VANISHING CRITERIA FOR CERESA CYCLES

JEF LAGA AND ARI SHNIDMAN

Abstract. Let C be a smooth projective curve, and let J be its Jacobian. We prove vanishing criteria for the Ceresa cycle $\kappa(C) \in \mathrm{CH}_1(J) \otimes \mathbb{Q}$ in the Chow group of 1-cycles on J. Namely,
(A) If $\mathrm{H}^3(J)^{\mathrm{Aut}(C)} = 0$, then $\kappa(C)$ vanishes;
(B) If $\mathrm{H}^0(J, \Omega^3_J)^{\mathrm{Aut}(C)} = 0$, then $\kappa(C)$ vanishes modulo algebraic equivalence,
with criterion (B) conditional on the Hodge conjecture.

We then study the first interesting case where (B) holds but (A) does not, namely the case of Picard curves $C\colon y^3 = x^4 + ax^2 + bx + c$. Using work of Schoen on the Hodge conjecture, we show that the Ceresa cycle of a Picard curve is torsion in the Griffiths group. Moreover, we determine exactly when it is torsion in the Chow group. As a byproduct, we show that there are infinitely many plane quartic curves over $\mathbb{Q}$ with torsion Ceresa cycle (in fact, there is a one parameter family of such curves). Finally, we determine which automorphism group strata are contained in the vanishing locus of the universal Ceresa cycle over $\mathcal{M}_3$.

1. Introduction

Let k be an algebraically closed field and C a smooth, projective, and connected curve over k of genus $g \geq 2$ with Jacobian variety J. Let e be a degree-1 divisor of C and let $\iota_e\colon C \hookrightarrow J$ be the Abel-Jacobi map based at e. We study the torsion behaviour of the Ceresa cycle

$$\kappa_{C,e} := [\iota_e(C)] - (-1)^*[\iota_e(C)] \in \mathrm{CH}_1(J) \tag{1.1}$$

in the Chow group modulo rational equivalence. If $\kappa_{C,e}$ is torsion, then $(2g-2)e = K_C$ in $\mathrm{CH}_0(C) \otimes \mathbb{Q}$, where K_C is the canonical divisor class. Moreover if $(2g-2)e$ is canonical, then the image of $\kappa_{C,e}$ in $\mathrm{CH}_1(J) \otimes \mathbb{Q}$ is independent of e and we denote it by $\kappa(C)$, see §2.4 for these claims. The class $\kappa(C)$ vanishes if and only if $\kappa_{C,e}$ is torsion for some degree-1 divisor e.

We also consider the image $\bar{\kappa}(C)$ of $\kappa(C)$ in the Griffiths group $\mathrm{Gr}_1(J) \otimes \mathbb{Q}$ of homologically trivial 1-cycles modulo algebraic equivalence. When $g = 2$, or more generally when C is hyperelliptic, it is easy to see that $\kappa(C) = 0$. On the other hand, Ceresa famously showed that $\bar{\kappa}(C) \neq 0$ for a very general curve C over $\mathbb{C}$ of genus $g \geq 3$ [13].

The vanishing of the Ceresa cycle is interesting for various reasons. For example, $\kappa(C) = 0$ if and only if the Chow motive $\mathfrak{h}(C)$ has a multiplicative Chow-Künneth decomposition (by [18, Proposition 3.1] and Proposition 2.7). Moreover $\bar{\kappa}(C) = 0$ if and only if the tautological subring modulo algebraic equivalence is generated by a theta divisor (by [5, Corollary 3.4]), in which case Poincaré's formula $[C] = \frac{\Theta^{g-1}}{(g-1)!}$ holds modulo algebraic equivalence. More generally, the Ceresa cycle over $\mathcal{M}_g$ serves as a testing ground for the study of homologically trivial algebraic cycles in codimension greater than 1.

1.1. **Vanishing criteria.** We prove cohomological vanishing criteria for Ceresa cycles of curves with nontrivial automorphisms. Let $\mathrm{H}^*(-)$ be a Weil cohomology functor, such as ℓ-adic cohomology with $\ell \neq \mathrm{char}\,(k)$ or singular cohomology when $k = \mathbb{C}$. Note that the finite group $\mathrm{Aut}(C)$ acts on $\mathrm{H}^*(J)$, by functoriality.

Theorem A. *If* $\mathrm{H}^3(J)^{\mathrm{Aut}(C)} = 0$*, then* $\kappa(C) = 0$.

This improves on a recent result of Qiu and Zhang stating that if $(\mathrm{H}^1(C)^{\otimes 3})^{\mathrm{Aut}(C)} = 0$, then $\kappa(C) = 0$ [34].[1] By contrast, Theorem A requires only the weaker condition that the subrepresentation $\mathrm{H}^3(J) \simeq \bigwedge^3 \mathrm{H}^1(C) \subset \mathrm{H}^1(C)^{\otimes 3}$ has no nontrivial $\mathrm{Aut}(C)$-fixed points.

Our analogous vanishing criterion for $\bar{\kappa}(C)$ is valid only in characteristic 0 and is conditional on the Hodge conjecture:

Theorem B. *Suppose* $\mathrm{char}(k) = 0$ *and assume the Hodge conjecture for abelian varieties. If* $\mathrm{H}^0(J, \Omega^3_J)^{\mathrm{Aut}(C)} = 0$*, then* $\bar{\kappa}(C) = 0$.

More precisely, we require the Hodge conjecture for $J \times A$, for a specific abelian variety A described in Proposition 3.3. Note that $\mathrm{H}^0(J, \Omega^3_J) \simeq \bigwedge^3 \mathrm{H}^0(C, \Omega^1_C)$, so the condition of Theorem B depends only on the abstract representation $(G, V) = (\mathrm{Aut}(C), \mathrm{H}^0(C, \Omega^1_C))$.

The proofs of Theorems A and B are quite direct, using the rich structure of motives of abelian varieties. Ingredients include the Beauville decomposition, the canonical Chow–Künneth decomposition of Deninger and Murre, and the Kimura finite-dimensionality of abelian varieties.

1.2. **Picard curves.** The condition $\mathrm{H}^3(J)^{\mathrm{Aut}(C)} = 0$ in Theorem A is only rarely satisfied, e.g. it holds for exactly two plane quartic curves over $\mathbb{C}$ (see Theorem D below). However, one can use Theorem B to study the vanishing behavior of $\kappa(C)$ in families $\mathcal{C} \to B$ of smooth projective curves over a complex base B with fixed automorphism group. The interesting case is where $\kappa(\mathcal{C}_b)$ is *not* identically zero but $\mathrm{H}^0(\mathcal{J}_b, \Omega^3_{\mathcal{J}_b})^{\mathrm{Aut}(\mathcal{C}_b)} = 0$ for all $b \in B(\mathbb{C})$. Using the Hodge conjecture, one can show that the vanishing of $\kappa(\mathcal{C}_b)$ is equivalent to the vanishing of $\sigma(b)$, where σ is a section of a certain abelian scheme $\mathcal{A} \to B$. The analytification of σ is the normal function associated to the universal Ceresa cycle. One can try to identify σ and hence determine the exact vanishing locus of $\kappa(\mathcal{C}_b)$ in the entire family. We work out the details for the family of Picard curves $y^3 = x^4 + ax^2 + bx + c$, where the relevant case of the Hodge conjecture is due to Schoen [36], who proved that Weil classes on abelian fourfolds with endomorphism algebra containing $\mathbb{Q}(\sqrt{-3})$ and signature $(2, 2)$ are algebraic.

Our main result in this direction is:

Theorem C. *Suppose* $\mathrm{char}(k) = 0$ *and let* C_f *be a smooth projective Picard curve with model*

$$C_f \colon y^3 = f(x) = x^4 + ax^2 + bx + c \tag{1.2}$$

for some $a, b, c \in k$*. Consider the point* $P_f = (a^2 + 12c, 72ac - 2a^3 - 27b^2)$ *on the elliptic curve*

$$E_f \colon y^2 = 4x^3 - 27 \cdot \mathrm{disc}(f).$$

Then $\bar{\kappa}(C_f) = 0$*, and* $\kappa(C_f) = 0$ *if and only if* $P_f \in E_f(k)$ *is torsion.*

Here, $\mathrm{disc}(f)$ is the discriminant of f and the coordinates of P_f are the usual I- and J-invariants of the binary quartic form $z^4 f(x/z)$; see §4.1. The fact that P_f defines a point on E_f follows from a classical relation between the invariants of a binary quartic. See also §4.2 for a more precise version of Theorem C, which relates order of $\kappa(C_f)$ to the order of P_f. This generalizes the results of [25] concerning *bielliptic* Picard curves, i.e. those with $b = 0$ in (1.2). As a corollary of Theorem C, there exist one-parameter families of plane quartic curves with vanishing $\kappa(C)$, corresponding to components of the torsion locus of P_f. Such families exist even over $\mathbb{Q}$, e.g. the family $y^3 = x^4 - 12x^2 + tx - 12$.

