Definite orthogonal modular forms

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Motivation

Definition

Let $r_4(n) = \#\{\lambda \in \mathbb{Z}^4 : \sum_{i=1}^{4} \lambda_i^2 = n\}$.

Theorem (Jacobi's four-square theorem)

$$r_4(n) = 8 \sum_{4 \nmid d | n} d.$$

Idea of Proof.

- Write $\theta(q) = \sum_{n=0}^{\infty} r_4(n) q^n$.
- Show that θ belongs to a finite dimensional vector space V.
- Find a basis for V.
- Represent θ in that basis and compare coefficients.

$\square$

Modular curves

- The **upper half plane** is $\mathfrak{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$.
- It admits an action of $\mathrm{GL}_2^+(\mathbb{R})$ by **Möbius transformations**

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathfrak{H} \to \mathfrak{H}, \quad z \mapsto \gamma z = \frac{az+b}{cz+d}$$

- For a discrete $\Gamma \leq \mathrm{GL}_2^+(\mathbb{R})$, can form $Y(\Gamma) = \Gamma \backslash \mathfrak{H}$.
- Specific groups Γ of interest

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a, d \equiv 1 \bmod N \right\}$$

- Note that $\gamma \mapsto d : \Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$ has kernel $\Gamma_1(N)$.
- Compactify using **cusps**

$$X(\Gamma) = Y(\Gamma) \cup (\Gamma \backslash \mathbb{P}^1(\mathbb{Q})), \quad X_0(N) = X(\Gamma_0(N))$$

Modular forms

- Fact: $X(\Gamma)$ is a compact Riemann surface.

Example

Local coordinate at ∞ for $X_0(N)$ is $q = e^{2\pi i z}$.

- For Γ torsion-free, let $M_2(\Gamma)$ be differentials on $X(\Gamma)$ holomorphic on $Y(\Gamma)$ with at most simple poles at the cusps.

Theorem (Riemann-Roch)

$$\dim M_2(\Gamma) = g(X(\Gamma)) + \#cusps - 1$$

Let $\pi : \mathfrak{H} \to X(\Gamma)$. For $\omega \in M_2(\Gamma)$ can consider $\pi^*(\omega) = f(z)dz$.

Example

For $X_0(N)$, if near ∞, $\omega = g(q)\ dq$, then $\pi^*(\omega) = 2\pi i\ q\ g(q)\ dz$

Modular forms, cusp forms and characters

Thus $\omega \mapsto f(z)$ identifies $M_2(\Gamma)$ with hol. functions $f : \mathfrak{H} \to \mathbb{C}$, s.t.

$$(cz+d)^{-2} f(\gamma z)\, dz = f(\gamma z)\, d(\gamma z) = f(z)\, dz \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

- $M_2(\Gamma)$ is the space of **modular forms** of level Γ (of weight 2).
- Write $S_2(\Gamma) \subseteq M_2(\Gamma)$ for the holomorphic differentials.
- The map $\omega \mapsto f(z)$ identifies $S_2(\Gamma)$ with the functions in $M_2(\Gamma)$ that vanish at the cusps, called **cusp forms**.
- $(\mathbb{Z}/N\mathbb{Z})^{\times} \simeq \Gamma_0(N)/\Gamma_1(N)$ acts on $M_2(\Gamma_1(N))$ via

$$f(z)d(z) \mapsto f(\gamma_0 z)d(\gamma_0 z)$$

- Write $M_2(N, \chi)$ (resp. $S_2(N, \chi)$) for the χ-isotypic component, so $f \in M_2(N, \chi)$ iff

