;
HG:=HeckeCharacterGroup(I);
DG:=DirichletGroup(I);
GR:=Grossencharacter(HG.0,DG.1*DG.2*DG.3,[[3,0],[2,1],[1,2]]);
```

We evaluated the L-function of this Grossencharacter at its center using the available tools in Magma. The sign in the functional equation turns out to be -1 and the order of vanishing at the center turns out to be 1. □

Remark 4.3. Beauville and Schoen [3] proved that $\kappa(C^9_{1,2})$ is torsion, but the order of vanishing of the L-function is 1 and not 0. What null-homologous 1-cycle on $J^9_{1,2}$ accounts for the vanishing of $L(M^9_{1,2}/\mathbb{Q}(\zeta_9)^+, 2)$, as predicted by Beilinson–Bloch?

4.3.2. *The L-function of $C^{12}_{1,3}$.* The computation in this case is somewhat similar, except that the Jacobian is not simple this time. In this subsection $C = C^{12}_{1,3}$ and $J = J^{12}_{1,3}$. Let ζ_3 be a third root of unity and define $\zeta_{12} = i\zeta_3$. Fix a primitive character $\chi : \mu_{12} \longrightarrow \mathbb{C}^\times$ by sending ζ_{12} to $e^{\frac{\pi i}{6}}$. Identify as usual $\mathrm{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q})$ with $(\mathbb{Z}/12\mathbb{Z})^\times = \{1, 5, 7, 11\}$ and order this set by conjugate pairs as $\{(1, 11), (5, 7)\}$. Note that $\mathbb{Q}(\zeta_{12}) = \mathbb{Q}(\zeta_3, i)$ is biquadratic.

Let $V = H^{1,0}(C)$ as before. A basis for f^*V is

$$L^{12}_{1,3} = \{w^{12}_{1,3}, w^{12}_{2,6}, w^{12}_{5,3}\}. \tag{16}$$

The automorphism $\sigma = \sigma_{1,4}$ of F^{12} descends to an order 12 automorphism of C. We have

$$\sigma^* w^{12}_{1,3} = \zeta_{12} w^{12}_{1,3}, \qquad \sigma^* w^{12}_{2,6} = \zeta^2_{12} w^{12}_{2,6}, \qquad \sigma^* w^{12}_{5,3} = \zeta^5_{12} w^{12}_{5,3}. \tag{17}$$

For each prime ℓ, the first ℓ-adic cohomology $H^1_{\mathrm{et}}(J_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell)$ inherits an action of $\mu_{12} \subset \mathrm{Aut}(C)$ and admits an eigenspace decomposition

$$H^1_{\mathrm{et}}(J_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell) = \bigoplus_\phi H^\phi,$$

where the sum is over (not necessarily primitive) characters $\phi : \mu_{12} \longrightarrow \mathbb{C}^\times$ and H^ϕ is the corresponding eigenspace. This decomposition carries an action of $\mathrm{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q}) = (\mathbb{Z}/12\mathbb{Z})^\times$ by $a(H^\phi) = H^{\phi^a}$. It follows from (17) that

$$H^{1,0}(C) = H^\chi \oplus H^{\chi^2} \oplus H^{\chi^5},$$

after base change to $\mathbb{C}$.

In general, the differential forms $w^m_{r,s}$ arise from the new part of the Jacobian of $C^m_{r,s}$ if and only if $\gcd(m, r, s) = 1$. In particular, $w^{12}_{1,3}$ and $w^{12}_{5,3}$ of (16) arise from the 2-dimensional abelian variety $A := A^{12}_{1,3}$ over $\mathbb{Q}$, while $2 \cdot w^{12}_{2,6}$ arises as the pullback of the form $w^6_{1,3}$ on F^6 via the map

$F^{12}\longrightarrow F^6$, $(x,y)\mapsto(x^2,y^2)$. Note that $w^6_{1,3}$ is the regular differential form of the elliptic curve $E := J^6_{1,3} = A^6_{1,3}$ over $\mathbb{Q}$. We have an isogeny $J \simeq A\times E$ over $\mathbb{Q}$.