In §3, we give two examples of genus 4 families of curves for which Theorem B applies. It would be interesting to try to completely characterize the vanishing of $\kappa(C)$ in these families as well, and to find more such families.

[1] This formulation is equivalent to theirs, using [41, Theorem 1.5.5].

1.3. **Vanishing loci in genus 3.** Let $V_g^{\mathrm{rat}} \subset \mathcal{M}_g$ be the subset consisting of all (geometric) isomorphism classes of curves with vanishing $\kappa(C)$. Let $V_g^{\mathrm{alg}} \subset \mathcal{M}_g$ be the analogous subset for $\bar{\kappa}(C)$. One can show (Lemma 5.1) that V_g^{rat} and V_g^{alg} are both countable unions of closed algebraic subvarieties. Moreover $V_g^{\mathrm{rat}} \subset V_g^{\mathrm{alg}}$ and Ceresa's result shows that $V_g^{\mathrm{alg}} \neq \mathcal{M}_g$ for all $g \geq 3$. What can be said about the irreducible components of V_g^{rat} and V_g^{alg} and their dimensions? Collino and Pirola have shown that V_3^{rat} does not contain subvarieties of dimension ≥ 4 that are not themselves contained in the hyperelliptic locus [15, Corollary 4.3.4]. Theorem C shows that V_3^{alg} contains the 2-dimensional Picard locus and that V_3^{rat} contains infinitely many 1-dimensional components.

Recall that there is a stratification of the non-hyperelliptic locus of $\mathcal{M}_3$ by locally closed subvarieties X_G, indexed by certain finite groups G, with the property that a non-hyperelliptic curve $[C] \in \mathcal{M}_3(\mathbb{C})$ lies in X_G if and only if $\mathrm{Aut}(C) \simeq G$. It turns out that the isomorphism class of the $\mathrm{Aut}(C)$-representation $\mathrm{H}^0(C, \Omega_C^1)$ does not depend on the choice of curve $[C]$ in $X_G(\mathbb{C})$. Our final theorem determines exactly which X_G are contained in V_3^{rat} or V_3^{alg}.

Theorem D. *Let G be a finite group isomorphic to the automorphism group of a plane quartic. Then $X_G \subset V_3^{\mathrm{rat}}$ if and only if $G = C_9$ or G_{48}, and $X_G \subset V_3^{\mathrm{alg}}$ if and only if X_G is contained in the Picard locus, in other words if and only if $G = C_3, C_6, C_9$ or G_{48}.*

Here, G_{48} is the group with GAP label $(48, 33)$. The strata X_{C_9} and $X_{G_{48}}$ are single closed points, represented by the curves $y^3 = x^4 - x$ and $y^3 = x^4 + 1$ respectively.

Theorem D shows that $X_G \subset V_3^{\mathrm{rat}}$ if and only if $\mathrm{H}^3(J)^G = 0$ for every (equivalently, some) curve in X_G, and that $X_G \subset V_3^{\mathrm{alg}}$ if and only if $\mathrm{H}^0(J, \Omega_J^3)^G = 0$ for every curve in X_G. Thus, for $g = 3$, the criteria of Theorems A and B exactly single out the strata X_G where $\kappa(C)$ or $\bar{\kappa}(C)$ identically vanishes.

1.4. **Structure of paper.** In §2 we collect some standard results on Chow groups, Chow motives and Ceresa cycles. In §3 we prove the cohomological vanishing criteria (Theorems A and B) and give some examples. In §4 we study the family of Picard curves in detail and prove Theorem C. Finally, in §5 we introduce the Ceresa vanishing loci $V_g^{\mathrm{rat}}, V_g^{\mathrm{alg}} \subset \mathcal{M}_g$ and determine which automorphism strata they contain in genus 3, proving Theorem D.

1.5. **Acknowledgements.** We thank Ben Moonen and Congling Qiu for helpful conversations. This research was carried out while the first author was a Research Fellow at St John's College, University of Cambridge, which he thanks for providing excellent working conditions. The second author was funded by the European Research Council (ERC, CurveArithmetic, 101078157).

2. Notation and background

2.1. **Chow groups.** Let k be field. A variety is by definition a separated scheme of finite type over k. We say a variety is nice if it is smooth, projective and geometrically integral. If X is a smooth and geometrically integral variety and $p \in \{0, \ldots, \dim X\}$, let $\mathrm{CH}^p(X)$ denote the Chow group (with $\mathbb{Z}$-coefficients) of codimension p cycles modulo rational equivalence. If $Z \subset X$ is a closed integral subscheme of codimension p (or a union of such subschemes), we denote its class in $\mathrm{CH}^p(X)$ by $[Z]$. If X is additionally projective, then $\mathrm{CH}^p(X)$ has a filtration by subgroups

$$\mathrm{CH}^p(X)_{\mathrm{alg}} \subset \mathrm{CH}^p(X)_{\mathrm{hom}} \subset \mathrm{CH}^p(X),$$

where $\mathrm{CH}^p(X)_{\mathrm{alg}}$ is the subgroup of algebraically trivial cycles (see [2, §3.1] for a definition over an arbitrary ground field) and $\mathrm{CH}^p(X)_{\mathrm{hom}}$ the subgroup of homologically trivial cycles (with respect to a fixed Weil cohomology theory for nice varieties over k). The Griffiths group is by definition the quotient $\mathrm{Gr}^p(X) = \mathrm{CH}^p(X)_{\mathrm{hom}}/\mathrm{CH}^p(X)_{\mathrm{alg}}$. We occasionally write $\mathrm{CH}_p(X) = \mathrm{CH}^{\dim X - p}(X)$

and $\mathrm{Gr}_p(X) = \mathrm{Gr}^{\dim X - p}(X)$. If R is a ring, we write $\mathrm{CH}^p(X)_R = \mathrm{CH}^p(X) \otimes_{\mathbb{Z}} R$ and $\mathrm{Gr}^p(X)_R = \mathrm{Gr}^p(X) \otimes_{\mathbb{Z}} R$.

2.2. **Specialization, base change and families.** Let R be a discrete valuation ring with fraction field K and residue field k. Let $X \to \mathrm{Spec}(R)$ be a smooth, projective morphism with geometrically integral fibers, so the generic and special fibers X_K and X_k are nice varieties over K and k respectively. In this setting (in fact, in a much more general setting that will not concern us here), Fulton has defined [19, §20.3] a specialization morphism $\mathrm{sp}\colon \mathrm{CH}^p(X_K) \to \mathrm{CH}^p(X_k)$ for every $0 \le p \le \dim(X_K)$. It has the property that if $Z \subset X$ is a closed integral subscheme of codimension p, smooth over R, then $\mathrm{sp}([Z_K]) = [Z_k]$.

Lemma 2.1. *In the above notation,* sp *sends* $\mathrm{CH}^p(X_K)_{\mathrm{alg}}$ *to* $\mathrm{CH}^p(X_k)_{\mathrm{alg}}$.