$$f(\gamma z) = \chi(d)(cz+d)^2 f(z) \quad \forall \gamma \in \Gamma_0(N)$$

$X_0(4)$

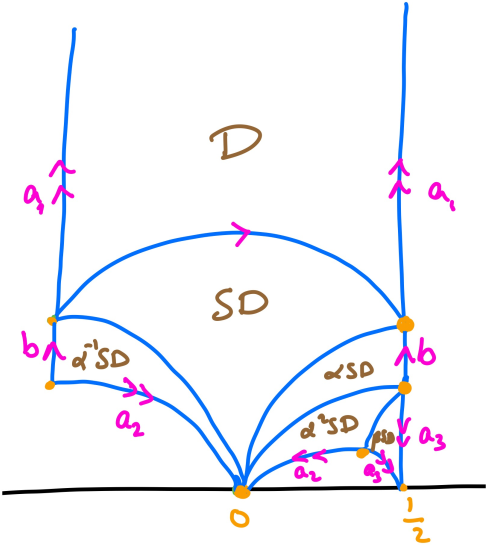
Example

One computes that

$$\Gamma_0(4)\backslash \mathrm{SL}_2(\mathbb{Z}) = \{1, \alpha, \alpha^2, \alpha^{-1}, \beta, s\}$$

where

$$\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$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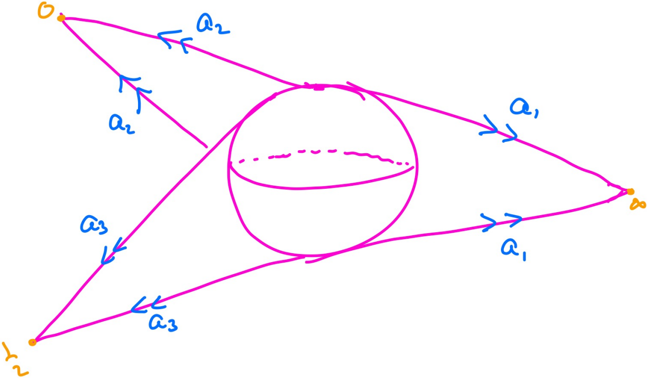

0
a_2
a_2
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∞
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$1/2$

$X_0(4)$

Example

We compute that

- $\Gamma_0(4)\backslash\mathbb{P}^1(\mathbb{Q}) = \{0, \frac{1}{2}, \infty\}$,
- $X_0(4) \simeq \mathbb{P}^1(\mathbb{C})$

so $\dim M_2(\Gamma_0(4)) = 2$.
The function $\theta(z) = \sum_{n=0}^{\infty} r_4(n)q^n$ is holomorphic, and

$$\theta(z+1) = \theta(z), \quad \theta\left(\frac{z}{4z+1}\right) = (4z+1)^2\theta(z),$$

hence $\theta \in M_2(\Gamma_0(4))$. [Also invariant under $z \mapsto -\frac{1}{4z}$]

Representation Numbers

Proof of Jacobi's four-square theorem.

- Construct $E_2(z) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) q^n$.
- $v_1 = E_2(z) - 2E_2(2z)$, $v_2 = E_2(2z) - 2E_2(4z) \in M_2(\Gamma_0(4))$.
- From first two terms deduce $\theta(z) = 8v_1 + 16v_2$, so

$$\sum r_4(n) q^n = 8(E_2(z) - 4E_2(4z)) = 8\left(\sum \sigma(n) q^n - \sum \sigma(n) q^{4n}\right)$$

yields $r_4(n) = \sum_{4 \nmid d | n} d$. □

More generally, if $Q(x) = \sum_{i \leq j} a_{ij} x_i x_j$ is a quadratic form with $a_{ij} \in \mathbb{Z}$, we may consider

$$r_Q(n) = \#\{\lambda \in \mathbb{Z}^4 : Q(\lambda) = n\}$$

and the function

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) q^n = \sum_{\lambda \in \mathbb{Z}^4} q^{Q(\lambda)}$$

is again a modular form.

Quadratic forms and Lattices

Let $Q : V \to \mathbb{Q}$ be a positive definite quaternary ($\dim_{\mathbb{Q}} V = 4$) quadratic space with associated bilinear form

$$T(x, y) := Q(x + y) - Q(x) - Q(y).$$

Let $\Lambda \subseteq V$ be an **integral** lattice, so that $Q(\Lambda) \subseteq \mathbb{Z}$.
Define $\Delta = \operatorname{disc}(\Lambda) = \det T \in \mathbb{Z}$.
Given a lattice, we may construct associated **theta series**

$$\theta_\Lambda(z) = \theta_{\Lambda,1}(z) = \sum_{\lambda \in \Lambda} q^{Q(\lambda)}, \quad q = e^{2\pi i z}$$

The **level** of Λ is the smallest N such that $N\Lambda^\sharp \subseteq \Lambda$.
Then $\theta_\Lambda(z) \in S_2(N, \chi_\Delta)$, where $\chi_\Delta(a) = \left(\frac{\Delta}{a}\right)$.
However, $\Lambda \mapsto \theta_\Lambda$ is not injective.