The elliptic curve E has CM by the imaginary quadratic field $\mathbb{Q}(\zeta_6)=\mathbb{Q}(\zeta_3)$ with CM type $\{1\}$. The L-function of E is given by the L-function of a Grossencharacter for $\mathbb{Q}(\zeta_3)$ of infinity type $[1,0]$, namely the Jacobi sum $\tau^{6,\chi}_{1,3}$. The Jacobi sum $\tau^{12,\chi^2}_{1,3} = \tau^{12,\chi}_{2,6}$ is a Grossencharacter for $\mathbb{Q}(\zeta_{12})$ and satisfies

$$\tau^{12,\chi}_{2,6} = \tau^{6,\chi}_{1,3}\circ \mathrm{Norm}^{\mathbb{Q}(\zeta_{12})}_{\mathbb{Q}(\zeta_3)} : I_{\mathbb{Q}(\zeta_{12})}\longrightarrow\mathbb{C}^\times.$$

Since $\mathrm{Gal}(\mathbb{Q}(\zeta_{12}/\mathbb{Q}(\zeta_3)))=\{1,7\}$, it follows that $\tau^{12,\chi}_{2,6}$ has infinity type $[[1,0],[0,1]]$. The submotive $H^{\chi^2}\oplus H^{\chi^{10}}$ of H is defined over $\mathbb{Q}(i)$ and arises from the elliptic curve E. Denoting by

$$\xi : \mathrm{Gal}(\mathbb{Q}(\zeta_{12}/\mathbb{Q}(\zeta_3))) = \mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})\longrightarrow\mathbb{C}^\times$$

the non-trivial character, we observe that

$$\begin{aligned} L(H^{\chi^2}/\mathbb{Q}(\zeta_{12}),s) &= L((H^{\chi^2}\oplus H^{\chi^{10}})/\mathbb{Q}(i),s) = L(E/\mathbb{Q}(i),s)\\ &= L(E/\mathbb{Q},s)L(E/\mathbb{Q},\xi,s) = L(\tau^{6,\chi}_{1,3}/\mathbb{Q}(\zeta_3),s)L(\tau^{6,\chi}_{1,3}/\mathbb{Q}(\zeta_3),\xi,s) = L(\tau^{12,\chi}_{2,6}/\mathbb{Q}(\zeta_{12}),s).\end{aligned}$$

The abelian variety $A = A^{12}_{1,3}$ is isogenous over $\mathbb{C}$ to the product of two isomorphic elliptic curves with CM by $\mathbb{Q}(i)$ and CM type $\{1\}$. We see that $H^{1,0}(A) = H^\chi\oplus H^{\chi^5}$ is defined over $\mathbb{Q}(i)$ and $H^{12}_{1,3}=\{1,5\}=\mathrm{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q}(i))$. We have

$$L(A/\mathbb{Q},s) = L(\tau^{12,\chi}_{1,3}/\mathbb{Q}(\zeta_{12}),s),$$

and the infinity type of $\tau^{12,\chi}_{1,3}$ is $[[1,0],[1,0]]$. The motive

$$M(A) := H^{2,0}(A)\oplus H^{0,2}(A) = (H^\chi\otimes H^{\chi^5})\oplus(H^{\chi^{11}}\otimes H^{\chi^7})$$

is defined over $\mathbb{Q}$ and $L(M(A)/\mathbb{Q},s) = L(\tau^{12,\chi}_{1,3}\tau^{12,\chi}_{5,3}/\mathbb{Q}(i),s)$.