Lemma 2.2. *Let X/k be a nice variety and K/k a (not necessarily finite) extension of fields.*

(1) The base change maps $\mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{CH}^p(X_K)_{\mathbb{Q}}$ *and* $\mathrm{Gr}^p(X)_{\mathbb{Q}} \to \mathrm{Gr}^p(X_K)_{\mathbb{Q}}$ *are injective.*

(2) If in addition k is algebraically closed, the base change maps $\mathrm{CH}^p(X) \to \mathrm{CH}^p(X_K)$ *and* $\mathrm{Gr}^p(X) \to \mathrm{Gr}^p(X_K)$ *are injective.*

Proof. The proof is a standard adaptation of [11, Lemma 1A.3, p. 22]. We first prove (2) for $\mathrm{CH}^p(X) \to \mathrm{CH}^p(X_K)$. Suppose $\alpha \in \mathrm{CH}^p(X)$ has trivial image in $\mathrm{CH}^p(X_K)$. Then α already has trivial image in $\mathrm{CH}^p(X_{K'})$, where $K' \subset K$ is a subfield that is finitely generated over k, since the data witnessing triviality in $\mathrm{CH}^p(X_K)$ can be defined over such a subfield. By spreading out, we can find a smooth integral variety U/k with function field K' such that α has trivial image in $\mathrm{CH}^p(X \times_k U)$. Since k is algebraically closed, there exists a k-point $u \in U(k)$. Pulling back along u defines a left-inverse $\mathrm{CH}^p(X \times_k U) \to \mathrm{CH}^p(X)$ to the map $\mathrm{CH}^p(X) \to \mathrm{CH}^p(X \times_k U)$. It follows that α is trivial in $\mathrm{CH}^p(X)$, as desired. The argument for $\mathrm{Gr}^p(X)$ is identical and omitted.

We now prove (1) for $\mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{CH}^p(X_K)_{\mathbb{Q}}$. There exists a field L containing both K and an algebraic closure $\bar{k}$ of k. It therefore suffices to prove the two base change maps $\mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{CH}^p(X_{\bar{k}})_{\mathbb{Q}} \to \mathrm{CH}^p(X_L)_{\mathbb{Q}}$ are both injective. The first one follows from the fact that for a finite extension k'/k the pushforward map $\mathrm{CH}^p(X_{k'}) \to \mathrm{CH}^p(X)$, when precomposed with the base change map, is multiplication by $[k' : k]$. The second follows from Part (2). The case of $\mathrm{Gr}^p(X)_{\mathbb{Q}}$ is again analogous. □

Lemma 2.3. *Let $X \to S$ be a smooth proper morphism of smooth varieties over a field k. Let α be a codimension p cycle on X. Then the locus of points $s \in S$ such that the fiber* $\alpha_s \in \mathrm{CH}^p(X_s)_{\mathbb{Q}}$ *is zero (resp. lies in* $\mathrm{CH}^p(X_s)_{\mathrm{alg},\mathbb{Q}}$*) is a countable union of closed algebraic subvarieties of S.*

Proof. We sketch the proof of this standard fact due to lack of reference. There exists a countable subfield $k_0 \subset k$, a smooth proper morphism $X_0 \to S_0$ of varieties over k_0 and a cycle α_0 on X_0 whose base change to k are $X \to S$ and α respectively. Since k_0 is countable, S_0 has only countably many closed subschemes. Let $\mathcal{F}$ be the collection of integral closed subschemes $Z \subset S_0$ such that α_0 is zero in $\mathrm{CH}^p(X_{0,\eta(Z)})_{\mathbb{Q}}$, where $\eta(Z) \in S_0$ denotes the generic point of Z. Then we claim that the locus of $s \in S$ for which $\alpha_s \in \mathrm{CH}^p(X_s)_{\mathbb{Q}}$ is zero is exactly the union $\bigcup_{Z \in \mathcal{F}} Z_k \subset S$. This follows from Lemmas 2.1 and 2.2; we omit the details, and the similar argument for algebraic triviality. □

2.3. **Chow motives.** We recall a few relevant facts about the category $\mathsf{Mot}(k)$ of (pure, contravariant) Chow motives M over k with $\mathbb{Q}$-coefficients; see [37] or [33, §2] for basic definitions.

Denote the Lefschetz motive by $\mathbb{L}$, its nth tensor power by $\mathbb{L}^n$ and write $M(n) = M \otimes \mathbb{L}^n$. Following [33, §2.5], the Chow group in codimension p of M is by definition $\mathrm{CH}^p(M) = \mathrm{Hom}_{\mathsf{Mot}(k)}(\mathbb{L}^p, M)$. If $M = \mathfrak{h}(X)$ where X is smooth projective over k, then $\mathrm{CH}^p(M) = \mathrm{CH}^p(X)_{\mathbb{Q}}$. (Beware that we need to take $\mathbb{Q}$-coefficients on the right hand side.) If C is a nice curve over k and $\varphi\colon \mathfrak{h}(C) \to M(p-1)$ a morphism in $\mathsf{Mot}(k)$, we obtain a homomorphism of abelian groups $\mathrm{CH}^1(\varphi)\colon \mathrm{CH}^1(C)_{\mathbb{Q}} \to \mathrm{CH}^p(M)$. We define $\mathrm{CH}^p(M)_{\mathrm{alg}}$ to be the union of the images of $\mathrm{CH}^1(\varphi)$ ranging over all pairs (C,φ) as above. This coincides with $\mathrm{CH}^p(X)_{\mathrm{alg}} \otimes \mathbb{Q}$ when $M = \mathfrak{h}(X)$, see [2, Corollary 3.13].

Let A/k be a g-dimensional abelian variety. Deninger and Murre [17, §3] have constructed a canonical Chow–Künneth decomposition

$$\mathfrak{h}(A) = \bigoplus_{i=0}^{2g} \mathfrak{h}^i(A), \tag{2.1}$$

uniquely characterized by the following property: if $(n)\colon A \to A$ denotes the multiplication-by-n, then $(n)^*$ acts on $\mathfrak{h}^i(A)$ via n^i for every integer n. On the other hand, Beauville [4] has shown that there exists a direct sum decomposition $\mathrm{CH}^p(A)_{\mathbb{Q}} = \bigoplus_{s=p-g}^{p} \mathrm{CH}^p_{(s)}(A)$, where

$$\mathrm{CH}^p_{(s)}(A) = \{\alpha \in \mathrm{CH}^p(A)_{\mathbb{Q}} \colon (n)^*\alpha = n^{2p-s}\alpha \quad \forall n \in \mathbb{Z}\}. \tag{2.2}$$

The two decompositions are linked by the formula $\mathrm{CH}^p(\mathfrak{h}^i(A)) = \mathrm{CH}^p_{(2p-i)}(A)$. Beauville has conjectured that $\mathrm{CH}^p_{(s)}(A) = 0$ when $s < 0$, and proved it when $p \in \{0, 1, g-2, g-1, g\}$, see [4, Proposition 3(a)].

Lemma 2.4. *If A/k is an abelian variety, then $\mathrm{CH}^p(\mathfrak{h}^1(A))_{\mathrm{alg}} = \mathrm{CH}^p(\mathfrak{h}^1(A))$ for all $p \geq 0$.*

Proof. By Lemma 2.2(1), we may assume k is algebraically closed. The only nonzero Chow group of $\mathfrak{h}^1(A)$ is $\mathrm{CH}^1(\mathfrak{h}^1(A)) = \mathrm{CH}^1_{(1)}(A)$, the set of anti-symmetric elements of $\mathrm{CH}^1(A)_{\mathbb{Q}}$. The lemma follows from the fact that $\mathrm{CH}^1_{(1)}(A) = \mathrm{CH}^1(A)_{\mathrm{hom}} \otimes \mathbb{Q}$ and that homological and algebraic equivalence coincide for codimension-1 cycles. □

Finally, we define motives of fixed points. If G is a finite group acting on a motive M, write M^G for the submotive cut out by the idempotent $\frac{1}{\#G}\sum_{g\in G} g_* \in \mathrm{End}(M)$. We have $\mathrm{CH}^p(M^G) = \mathrm{CH}^p(M)^G$. If G acts on a nice variety X, then G acts on $\mathfrak{h}(X)$ and $\mathrm{CH}^p(X)$, and $\mathrm{CH}^p(\mathfrak{h}(X)^G) = \mathrm{CH}^p(X)^G \otimes \mathbb{Q}$.

2.4. **Ceresa cycles.** Let C/k be a nice curve with Jacobian J. Let e be a degree-1 divisor on C and let $\iota_e\colon C \to J$ be the Abel–Jacobi map based at e, sending $x \in C$ to the divisor class of $x - e$. We define $\kappa_{C,e} \in \mathrm{CH}_1(J)$ using the formula (1.1) from the introduction. Let $K_C \in \mathrm{CH}_0(C)$ denote the canonical divisor class.