Isometry and genus

We define the orthogonal group

$$\begin{aligned} \mathrm{O}(V) &= \{g \in \mathrm{GL}(V) : Q(gv) = Q(v)\} \\ \mathrm{O}(\Lambda) &= \{g \in \mathrm{O}(V) : g\Lambda = \Lambda\} \end{aligned}$$

and write $\mathrm{SO}(V)$ and $\mathrm{SO}(\Lambda)$ for those with $\det(g) = 1$. Lattices Λ, Π are **isometric**, written $\Pi \simeq \Lambda$, if there exists $g \in \mathrm{O}(V)$ such that $g\Lambda = \Pi$. The **genus** of Λ is

$$\mathrm{gen}(\Lambda) := \{\Pi \subseteq V : \Lambda_p \simeq \Pi_p \text{ for all } p\}.$$

The **class set** $\mathrm{cls}(\Lambda) = \mathrm{gen}(\Lambda)/\simeq$ is the set of (global) isometry classes in $\mathrm{gen}(\Lambda)$. It is finite, by geometry of numbers.

Neighbors

Kneser's theory of p-neighbors gives an effective method to compute the class set; it also gives a Hecke action!

Let $p \nmid \operatorname{disc}(\Lambda)$ be a prime; $p = 2$ is OK.

We say that a lattice $\Pi \subseteq V$ is a ***p*-neighbor** of Λ, and write $\Pi \sim_p \Lambda$ if

$$[\Lambda : \Lambda \cap \Pi] = [\Pi : \Lambda \cap \Pi] = p.$$

If $\Lambda \sim_p \Pi$ then:

- $\operatorname{disc}(\Lambda) = \operatorname{disc}(\Pi)$,
- Π is integral, and
- $\Pi \in \operatorname{gen}(\Lambda)$.

Moreover, there exists S such that every $[\Pi] \in \operatorname{cls}(\Lambda)$ is an **iterated *S*-neighbor** of Λ.

$$\Lambda \sim_{p_1} \Lambda_1 \sim p_2 \cdots \sim_{p_r} \Lambda_r \simeq \Pi$$

with $p_i \in S$. Typically may take $S = \{p\}$.

Example - Computing the class set

Let $\Lambda = \mathbb{Z}^4$ with the quadratic form

$$Q(x_1, x_2, x_3, x_4) = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_1x_4 + x_3x_4 + 3x_4^2$$

and bilinear form given by

$$[T_\Lambda] = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 6 \end{pmatrix}$$

Thus $\mathrm{disc}(\Lambda) = 29$.

$$\Lambda' = \frac{1}{2}\mathbb{Z}(e_2 + e_4) + 2\mathbb{Z}e_3 + \mathbb{Z}e_1 + \mathbb{Z}e_4$$

with corresponding quadratic form

$$Q(x) = x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 3x_1x_4 + 2x_2x_4 + x_3x_4 + 3x_4^2$$

Orthogonal modular forms

The space of **orthogonal modular forms** of level Λ is

$$M(\mathrm{O}(\Lambda)) := \{f : \mathrm{cls}(\Lambda) \to \mathbb{C}\} \simeq \mathbb{C}^{h(\Lambda)}$$

For $p \nmid \mathrm{disc}(\Lambda)$ define the **Hecke operator**

$$T_p : M(\mathrm{O}(\Lambda)) \to M(\mathrm{O}(\Lambda))$$
$$f \mapsto \left([\Lambda'] \mapsto \sum_{\Pi' \sim_p \Lambda'} f([\Pi']) \right)$$

The Hecke operators commute and are self-adjoint, hence there is a basis of simultaneous eigenvectors - **eigenforms**. (Gross, 1999)