Let H denote the motive $H^1(J)$. The motive of interest is the $(3,0)+(0,3)$ part of $\bigwedge^3 H$,

$$M = M^{12}_{1,3} = (H^\chi\otimes H^{\chi^2}\otimes H^{\chi^5})\oplus(H^{\chi^{11}}\otimes H^{\chi^{10}}\otimes H^{\chi^7}),$$

which is defined over $\mathbb{Q}(\zeta_{12})^+ = \mathbb{Q}(\sqrt{3})$. The Beilinson–Bloch conjecture (11) predicts that

$$\mathrm{rank}_{\mathbb{Z}}\,\mathrm{Gr}^2(J_{\mathbb{Q}(\sqrt{3})}) = \mathrm{ord}_{s=2}\,L(M/\mathbb{Q}(\sqrt{3}),s). \tag{18}$$

Proposition 4.2. *We have $L(M/\mathbb{Q}(\sqrt{3}),s) = L((M(A)\otimes H^1(E))/\mathbb{Q},s)$ and the order of vanishing at $s=2$ is equal to 0.*

Proof. The motive

$$\begin{aligned} M(A)\otimes H^1(E) = (H^\chi\otimes H^{\chi^2}\otimes H^{\chi^5})\oplus(H^\chi\otimes H^{\chi^5}\otimes H^{\chi^{10}})\\ \oplus(H^{\chi^2}\otimes H^{\chi^{11}}\otimes H^{\chi^7})\oplus(H^{\chi^{10}}\otimes H^{\chi^{11}}\otimes H^{\chi^7}),\end{aligned}$$

is defined over $\mathbb{Q}$ and the equality of L-functions is clear.

For the order of vanishing, we proceed computationally as in the proof of Proposition 4.1. Define the Grossencharacter $\psi = \tau^{12,\chi}_{1,3}\tau^{12,\chi}_{5,3}\tau^{12,\chi}_{2,6}$. Then we have

$$L(M/\mathbb{Q}(\sqrt{3}),s) = L(\psi/\mathbb{Q}(\zeta_{12}),s),$$

and the infinity type of ψ is $[[3,0],[2,1]]$. We identify ψ in Magma by comparing values at all primes above $p\in\{5,7,11,13,17,19,23,37,73\}$. In Magma, the character is given by the commands:

```
K:=CyclotomicField(12);
I:=Factorization(2*IntegerRing(K))[1][1];
HG:=HeckeCharacterGroup(I);
DG:=DirichletGroup(I);
GR:=Grossencharacter(HG.0,DG.1^2,[[3,0],[2,1]]);
```

Using Magma the sign of the functional equation of $L(M/\mathbb{Q}(\sqrt{3}), s)$ is $+1$ and

$$L(M/\mathbb{Q}(\sqrt{3}), 2) = L(\psi/\mathbb{Q}(\zeta_{12}), s) \approx 0.724.$$

□

By Proposition 4.2, the Beilinson–Bloch conjecture (11) predicts that the group $\mathrm{Gr}^2(J_{\mathbb{Q}(\sqrt{3})})$ is torsion. In particular, it predicts that the class $\kappa(C)$ of the Ceresa cycle should be torsion. This would follow from Proposition 3.2, if it were known that the complex Abel–Jacobi map is injective on cycles defined over $\bar{\mathbb{Q}}$. In fact, since C has genus 3, one can prove *unconditionally* that $\kappa(C)$ is torsion using the argument of [3]. Indeed, Reid's criterion shows that the quotient J/μ_{12} is uniruled, and proceeding as in [3], one deduces that $\kappa(C)$ is torsion in the Griffiths group. This example therefore gives new evidence for the Beilinson–Bloch conjecture (11).

Remark 4.4. The conductor of the Grossencharacter ψ is the prime ideal of $\mathbb{Z}[\zeta_{12}]$ of norm 4, which we observe is remarkably small. Thus, the curve $C^{12}_{1,3}$ shows that

(a) the group $\mathrm{Gr}_1(J)$ can (and assuming the Beilinson–Bloch conjecture, does) have rank 0 sometimes, even for non-hyperelliptic Jacobians, and

(b) we need only look at one of the first non-hyperelliptic curves to find such rank 0 examples!

To make (b) more precise, we first recall a phenomenon which can be observed in tables of elliptic curves: if N_r is the minimal conductor of an elliptic curve $E/\mathbb{Q}$ with Mordell–Weil rank r, then $N_r < N_{r+1}$. One can (conjecturally) define analogous integers N_r for the ranks of the Griffiths group $\mathrm{Gr}_1(J)$ of Jacobians J of genus $g \geq 3$ curves $C/\mathbb{Q}$, using the Galois representation F^0/F^1 from Section 4.1 to define a notion of "Griffiths group conductor". Then D^{12} is one of the first non-hyperelliptic curves in the sense that it has small Griffiths group conductor. Does the curve D^{12} have minimal Griffiths group conductor among all non-hyperelliptic genus 3 curves over $\mathbb{Q}$?