Lemma 2.5. *If $\kappa_{C,e}$ is torsion, then $(2g-2)e - K_C$ is torsion.*

Proof. If $\kappa_{C,e}$ is torsion, then $[\iota_e(C)] = (-1)_*[\iota_e(C)]$ in $\mathrm{CH}_1(J)_{\mathbb{Q}}$. Taking the $(g-1)$-fold Pontryagin product of this equality and dividing by $(g-1)!$, we get $[\Theta_{(g-1)e}] = (-1)_*[\Theta_{(g-1)e}]$ in $\mathrm{CH}_1(J)_{\mathbb{Q}}$, where for a degree $g-1$ divisor class D we write Θ_D for the image of the map $\mathrm{Sym}^{g-1}(C) \to J$ defined by $x \mapsto [x] - D$. By Riemann–Roch, $(-1)_*[\Theta_{(g-1)e}] = [\Theta_{K_C-(g-1)e}]$. Since $\Theta_{(g-1)e}$ defines a principal polarization, the map $\varphi\colon J(k) \to \mathrm{CH}^1(J)_{\mathrm{hom}}$ defined by $x \mapsto [\Theta_{(g-1)e+x}] - [\Theta_{(g-1)e}]$ is an isomorphism. Since $\varphi(K_C - (2g-2)e) = [\Theta_{K_C-(g-1)e}] - [\Theta_{(g-1)e}] = (-1)_*[\Theta_{(g-1)e}] - [\Theta_{(g-1)e}]$ is torsion in $\mathrm{CH}^1(J)_{\mathrm{hom}}$, the divisor class $K_C - (2g-2)e$ is torsion. □

Lemma 2.6. *If $e, e' \in \mathrm{CH}_0(C)$ are degree-1 divisors such that $e - e'$ is torsion, then $[\iota_e(C)] = [\iota_{e'}(C)]$ and $\kappa_{C,e} = \kappa_{C,e'}$ in $\mathrm{CH}_1(J)_{\mathbb{Q}}$.*

Proof. Suppose $e - e'$ has order n in $\mathrm{CH}_0(C)$. Then $(n)\circ\iota_e = (n)\circ\iota_{e'}$, hence $(n)_*([\iota_e(C)] - [\iota_{e'}(C)]) = 0$ in $\mathrm{CH}_1(J)$. On the other hand, the Beauville decomposition of §2.3 shows that $(n)_*\colon \mathrm{CH}_1(J)_{\mathbb{Q}} \to \mathrm{CH}_1(J)_{\mathbb{Q}}$ is an isomorphism. Therefore $[\iota_e(C)] - [\iota_{e'}(C)]$ is torsion. Hence $\kappa_{C,e} - \kappa_{C,e'}$ is torsion too. □

Let $\kappa(C)$ be the image of $\kappa_{C,e}$ in $\mathrm{CH}_1(J)_{\mathbb{Q}}$ for any choice of degree-1 divisor e on C such that $(2g-2)e = K_C$ in $\mathrm{CH}_0(C)_{\mathbb{Q}}$. Lemma 2.6 shows that this class is independent of the choice of e. (If k is not algebraically closed and no such e exists over k, there exists a unique class $\kappa(C) \in \mathrm{CH}_1(J)_{\mathbb{Q}}$ such that $\kappa(C)_{\bar{k}} = \kappa(C_{\bar{k}})$ in $\mathrm{CH}_1(J_{\bar{k}})_{\mathbb{Q}}$, since Chow groups with $\mathbb{Q}$-coefficients satisfy Galois descent.) We let $\bar{\kappa}(C)$ be the image of $\kappa(C)$ in $\mathrm{Gr}_1(J)_{\mathbb{Q}}$. Since all degree-1 divisors e on C are algebraically equivalent, $\bar{\kappa}(C)$ is also the image of $\kappa_e(C)$ in $\mathrm{Gr}_1(J)_{\mathbb{Q}}$ for *any* degree-1 divisor, not necessarily with the property that $(2g-2)e = K_C$ in $\mathrm{CH}_0(C)_{\mathbb{Q}}$.

Now suppose $(2g-2)e = K_C$ in $\mathrm{CH}_0(C)_{\mathbb{Q}}$ and use the Beauville decomposition (2.2) to write $[\iota_e(C)] = \sum_{s=0}^{g-1}[C]_s$ with $[C]_s \in \mathrm{CH}^{g-1}_{(s)}(J)$. Since $[\iota_e(C)] \in \mathrm{CH}_1(J)_{\mathbb{Q}}$ is independent of the choice of e, the same is true for the classes $[C]_s$. We have $\kappa(C) = 2[C]_1 + 2[C]_3 + \cdots + 2[C]_{2\lfloor\frac{g-2}{2}\rfloor+1}$.

Proposition 2.7. *In the above notation, $\kappa(C) = 0$ in $\mathrm{CH}_1(J)_{\mathbb{Q}}$ if and only if $[C]_1 = 0$ in $\mathrm{CH}_1(J)_{\mathbb{Q}}$ if and only if $[C]_s = 0$ for all $s \geq 1$. Moreover $\bar{\kappa}(C) = 0$ in $\mathrm{Gr}_1(J)_{\mathbb{Q}}$ if and only if $[C]_1 \in \mathrm{Gr}_1(J)_{\mathbb{Q}}$ if and only if $[C]_s = 0$ in $\mathrm{Gr}_1(J)_{\mathbb{Q}}$ for all $s \geq 1$.*

Proof. To prove the claims in the first sentence, it suffices to prove $[C]_1 = 0$ implies $[C]_s = 0$ for all $s \geq 1$. This follows from the third centered equation of [41, Theorem 1.5.5]. The proof for $\bar{\kappa}(C)$ is identical. □

3. Vanishing criteria in the Chow and Griffiths group

The proof of Theorem A is quite short, but uses in a crucial way Kimura's notion of finite dimensional Chow motives [24].[2]

Proof of Theorem A. We use the definitions and notations of §2.3 and §2.4. Choose a degree-1 divisor e such that $(2g-2)e = K_C$ in $\mathrm{CH}_0(C)_{\mathbb{Q}}$ and decompose $[\iota_e(C)] = \sum_{s=-1}^{g-1}[C]_s$ with $[C]_s \in \mathrm{CH}^{g-1}_{(s)}(J)$. The finite group $G = \mathrm{Aut}(C)$ acts on J and hence on the motive $\mathfrak{h}(J)$. Since the G-action on J commutes with (n) for every n, we obtain an induced G-action on $\mathfrak{h}^{2g-3}(J)$. Since the class $[C]_1$ is independent of the choice of e (by Lemma 2.6), it is G-invariant. Therefore $[C]_1 \in \mathrm{CH}^{g-1}_{(1)}(J)^G = \mathrm{CH}^{g-1}(\mathfrak{h}^{2g-3}(J))^G = \mathrm{CH}^{g-1}(\mathfrak{h}^{2g-3}(J)^G)$.

By the hypothesis and the Lefschetz isomorphism, $\mathrm{H}^*(\mathfrak{h}^{2g-3}(J)^G) = \mathrm{H}^{2g-3}(J)^G = 0$. Since any summand of the motive of an abelian variety is finite-dimensional in the sense of Kimura [24, Example 9.1], it follows from Kimura's [24, Corollary 7.3] that $\mathfrak{h}^{2g-3}(J)^G = 0$ and hence $[C]_1 = 0$ in $\mathrm{CH}^{g-1}(J)_{\mathbb{Q}}$. By Proposition 2.7, we conclude that $\kappa(C) = 0$. □

Example 3.1. There is exactly one non-hyperelliptic genus 3 curve over $\mathbb{C}$ for which the criterion of [34] applies, namely the curve $y^3 = x^4 + 1$. The curve $y^3 = x^4 + x$ satisfies the weaker hypothesis of Theorem A (as was observed in [6]), so we deduce that $\kappa(C) = 0$ for this curve as well. Beauville and Schoen studied the specific geometry of this curve and showed that $\bar{\kappa}(C) = 0$ [7].

Example 3.2. The genus 4 curve $y^3 = x^5 + 1$ satisfies $\mathrm{H}^3(J)^{\mathrm{Aut}(C)} = 0$ [27, proof of Theorem 3.3], so $\kappa(C) = 0$.