Example - square discriminant

Let Λ have the Gram matrix

$$[T_\Lambda] = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 6 & 0 \\ 1 & 0 & 0 & 6 \end{pmatrix}$$

so that $\text{disc}(\Lambda) = \det T = 11^2$. Then $h(\Lambda) = 3$.
Write $\text{cls}(\Lambda) = \{[\Lambda] = [\Lambda_1], [\Lambda_2], [\Lambda_3]\}$. Then a basis of eigenforms is given by

$$f_1 = [\Lambda_1] + [\Lambda_2] + [\Lambda_3], \qquad f_2 = 4[\Lambda_1] - 6[\Lambda_2] + 9[\Lambda_3]$$
$$f_3 = 4[\Lambda_1] + [\Lambda_2] - 6[\Lambda_3],$$

and we have

$$\theta(f_1) = \frac{5}{12} + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + O(q^7) \in E_2(11)$$
$$\theta(f_2) = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + O(q^9) \in S_2(11)$$

where $T_p(f_2) = a_p(f_2)$ with $a_2 = 4, a_3 = 1, a_5 = 1, a_7 = 4, \ldots$

Example - Hecke action

Let Λ be as before with discriminant 29. By checking isometry we compute w.r.t. basis $[\Lambda'], [\Lambda]$

$$[T_2] = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, [T_3] = \begin{pmatrix} 4 & 3 \\ 6 & 7 \end{pmatrix}, [T_5] = \begin{pmatrix} 18 & 9 \\ 18 & 27 \end{pmatrix}, \ldots$$

The constant function $e = [\Lambda] + [\Lambda']$ is an **Eisenstein series** with $T_p(e) = (p^2 + (1 + \chi_{29}(p)) + 1)e$. Another eigenvector is $f = [\Lambda] - 2[\Lambda']$, with $T_p(f) = a_p(f)$

$$a_2 = -1, a_3 = 1, a_5 = 9, a_7 = 4, a_{11} = 17, \ldots$$

We match them with the **Hilbert modular form** labeled 2.2.29.1-1.1-a in the LMFDB.

Hilbert modular forms

Let K be a real quadratic field.

- K has two real embeddings $v_1, v_2 : K \to \mathbb{R}$.
- For $a \in K^\times$ write $a_i = v_i(a)$.
- $a \in K^\times$ is **totally positive** if $a_1 > 0$ and $a_2 > 0$.
- Write $K_{>0}^\times$ for the group of totally positive elements.
- Denote $\mathrm{GL}_2^+(K) = \{\gamma \in \mathrm{GL}_2(K) : \det\gamma \in K_{>0}^\times\}$
- $\mathrm{GL}_2^+(K)$ acts on $\mathfrak{H} \times \mathfrak{H}$ by

$$z = (z_1, z_2) \mapsto \gamma z = (\gamma_1 z_1, \gamma_2 z_2) = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2} \right)$$

- A **Hilbert modular form** of weight (k_1, k_2) and level $\Gamma \subseteq \mathrm{GL}_2^+(K)$ is a holomorphic $f : \mathfrak{H} \times \mathfrak{H} \to \mathbb{C}$ such that

$$f(\gamma z) = \frac{(c_1 z_1 + d_1)^{k_1}}{\det(\gamma_1)^{k_1/2}} \frac{(c_2 z_2 + d_2)^{k_2}}{\det(\gamma_2)^{k_2/2}} f(z) \quad \forall \gamma \in \Gamma$$

Towards a bijection?

Would like to have a bijection between **orthogonal modular forms** and **Hilbert modular forms**, but... Consider $Q(x) = x_1^2 + x_2^2 + x_3^2 + x_1x_4 + x_2x_4 + 3x_4^2$ with Gram matrix

$$[T_\Lambda] = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 6 \end{pmatrix}$$

and $\operatorname{disc}(\Lambda) = 40$.

- Then $\dim S(O(\Lambda)) = 1 \neq 2 = \dim S_2(\mathbb{Z}[\sqrt{10}])$.
- This is because of the lattice Λ_2 with form $Q_2(x) = x_1^2 + x_2^2 + 2x^3 + x_2x_4 + 2x_3x_4 + 2x_4^2$.
- Although $\Lambda_2 \notin \operatorname{gen}(\Lambda_1)$, it is everywhere locally **similar** to Λ_1.