4.3.3. *The L-function of* $C^{15}_{1,5}$. We attempted a similar computation for the third curve $C^{15}_{1,5}$, which has genus 4. Since $H^{4,1}(J)$ is 4-dimensional, the relevant L-function in this case is a product of four Hecke L-functions for the degree 8 field $\mathbb{Q}(\zeta_{15})$, of norm conductor as large as 2025. With these large parameters, Magma was not able to compute the central values of these L-functions on our laptops.

5. Hyperelliptic Isogenies

It is of course desirable to prove that the Ceresa cycles $\kappa_{(0,0,1)}$ for D^9, D^{12}, and D^{15} are themselves torsion (not just their images in the Griffiths group and intermediate Jacobian). We would also like to understand, more generally, why certain non-hyperelliptic curves have torsion Ceresa cycles, and whether there is a way to characterize them. During our computations, we noticed that these curves share a certain property that seems relevant for these questions. Namely, each of these curves admits a non-zero algebraic correspondence to the hyperelliptic curve $C^m_{1,1} \colon y^m = x(1-x)$. In fact, this property nearly characterizes the curves D^9, D^{12}, and D^{15}, among all non-hyperelliptic cyclic Fermat quotients, as we will explain.

Let $J^m_{a,b}$ denote the Jacobian of $C^m_{a,b}$ and recall the new part $A^m_{a,b}$, which has CM by $\mathbb{Z}[\zeta_m]$. We say two cyclic quotients $C^m_{a,b}$ and $C^m_{a',b'}$ are *isogenous* if their corresponding new parts are isogenous. The isogeny relation among the curves $C^m_{a,b}$ is a coarser relation than the equivalence relation $\sim_m$ introduced in Section 2.2. To check whether two such curves are isogenous, it is enough to check that the corresponding new parts have the same CM types, up to the action of $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. Recall from Section 4.2 that the CM type of $A^m_{a,b}$ can be identified with the set

$$H^m_{a,b} := \{h \in (\mathbb{Z}/m\mathbb{Z})^\times : \langle ha\rangle + \langle hb\rangle < m\}$$

The group $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$ acts on the set of CM types by scaling.

Lemma 5.1. *Up to isogeny, the unique hyperelliptic cyclic quotient of F^m is $C^m_{1,1}$.*

Proof. For m odd this follows from Proposition 2.1. For m even, we observe that $A^m_{1,1}$ is isogenous to $A^m_{1,m/2}$ since they both have CM type $\{t \in (\mathbb{Z}/m\mathbb{Z})^\times : \langle t\rangle < m/2\}$, by direct computation. Thus this case also follows from Proposition 2.1. □

We can classify those non-hyperelliptic curves $C^m_{a,b}$ which are isogenous to a hyperelliptic one. A result of Koblitz and Rohrlich implies that this can only happen if $\gcd(m, 6) > 1$ [16]. In fact, there are only finitely many examples with $m \not\equiv 2 \pmod 4$:

Proposition 5.2. *Suppose $C = C^m_{a,b}$ is non-hyperelliptic and $m \not\equiv 2 \pmod 4$. Then C is isogenous to $C^m_{1,1}$ if and only if (m, a, b) is equivalent to one of the following:*

$$\{(9, 1, 2), (12, 1, 3), (15, 1, 5), (21, 1, 2), (21, 1, 3), (24, 1, 5), (24, 1, 7), (60, 1, 10), (60, 1, 19)\}.$$

Proof. That there are no examples with $m > 180$ is a special case of a result of Aoki [1, Thm. 0.1], who determined when two cyclic Fermat quotients are isogenous to each other. From staring at his classification, we quickly rule out any "exceptional isogenies" from $C^m_{1,1}$, with $m \not\equiv 2 \pmod 4$, to a non-hyperelliptic $C^m_{a,b}$. For $m \le 180$, it is enough to check which of the finitely many equivalence classes of curves $C^m_{a,b}$ has new part $A^m_{a,b}$ with the same CM type as $A^m_{1,1}$, up to automorphisms of $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$. It turns out that up to equivalence the only non-hyperelliptic curves $C^m_{a,b}$ with this CM type are those listed. □

Notice that the first three curves on this list are D^9, D^{12}, and D^{15}. When $m \equiv 2 \pmod 4$, the situation is somewhat different, since there are actually infinitely many examples.