Theorem B will follow from the next proposition. In that proposition and its proof, if X is a nice variety over $\mathbb{C}$ we write $\mathrm{H}^*(X)$ for the singular cohomology of $X(\mathbb{C})$ with $\mathbb{Q}$-coefficients, seen as an object in the category of Hodge structures.

Proposition 3.3. *Let C be a smooth, projective, integral curve over $\mathbb{C}$ with Jacobian J and let $G \subset \mathrm{Aut}(C)$ be a subgroup with $\mathrm{H}^0(J, \Omega^3_J)^G = 0$. Then there exists an abelian variety $A/\mathbb{C}$ such*

[2] Congling Qiu and Wei Zhang have found a very similar proof (personal communication).

that $\mathrm{H}^3(J)^G \simeq \mathrm{H}^1(A)(-1)$*. If the Hodge conjecture holds for* $J \times A$*, then* $\mathfrak{h}^3(J)^G \simeq \mathfrak{h}^1(A)(-1)$ *and* $\bar{\kappa}(C) = 0$*.*

Proof. Let $N^1\,\mathrm{H}^3(J)$ be the largest sub-Hodge structure of $\mathrm{H}^3(J)$ of type $(1,2)+(2,1)$. The polarization on $\mathrm{H}^1(J)$ induces a polarization on $N^1\,\mathrm{H}^3(J)$ and so $N^1\,\mathrm{H}^3(J) \simeq \mathrm{H}^1(B)(-1)$ for some abelian variety $B/\mathbb{C}$. The assumptions and the Hodge decomposition imply that $\mathrm{H}^3(J)^G$ is a Hodge structure of type $(1,2)+(2,1)$. It follows that $\mathrm{H}^3(J)^G \subset N^1\,\mathrm{H}^3(J)$, so there exists an abelian subvariety $A \subset B$ with $\mathrm{H}^3(J)^G \simeq \mathrm{H}^1(A)(-1)$.

We now show that the Hodge conjecture for $J \times A$ implies the claims of the final sentence. Fix mutually inverse isomorphisms of Hodge structures $\phi\colon \mathrm{H}^3(J)^G \to \mathrm{H}^1(A)(-1)$ and $\psi\colon \mathrm{H}^1(A)(-1) \to \mathrm{H}^3(J)$. By the Hodge conjecture, there exist morphisms of motives $\Phi\colon \mathfrak{h}^3(J)^G \to \mathfrak{h}^1(A)(-1)$ and $\Psi\colon \mathfrak{h}^1(A)(-1) \to \mathfrak{h}^3(J)^G$ (in other words, cycles on $J \times A$ with certain properties) with $\mathrm{H}^*(\Phi) = \phi$ and $\mathrm{H}^*(\Psi) = \psi$. Since $\mathfrak{h}^1(A)(-1)$ and $\mathfrak{h}^3(J)^G$ are Kimura finite-dimensional [24, Example 9.1] and $\mathrm{H}^*(\Phi) \circ \mathrm{H}^*(\Psi)$ and $\mathrm{H}^*(\Psi) \circ \mathrm{H}^*(\Phi)$ are the identity, it follows from [24, Proposition 7.2(ii)] (see also [3, Corollaire 3.16]) that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are themselves isomorphisms. Hence Φ and Ψ are isomorphisms too and we conclude that $\mathfrak{h}^3(J)^G \simeq \mathfrak{h}^1(A)(-1)$.

Using the Lefschetz isomorphism $\mathfrak{h}^{2g-3}(J) \simeq \mathfrak{h}^3(J)(-g+3)$ of [37, Theorem 5.2(iii)], we obtain an isomorphism $\mathfrak{h}^{2g-3}(J)^G \simeq \mathfrak{h}^1(A)(-g+2)$. Similarly to the proof of Theorem A, we decompose $[\iota_e(C)] = \sum_{s=0}^{g-1}[C]_s$ with $[C]_s \in \mathrm{CH}^{g-1}_{(s)}(J)$ and observe that the class $[C]_1$ lies in $\mathrm{CH}^{g-1}_{(1)}(J)^G = \mathrm{CH}^{g-1}(\mathfrak{h}^{2g-3}(J)^G)$. Lemma 2.4 combined with the isomorphism $\mathfrak{h}^{2g-3}(J)^G \simeq \mathfrak{h}^1(A)(-g+2)$ shows that $\mathrm{CH}^{g-1}(\mathfrak{h}^{2g-3}(J)^G) = \mathrm{CH}^{g-1}(\mathfrak{h}^{2g-3}(J)^G)_{\mathrm{alg}}$, hence every element of $\mathrm{CH}^{g-1}_{(1)}(J)^G$ lies in $\mathrm{CH}^{g-1}(J)_{\mathrm{alg},\mathbb{Q}}$. Therefore $[C]_1 \in \mathrm{CH}^{g-1}(J)_{\mathrm{alg},\mathbb{Q}}$, hence the image of $[C]_1$ in $\mathrm{Gr}_1(J)_{\mathbb{Q}}$ vanishes. Proposition 2.7 then implies that $\bar{\kappa}(C)$ vanishes too. □

Proof of Theorem B. Since C can be defined over a countable field and since such a field can be embedded in $\mathbb{C}$, Lemma 2.2 shows that we may assume $k = \mathbb{C}$. We conclude by Proposition 3.3 applied to $G = \mathrm{Aut}(C)$. □

Example 3.4. Let $C\colon y^3 = x^4 + ax^2 + bx + c$ be a Picard curve. Then $\mathrm{H}^0(C, \Omega^1_C) \simeq \chi \oplus \chi \oplus \chi^2$ as C_3-representations, where χ is a character of order 3. It follows that $\mathrm{H}^0(J, \Omega^3_J)^{\mathrm{Aut}(C)} = 0$, so that the condition of Theorem B is satisfied. We consider these curves in detail in the next section.

We exhibit two families of examples of Theorem B in genus 4. It would be interesting to study these families in more detail (along the lines of what we do in the next section for Picard curves).

Example 3.5. Consider for every $t \in \mathbb{C} \setminus \{0, 1, -1\}$ the μ_5-cover of $\mathbb{P}^1$ with equation

$$C\colon y^5 = x^4(x+1)^3(x-1)^2(x-t).$$

By [29, Lemma 2.7], the μ_5-representation $\mathrm{H}^0(C, \Omega_C)$ is the direct sum of the four nontrivial characters. It follows that $\mathrm{H}^0(J, \Omega^3_J)^{\mu_5} = 0$ and $\mathrm{H}^3(J)^{\mu_5}$ is isomorphic to $\mathrm{H}^1(A)(-1)$, for some abelian surface A (using Proposition 3.3). We have $D_5 \subset \mathrm{Aut}(C)$ by [21, Corollary 1]. By the Kani-Rosen formula [22, Theorem B], the Jacobian J is isogenous to a product of abelian surfaces of the form $\mathrm{Jac}(C/\tau)$, for some involution τ. The Hodge conjecture in any codimension is known for products of abelian surfaces [35, Theorem 3.15], so $\bar{\kappa}(C) = 0$ by Proposition 3.3.

Example 3.6. The general μ_3-cover of $\mathbb{P}^1$ with equation

$$y^3 = (x^2-1)^2(x^3 + ax^2 + bx + c),$$

has genus 4 and there is an isomorphism of μ_3-representations $\mathrm{H}^0(C, \Omega_C) \simeq \chi \oplus \chi \oplus \chi^2 \oplus \chi^2$. It follows that $\mathrm{H}^0(J, \Omega^3_J)^{\mu_3} = 0$, and the abelian variety A from Proposition 3.3 is 4-dimensional. If the Hodge conjecture holds for the 8-dimensional $J \times A$, then $\bar{\kappa}(C) = 0$.

4. Picard curves

4.1. **Generalities.** Let k be a field of characteristic zero. A Picard curve over k is by definition a nice curve with an affine model $y^3 = f(x) = x^4 + ax^2 + bx + c$ for some $a, b, c \in k$. Conversely, given such a polynomial $f(x) \in k[x]$ of nonzero discriminant, the projective closure of $y^3 = f(x)$ in $\mathbb{P}^2_k$ is a nice curve denoted by C_f. It has a unique point at infinity P_∞, which is k-rational.