Similitude classes

We define the general orthogonal group

$$\mathrm{GO}(V) = \{g \in \mathrm{GL}(V) : Q(gv) = \mu(g)Q(v), \quad \mu(g) \in \mathbb{Q}^\times\}$$
$$\mathrm{GO}(\Lambda) = \{g \in \mathrm{GO}(V) : g\Lambda = \Lambda\}$$

and write $\mathrm{GSO}(V)$ and $\mathrm{GSO}(\Lambda)$ for those with $\det(g) > 0$. Lattices Λ, Π are **similar**, written $\Pi \sim \Lambda$, if there exists $g \in \mathrm{GO}(V)$ such that $g\Lambda = \Pi$. The **similitude genus** of Λ is

$$\mathrm{sgen}(\Lambda) := \{\Pi \subseteq V : \Lambda_p \sim \Pi_p \text{ for all } p\}.$$

The **similitude class set** $\mathrm{scls}(\Lambda) = \mathrm{sgen}(\Lambda)/\sim$ is the set of (global) similitude classes in $\mathrm{sgen}(\Lambda)$. It is finite, by geometry of numbers.

Main Theorem

Theorem (A., Fretwell, Ingalls, Logan, Secord, and Voight (2022))

Assume $\mathrm{disc}(\Lambda) = D_0 N^2$, $K = \mathbb{Q}[\sqrt{D_0}]$. *Then*

$$S(GO(\Lambda)) \hookrightarrow G_K \backslash S_2(N\mathbb{Z}_K)$$

with image the orbits in $S_2(N\mathbb{Z}_K; W = \epsilon)^{D\text{-}new}$

- $G_K = \mathrm{Gal}(K|\mathbb{Q})$ acts naturally on the space of Hilbert modular forms.
- D is the product of the anisotropic primes.
- For $p \mid N$, we set $\epsilon_p = -1$ if $p \mid D$, else $\epsilon_p = 1$.
- W_p is the Atkin-Lehner involution at $p\mathbb{Z}_K \mid N\mathbb{Z}_K$.

The other forms

- The space of orthogonal modular forms of **weight** (k, j) is

$$M_{k,j}(\mathrm{GO}(\Lambda)) = \{f : \mathrm{scls}(\Lambda) \to W_{k,j} : f(gx) = \rho_{k,j}(g)f(x)\}.$$

- Twisting by the spinor norm, we obtain all the spaces

$$S_{k_1,k_2}(N\mathbb{Z}_K, W = \epsilon)^{D\text{-new}}$$

- The space $S(O(\Lambda))$ is identified as the forms invariant under twists by Hecke characters.
- If $D_0 = 1$, $K = \mathbb{Q} \times \mathbb{Q}$, so $M_{k_1,k_2}(N\mathbb{Z}_K) = M_{k_1}(N) \otimes M_{k_2}(N)$, this case was proved by Böcherer and Schulze-Pillot (1991).

Key Ideas

Theorem (Auel and Voight)

The even Clifford functor with descent data induces an equivalence

$$\left\{ \begin{array}{c} \textit{lattices } \Lambda \subseteq V \\ \textit{under oriented similarities} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \mathbb{Z}_K\textit{-orders } \mathcal{O} \subseteq C_0(V) \\ \textit{under isomorphisms*} \end{array} \right\}$$

- It also induces an isomorphism $C_0 : \mathrm{GSO}(V)/\mathbb{Q}^\times \xrightarrow{\simeq} B_K^\times / K^\times$.
- Compatible, hence $C_0^* : M(\mathrm{Typ}_s(\mathcal{O}), \rho) \to M(\mathrm{GSO}(\Lambda), \rho \circ C_0)$.
- Description of $\mathrm{Typ}_s(\mathcal{O})$ based on Ponomarev (1976).
- Sends p-neighbors to $p\mathbb{Z}_K$-neighbors.
- Characterize $M(\mathrm{Typ}_s(\mathcal{O}), \rho)$ as a subspace of $M(\mathcal{O}^\times, \rho))$ using AL, as in Hein (2016).