Proposition 5.3. *Suppose $C = C^m_{a,b}$ is non-hyperelliptic and $m = 4k + 2$, for some $k \ge 0$. Then C is isogenous to $C^m_{1,1}$ if and only if (m, a, b) is equivalent to $(m, 1, k)$ or one of the following:*

$$(14, 1, 2), (18, 1, 2), (18, 1, 5), (30, 1, 2), (30, 1, 3), (30, 1, 4), (30, 1, 8), (30, 2, 3),$$
$$(42, 1, 2), (42, 1, 4), (42, 1, 5), (42, 1, 8), (42, 1, 11), (42, 1, 15), (78, 1, 16).$$

Proof. The proof is similar. The infinite family $C^m_{1,k}$ can again be read off of Aoki's result. □

We say a non-hyperelliptic $C^m_{a,b}$ is *minimal* if it does not cover any non-hyperelliptic cyclic Fermat quotient of lower genus. For our purposes, the minimal non-hyperelliptic $C^m_{a,b}$ are the first curves to study, since if the Ceresa cycle of a curve has infinite order, so will the Ceresa cycle of any cover. For example, the curve $C^{21}_{1,2}$ listed above is isogenous to a hyperelliptic curve, but it also covers the Klein quartic curve $C^7_{1,2}$. Since the latter has infinite order Ceresa cycle [15, 21], so does $C^{21}_{1,2}$.

Proposition 5.4. *If $C = C^m_{a,b}$ is a minimal non-hyperelliptic cyclic Fermat quotient which is isogenous to $C^m_{1,1}$, then C is equivalent to one of the following curves:*

$$\{(9, 1, 2), (12, 1, 3), (15, 1, 5), (21, 1, 3)\} \cup \{(4k + 2, 1, k) : k \in \mathbb{Z}\}.$$

Proof. For each curve (m, a, b) in Proposition 5.2, we must determine whether there exists a divisor $d \mid m$, such that $C^m_{da,db}$ is non-hyperelliptic. We have $C^m_{da,db} \simeq C^{m/g}_{a',b'}$ where $g = \gcd(\langle da\rangle, \langle db\rangle, \langle d(m-a-b)\rangle)$, $a' = \langle da\rangle/g$ and $b' = \langle db\rangle/g$. For example, the only interesting subcover of $C^{21}_{1,3}$ is $C^7_{1,3} \simeq C^7_{1,1}$, which is hyperelliptic, so it is minimal. We must do the same to the 15 sporadic examples with $m \equiv 2 \pmod 4$, but we find that they are all non-minimal. □

Thus, the curves D^9, D^{12},D^{15}, and $C_{1,3}^{21}$ are characterized as the minimal non-hyperelliptic cyclic Fermat quotients $C_{a,b}^m$ with $m \not\equiv 2 \pmod 4$ that are isogenous to a hyperelliptic one.

Remark 5.1. Not all curves of the form $C_{1,k}^{4k+2}$ are minimal, but infinitely many of them are. For example, if $4k+2 = 2p$ for some prime p, then the curve is minimal.

5.1. **Future directions.** It is natural to wonder whether there is an algebraic correspondence from $C_{1,1}^m$ to D^m which sends the Ceresa class of $C_{1,1}^m$ (in the Chow group of $J_{1,1}^m$) to that of D^m or to some multiple of it. This would prove that the Ceresa cycle is torsion in these cases, and it would give a nice geometric explanation as well. In other words, even though the curves D^m are not hyperelliptic, perhaps the motives $H^{2g-3}(\mathrm{Jac}(D^m))$ are hyperelliptic in a certain sense. We hope to study this question in future work. It would also be interesting to compute numerically the Abel–Jacobi image of the Ceresa cycle for the genus 9 curve $C_{1,3}^{21}$, to see whether it too is torsion, despite the fact that Beauville's method does not apply.

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