For every third root of unity $\omega \in \bar{k}$, the map $(x, y) \mapsto (x, \omega y)$ defines an automorphism of $C_{f,\bar{k}}$. We view μ_3 as a subgroup of $\mathrm{Aut}(C_{f,\bar{k}})$ in this way. Then μ_3 also acts on the Jacobian $J_{f,\bar{k}}$ by taking images of divisors.

The discriminant of f has the following expression:

$$\mathrm{disc}(f) = -4a^3b^2 - 27b^4 + 16a^4c + 144ab^2c - 128a^2c^2 + 256c^3. \tag{4.1}$$

We view f as the dehomogenization $F(x, 1)$ of the quartic form $F(X, Z) = X^4+aX^2Z^2+bXZ^3+cZ^4$, and we define $I(f)$ and $J(f)$ to be the usual I- and J-invariants attached to F, as in [9, §2]. Their explicit formulae in our case are:

$$\begin{aligned} I(f) &= a^2 + 12c, \\ J(f) &= 72ac - 2a^3 - 27b^2. \end{aligned}$$

The 19th century invariant theorists observed the identity $J(f)^2 = 4I(f)^3 - 27 \cdot \mathrm{disc}(f)$, which can be verified by direct computation. Therefore $P_f := (I(f), J(f))$ is a k-point on the elliptic curve

$$E_f \colon y^2 = 4x^3 - 27 \cdot \mathrm{disc}(f).$$

4.2. **Ceresa vanishing criteria.** Since $(2g-2)P_\infty = 4P_\infty$ is canonical, we may use P_∞ to embed C_f in its Jacobian J_f and define the Ceresa cycle $\kappa_{C_f,P_\infty} \in \mathrm{CH}_1(J_f)$ as in the introduction; we denote it by κ_f for simplicity. Recall that $\kappa(C_f)$ denotes the image of κ_f in $\mathrm{CH}_1(J_f)_\mathbb{Q}$ and $\bar{\kappa}(C_f)$ its image in $\mathrm{Gr}_1(J_f)_\mathbb{Q}$. Theorem C follows from the following slightly stronger theorems, whose proofs will take up the rest of this section.

Theorem 4.1. *There exists an integer $N \geq 1$ (depending neither on f nor k) such that $N \cdot \kappa_f \in \mathrm{CH}^2(J_f)_{\mathrm{alg}}$ for every Picard curve C_f over every algebraically closed field k of characteristic zero.*

Theorem 4.2. *The Ceresa cycle $\kappa_f \in \mathrm{CH}_1(J)$ is torsion if and only if $P_f \in E_f(k)$ is torsion. Moreover, if k is algebraically closed, there exists an integer $M \geq 1$ with the following property: if C_f is a Picard curve and κ_f is torsion, then $\mathrm{ord}(\kappa_f)$ divides $M \cdot \mathrm{ord}(P_f)$ and $\mathrm{ord}(P_f)$ divides $M \cdot \mathrm{ord}(\kappa_f)$.*

Theorem 4.1 will be proven in §4.4, and Theorem 4.2 will be proven in §4.7. A standard argument using Lemma 2.2 shows that we may assume $k = \mathbb{C}$. So in the remainder of §4, all varieties will be over $\mathbb{C}$, and cohomology will be singular cohomology.

Remark 4.3. Theorem 4.2 generalizes [25, Theorem 5.16], which considered the special case where $b = 0$. There, we exploited the bielliptic cover to show that the Ceresa cycle maps via a correspondence to a multiple of the point $Q_f = (a^2 - 4c, a(a^2 - 4c))$ on the elliptic curve $E'_f \colon y^2 = x^3 + 16 \cdot \mathrm{disc}(f)$. This is compatible with the general case since there is a 3-isogeny $\phi_f \colon E'_f \to E_f$, and one checks using the explicit formula for ϕ_f [8, Equation (2)] that $\phi_f(Q_f) = P_f$.

Remark 4.4. Is it always the case that $\kappa_f \neq 0$? (Recall that κ_f lies in the Chow group with $\mathbb{Z}$-coefficients.) We cannot conclude this from our proof of Theorem 4.2 below since we have worked with $\mathbb{Q}$-coefficients, and we make use of various isogenies whose degrees we do not control.

4.3. **Multilinear algebra.** Our first goal (Proposition 4.7) is to explicitly identify the abelian variety A of Proposition 3.3 for Picard curves.

Write $\mathcal{O} = \mathbb{Z}[\omega]$ for the ring of Eisenstein integers with $\omega^2 + \omega + 1 = 0$ and let $K = \mathbb{Q}(\sqrt{-3})$ be its fraction field. Let C be a Picard curve over $\mathbb{C}$ with Jacobian variety J. The μ_3-action on C extends to an embedding $\mathcal{O} \subset \mathrm{End}(J)$. Using this action, the singular cohomology group $\mathrm{H}^1(J;\mathbb{Z})$ is a free $\mathcal{O}$-module of rank 3, and $\mathrm{H}^1(J;\mathbb{Q})$ is a 3-dimensional K-vector space. The next lemma says that the criterion of Theorem B is always satisfied for Picard curves.

Lemma 4.5. $\mathrm{H}^0(J, \Omega^3_J)^{\mu_3} = 0$ *and the Hodge structure* $\mathrm{H}^3(J;\mathbb{Q})^{\mu_3}$ *is of type* $(1,2)+(2,1)$.

Proof. Since $\mathrm{H}^0(J,\Omega^3_J) \simeq \bigwedge^3 \mathrm{H}^0(C,\Omega^1_C)$, the first claim follows from a calculation with differentials (Example 3.4). The second claim follows from the Hodge decomposition for $\mathrm{H}^3(J;\mathbb{Q})$. □

We may view $\mathrm{H}^1(J;\mathbb{Q})$ either as a K-vector space or $\mathbb{Q}$-vector space; when we perform tensor operations, we will add the subscript K when we view it as a K-vector space, and add no subscript otherwise. For example, cup product induces an isomorphism $\bigwedge^3 \mathrm{H}^1(J;\mathbb{Q}) \simeq \mathrm{H}^3(J;\mathbb{Q})$, and we will use this identification without further mention.

The universal property of exterior powers induces a canonical $\mathbb{Q}$-linear surjection $\bigwedge^3 \mathrm{H}^1(J;\mathbb{Q}) \to \bigwedge^3_K \mathrm{H}^1(J;\mathbb{Q})$. It is well known (see [30, Lemma 12(i)] or [16, Lemma 4.3]) that this map admits a canonical splitting, which we use to view $\bigwedge^3_K \mathrm{H}^1(J;\mathbb{Q})$ as a direct summand of $\mathrm{H}^3(J;\mathbb{Q})$.

Lemma 4.6. *We have* $\mathrm{H}^3(J;\mathbb{Q})^{\mu_3} = \bigwedge^3_K \mathrm{H}^1(J;\mathbb{Q})$ *inside* $\mathrm{H}^3(J;\mathbb{Q})$. *Moreover* $\dim_{\mathbb{Q}} \mathrm{H}^3(J;\mathbb{Q})^{\mu_3} = 2$.

Proof. It suffices to prove the statements after tensoring with $\mathbb{C}$. Let $g \in \mu_3 \subset \mathrm{Aut}(C)$ be a nontrivial element. The action of g on $\mathrm{H}^1(J;\mathbb{Q})$ has eigenvalues (with multiplicity) $\omega,\omega,\omega,\omega^2,\omega^2,\omega^2$ so we can write $\mathrm{H}^1(J;\mathbb{C}) = V_1 \oplus V_2$ where V_i is the ω^i-eigenspace. Since a three element subset of these eigenvalues have product 1 if and only if they are all equal, $\mathrm{H}^3(J;\mathbb{C})^{\mu_3} = (\bigwedge^3 V_1) \oplus (\bigwedge^3 V_2)$. On the other hand, an argument similar to the proof of [16, Proposition 4.4] shows $(\bigwedge^3_K \mathrm{H}^1(J;\mathbb{Q})) \otimes_K \mathbb{C} = \bigwedge^3_{K\otimes\mathbb{C}} \mathrm{H}^1(J;\mathbb{C}) = (\bigwedge^3 V_1) \oplus (\bigwedge^3 V_2)$, proving the equality. The explicit description of this subspace shows that it is 2-dimensional over $\mathbb{Q}$. □

Let E be the elliptic curve with Weierstrass equation $y^2 = x^3 + 1$.