$\bullet\ \bullet \cong \begin{smallmatrix}\bullet \\ \\ \bullet\end{smallmatrix}$ [$A_1 \times A_1 = D_2$, equiv. $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \cong \mathfrak{so}_4$]

Applications

We obtain commutative diagrams of Hecke modules

$$\begin{array}{ccc} S(\mathrm{Typ}_s(O))_{G_K} & \overset{C_0^*}{\longleftrightarrow} & S(\mathrm{GO}(\Lambda)) \\ \updownarrow {\scriptstyle JL} & & \downarrow {\scriptstyle \theta_2} \\ S(N\mathbb{Z}_K, W = \epsilon)_{G_K}^{D\text{-new}} & \longrightarrow & S(\mathrm{Sp}_4(\mathbb{Z}), \Gamma_0(N), \chi_{D_0}) \end{array}$$

The bottom line is:

- Yoshida lift when $K = \mathbb{Q} \times \mathbb{Q}$ and f, g are both cuspidal.
- Saito-Kurokawa lift when $K = \mathbb{Q} \times \mathbb{Q}$ otherwise.
- Asai lift when K is a quadratic field.

Higher rank

We can also do that for lattices of higher even rank. For example, this yields the following theorem

Theorem (A., Fretwell, Ingalls, Logan, Secord, and Voight (2022))

Let $\Lambda = A_6 \oplus A_2$, *and let* $f_1, f_2 \in S_4(\Gamma_0(21), \chi_{21})$ *be representatives for the two Galois orbits of newforms. Write*

$$\lambda_{p,1}^{(i)} = a_p(f_i)^2 - \chi_{21}(p)p^3 + p \cdot \frac{p^5 - 1}{p - 1}, \quad \lambda_{p,1}^{(3)} = \frac{p^7 - 1}{p - 1} + \chi_{21}(p)p^3$$

There are $A_i \in M_{3\times 3}(\mathbb{Q})$ *such that the* p*-neighbor adjacency matrix in* $\operatorname{cls}(\Lambda)$ *is*

$$A_1\lambda_{p,1}^{(1)} + A_2\lambda_{p,1}^{(2)} + A_3\lambda_{p,1}^{(3)}$$

Eisenstein congruences

This method allows us also to prove Eisenstein congruences.

Theorem (A., Fretwell, Ingalls, Logan, Secord, and Voight (2022))

For all $p \neq 53$, we have

$$a_{1,p^2}(F) \equiv a_p(f)^2 - (1 + \chi_{53}(p))p^3 + p^5 + p \pmod{\mathfrak{q}}$$

where $F \in S_4(\Gamma_0^{(2)}(53), \chi_{53})$, $\mathfrak{q} \mid 397$ in $K = \mathbb{Q}(F)$, and $f \in S_4(\Gamma_0(53), \chi)$.

Why 397?
The numerator of the norm of

$$\frac{L(\mathrm{Sym}^2(f), 1)}{\pi^2 L(\mathrm{Sym}^2(f), 3)}$$

is divisible by 397, hence

$$\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(\mathrm{Sym}^2(f), 6)) > 0.$$

Future research

Conjecture (A. et al. (2022))

Let Λ be of rank 6 and discriminant p.

$$\dim \ker \theta_2 = \# \operatorname{cls}^+(\Lambda) - \operatorname{cls}(\Lambda) = \dim M(O(\Lambda), \det)$$

Conjecture (A. et al. (2022))

If $f \in S_{j+k}(\Gamma_0(N), \chi)$ is an eigenform with

$$\operatorname{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(\operatorname{Sym}^2(f), j + 2k - 2)) > 0$$

and $\mathfrak{q}$ lies above $q > 2(j+k) - 1$. Then there are $F \in S_{j,k}(\Gamma_0^{(2)}(N), \chi)$ and $\mathfrak{q}' \mid \mathfrak{q}$ such that

$$b_{1,p^2} \equiv a_p^2 - \chi(p)p^{j+k-1} - p^{j+2k-5} + p^{j+2k-3} + p^{j+1} \pmod{\mathfrak{q}'}$$

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