Proposition 4.7. *There is an isomorphism of Hodge structures* $\mathrm{H}^3(J;\mathbb{Q})^{\mu_3} \simeq \mathrm{H}^1(E,\mathbb{Q})(-1)$.

Proof. Since there exists a unique elliptic curve up to isogeny with endomorphism algebra K, there exists a unique Hodge structure of dimension 2, type $(0,1)+(1,0)$ and carrying an action of K. Since both $\mathrm{H}^3(J;\mathbb{Q})^{\mu_3}$ and $\mathrm{H}^1(E;\mathbb{Q})$ have these properties (the former by Lemma 4.6), they must be isomorphic, and we conclude using Lemma 4.6. □

4.4. **Weil classes and Schoen's theorem.** Our next goal is to upgrade the isomorphism of Proposition 4.7 to an isomorphism in the category of Chow motives, and hence deduce (using Proposition 3.3) the vanishing of $\bar{\kappa}(C)$ in the Griffiths group. We use the following special case of a result of Schoen, which crucially uses the assumption that the endomorphism algebra of $J \times E$ contains $K = \mathbb{Q}(\omega)$:

Theorem 4.8 (Schoen)**.** *The Hodge conjecture holds for* $J \times E$.

Proof. Since $J \times E$ is four-dimensional, it suffices to prove Hodge classes in $\mathrm{H}^4(J \times E;\mathbb{Q})$ are algebraic. By [31, Theorem (0.1), Part (i) and (iv)], such Hodge classes are sums of products of divisor classes (which are algebraic) and Weil classes $W_K := \bigwedge^4_K \mathrm{H}^4(J \times E;\mathbb{Q}) \subset \mathrm{H}^4(J \times E;\mathbb{Q})$. Since $K = \mathbb{Q}(\omega)$ and the embedding $K \subset \mathrm{End}(J \times E) \otimes \mathbb{Q}$ can be chosen to have signature $(2,2)$, Schoen has shown in [36] that the classes in W_K are algebraic, concluding that all Hodge classes of $\mathrm{H}^4(J \times E; \mathbb{Q})$ are algebraic. ☐

Recall from §2.3 our conventions on motives, the canonical Chow–Künneth components $\mathfrak{h}^i(J)$ and $\mathfrak{h}^j(E)$, and the motive of fixed points of a finite group action.

Corollary 4.9. *There is an isomorphism $\mathfrak{h}^3(J)^{\mu_3} \simeq \mathfrak{h}^1(E)(-1)$.*

Proof. This follows from Proposition 3.3, using Proposition 4.7 and Theorem 4.8. ☐

Corollary 4.10. *If C is a Picard curve over $\mathbb{C}$, then $\bar{\kappa}(C) = 0$.*

Proof. This follows from Proposition 3.3, using Proposition 4.7 and Theorem 4.8. ☐

Proof of Theorem 4.1. Consider the parameter space $S_0 = \{(a, b, c) \mid \mathrm{disc}(f) \neq 0\} \subset \mathbb{A}^3_{\mathbb{Q}}$ of Picard curves. Let η be the generic point of S_0 with function field $k(\eta) = \mathbb{Q}(a, b, c)$, let C_η be the generic Picard curve over $k(\eta)$ with Jacobian J_η, and let $\kappa_{C_\eta} \in \mathrm{CH}_1(J_\eta)$ be the Ceresa cycle of C_η based at the point at infinity. Fix an embedding $j\colon k(\eta) \to \mathbb{C}$. By Corollary 4.10, the base change of $\kappa_{C\eta}$ along j is torsion in the Griffiths group. By Lemma 2.2, this implies κ_{C_η} is itself torsion in $\mathrm{Gr}_1(J_\eta)$. Let N be the finite order of κ_{C_η} in $\mathrm{Gr}_1(J_\eta)$. A specialization argument (using Lemmas 2.1 and 2.2 and [39, Tag 054F]) shows that $N \cdot \kappa_C$ is algebraically trivial for every Picard curve over every algebraically closed field of characteristic zero, as required. ☐

The remainder of the section is devoted to proving Theorem 4.2. To this end, we will analyze the Abel–Jacobi image of κ_f (the "normal function" associated to κ_f) in the next two subsections.

4.5. **The Abel–Jacobi map.** If X is a smooth variety over $\mathbb{C}$, we will use the notion of an integral (respectively rational) variation of (pure) Hodge structures over the complex manifold $X(\mathbb{C})$, called a $\mathbb{Z}$-VHS (respectively $\mathbb{Q}$-VHS) for short; see [40, §5.3.1] for definitions. If V is a Hodge structure of odd weight $2k-1$, its intermediate Jacobian $\mathrm{J}(V)$ is the complex torus

$$(V \otimes_{\mathbb{Z}} \mathbb{C})/(F^k + V_\tau),$$

where $F^k \subset V \otimes_{\mathbb{Z}} \mathbb{C}$ is a part of the descending Hodge filtration and V_τ denotes the quotient of V by its torsion subgroup. More generally, if $\mathbb{V}$ is a $\mathbb{Z}$-VHS of weight $2k-1$ over $X(\mathbb{C})$, we can define its intermediate Jacobian $\mathrm{J}(\mathbb{V}) \to X(\mathbb{C})$, a relative complex torus whose fibers over points $x \in X(\mathbb{C})$ are the classical intermediate Jacobians $\mathrm{J}(\mathbb{V}_x)$, see [40, §7.1.1]. If $\mathbb{V}, \mathbb{W}$ are two $\mathbb{Z}$-VHS of odd weight, then $\mathrm{J}(\mathbb{V}(p)) = \mathrm{J}(\mathbb{V})$ for all $p \in \mathbb{Z}$ and a morphism $\mathbb{V} \to \mathbb{W}$ of $\mathbb{Z}$-VHS induces a homomorphism of (relative) complex tori $\mathrm{J}(\mathbb{V}) \to \mathrm{J}(\mathbb{W})$.

If $X/\mathbb{C}$ is a nice variety and $0 \leq p \leq \dim(X)$, we write $\mathrm{J}^p(X) = \mathrm{J}(\mathrm{H}^{2p-1}(X(\mathbb{C}); \mathbb{Z}))$. In this situation there is an Abel–Jacobi map

$$\mathrm{AJ}^p_X\colon \mathrm{CH}^p(X)_{\mathrm{hom}} \to \mathrm{J}^p(X), \tag{4.2}$$

defined in [40, §7.2.1]. Moreover, if $S/\mathbb{C}$ is a smooth variety, $\pi\colon X \to S$ a smooth projective morphism with geometrically integral fibres and $p \in \mathbb{Z}_{\geq 0}$, then $R^p\pi_*\mathbb{Z}$ (pushforward of the constant sheaf in the analytic topology) has the structure of a $\mathbb{Z}$-VHS over $S(\mathbb{C})$. If Z is a codimension p cycle on X all of whose components are flat over S, and such that $Z_s \in \mathrm{CH}^p(X_s)_{\mathrm{hom}}$ for every $s \in S(\mathbb{C})$, then Griffiths has shown that there exists a holomorphic section $\mathrm{AJ}(Z)$ of the relative complex torus $\mathrm{J}(R^p\pi_*\mathbb{Z}) \to S(\mathbb{C})$ with the property that $\mathrm{AJ}(Z)_s = \mathrm{AJ}^p_X(Z_s)$ for all $s \in S(\mathbb{C})$; this is called the normal function associated to Z.

We record the fact that Abel–Jacobi maps are compatible with correspondences. Let X, Y be nice varieties over $\mathbb{C}$ and let $\gamma \in \mathrm{CH}^{r+\dim(X)}(X \times Y)$ be a correspondence of degree r. This induces for every $p \geq 0$ a homomorphism $\gamma_*\colon \mathrm{CH}^p(X) \to \mathrm{CH}^{p+r}(Y)$ via the formula $\alpha \mapsto \pi_{Y,*}(\pi_X^*(\alpha) \cdot \gamma)$, where

$\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ denote the projections. The same formula defines morphism of Hodge structures $\mathrm{H}^p(X;\mathbb{Z}) \to \mathrm{H}^{p+2r}(Y)(r)$ for every p, hence a homomorphism of complex tori $\gamma_*\colon \mathrm{J}^p(X) \to \mathrm{J}^{p+r}(Y)$.

Lemma 4.11. *In the above notation, γ_* sends $\mathrm{CH}^p(X)_{\mathrm{hom}}$ to $\mathrm{CH}^{p+r}(Y)_{\mathrm{hom}}$. Moreover for every $\alpha \in \mathrm{CH}^p(X)_{\mathrm{hom}}$, $\gamma_*(\mathrm{AJ}^p_X(\alpha)) = \mathrm{AJ}^{p+r}_Y(\gamma_*(\alpha))$.*

Proof. The first sentence follows from the compatibility of the cycle class map with correspondences [40, Proposition 9.21]. To prove the compatibility of the Abel–Jacobi map with correspondences, it suffices to prove the compatibility with pullbacks, pushforwards and intersection product. The case of pullback is elementary, using the definition of AJ^p_X in terms of extensions of Hodge structures, see [14, §2.2]. The case of pushforward follows from that of pullback and Poincare duality. Finally, compatibility with intersection product follows from [40, Proposition 9.23]. □

Let $S = \{(a,b,c) \mid \mathrm{disc}(f) \neq 0\} \subset \mathbb{A}^3_{\mathbb{C}}$ be the parameter space of Picard curves over $\mathbb{C}$. We will identify $\mathbb{C}$-valued points of S with polynomials $f = x^4 + ax^2 + bx + c \in \mathbb{C}[x]$ of nonzero discriminant. Let $\mathcal{C} \to S$ be the universal Picard curve, and let $\pi\colon \mathcal{J} \to S$ its relative Jacobian variety. The point at infinity defines a section P_∞ of $\mathcal{C}$. Let $\kappa_{\mathcal{C}} \in \mathrm{CH}_1(\mathcal{J})$ be the universal Ceresa cycle with respect to this section. (Comparing with our earlier notation, we have $\mathcal{C}_f = C_f$ and $\kappa_{\mathcal{C},f} = \kappa_f$ for every $f \in S(\mathbb{C})$.)

Let $\mathbb{V} = R^3\pi_*\mathbb{Z}$ be the $\mathbb{Z}$-VHS on $S(\mathbb{C})$ interpolating the cohomology groups $\mathrm{H}^3(\mathcal{J}_f;\mathbb{Z})$ for $f \in S(\mathbb{C})$. Then the normal function $\mathrm{AJ}(\kappa_{\mathcal{C}})$ is a section of $\mathrm{J}(\mathbb{V}) \to S(\mathbb{C})$ interpolating $\mathrm{AJ}^2_{\mathcal{J}_f}(\kappa_f)$ for all $f \in S(\mathbb{C})$.

Proposition 4.12. *If $f \in S(\mathbb{C})$, then $\kappa_f \in \mathrm{CH}^2(J_f)_{\mathrm{hom}}$ is torsion if and only if $\mathrm{AJ}^2_{J_f}(\kappa_f) \in \mathrm{J}^2(J_f)$ is torsion. The torsion order of κ_f, if finite, equals the torsion order of $\mathrm{AJ}^2_{J_f}(\kappa_f)$.*

Proof. Let E be the elliptic curve $y^2 = x^3+1$. Corollary 4.9 shows that there exists a correspondence $\gamma \in \mathrm{CH}^2(E \times J)$ such that γ_* induces isomorphisms $\mathrm{CH}^1(E)_{\mathbb{Q}} \to \mathrm{CH}^2(J)^{\mu_3}_{\mathbb{Q}}$ and $\mathrm{H}^1(E;\mathbb{Q}) \to \mathrm{H}^3(J;\mathbb{Q})^{\mu_3}(1)$. By Lemma 4.11 these form a commutative diagram:

$$\begin{tikzcd}
\mathrm{CH}^1(E)_{\mathrm{hom},\mathbb{Q}} \arrow[r, "\mathrm{AJ}"] \arrow[d, "\gamma_*"] & \mathrm{J}^1(E) \otimes \mathbb{Q} \arrow[d, "\gamma_*"] \\
\mathrm{CH}^2(J_f)^{\mu_3}_{\mathrm{hom},\mathbb{Q}} \arrow[r, "\mathrm{AJ}"] & \mathrm{J}^2(J_f)^{\mu_3} \otimes \mathbb{Q}
\end{tikzcd}$$

All arrows except the horizontal bottom one are isomorphisms of abelian groups. The same is therefore true for the bottom one $\mathrm{AJ}\colon \mathrm{CH}^2(J_f)^{\mu_3}_{\mathrm{hom},\mathbb{Q}} \to \mathrm{J}^2(J_f) \otimes \mathbb{Q}$. Since $\kappa_f \in \mathrm{CH}^2(J_f)^{\mu_3}_{\mathrm{hom}}$, we conclude that κ_f is torsion if and only if $\mathrm{AJ}^2_{J_f}(\kappa_f)$ is. The last claim follows from the fact that $\mathrm{AJ}^2_{J_f}$ is injective on torsion subgroups by a result of Murre [32, Theorem 10.3]. □

4.6. **Identifying the complex torus $\mathrm{J}(\mathbb{V}^{\mu_3})$.** The μ_3-action on $\mathcal{C}$ induces, via functoriality, a μ_3-action on $\mathcal{J}$, $\mathbb{V}$ and $\mathrm{J}(\mathbb{V})$. The subsheaf of fixed points $\mathbb{V}^{\mu_3}$ has the structure of a $\mathbb{Z}$-VHS. The connected component of the identity $\mathrm{J}(\mathbb{V})^{\mu_3,\circ}$ of $\mathrm{J}(\mathbb{V})^{\mu_3}$ is a relative complex torus over $S(\mathbb{C})$. Moreover, the natural homomorphism $\mathrm{J}(\mathbb{V}^{\mu_3}) \to \mathrm{J}(\mathbb{V})$ induces an isomorphism onto $\mathrm{J}(\mathbb{V})^{\mu_3,\circ}$.

Lemma 4.13. *The multiple $3 \cdot \mathrm{AJ}(\kappa_{\mathcal{C}})$ lands in $\mathrm{J}(\mathbb{V})^{\mu_3,\circ}$.*

Proof. Since the Ceresa cycle $\kappa_{\mathcal{C}}$ is μ_3-invariant, $\mathrm{AJ}(\kappa_{\mathcal{C}})$ lands in $\mathrm{J}(\mathbb{V})^{\mu_3}$. The norm map $\mathrm{N}\colon \mathrm{J}(\mathbb{V}) \to \mathrm{J}(\mathbb{V})$, defined by $x \mapsto x + \omega \cdot x + \omega^2 \cdot x$, lands in $\mathrm{J}(\mathbb{V})^{\mu_3,\circ}$, since the image must be connected and μ_3-invariant. We conclude that $\mathrm{N}(\mathrm{AJ}(\kappa_{\mathcal{C}})) = 3 \cdot \mathrm{AJ}(\kappa_{\mathcal{C}})$ lands in $\mathrm{J}(\mathbb{V})^{\mu_3,\circ}$. □

Therefore $3 \cdot \mathrm{AJ}(\kappa_{\mathcal{C}})$ defines a section of $\mathrm{J}(\mathbb{V})^{\mu_3,\circ}$, hence we may view it as a section of the complex torus $\mathrm{J}(\mathbb{V}^{\mu_3}) \to S(\mathbb{C})$ in what follows. We study this relative complex torus (up to isogeny) in the next two propositions.

Let $\mathcal{E} \to S$ be the relative elliptic curve with Weierstrass equation $y^2 = 4x^3 - 27 \cdot \mathrm{disc}(f)$. Recall that we write $\mathcal{O} = \mathbb{Z}[\omega]$ and $K = \mathbb{Q}(\omega)$.

Proposition 4.14. *The relative complex torus $\mathrm{J}(\mathbb{V}^{\mu_3}) \to S$ is isogenous to the relative complex torus $\mathcal{E}(\mathbb{C}) \to S(\mathbb{C})$.*

Proof. The $\mathbb{Z}$-VHS $R^1\pi_*\mathbb{Z}$ interpolating the cohomology groups $\mathrm{H}^1(\mathcal{J}_f;\mathbb{Z})$ comes equipped with an action of $\mathcal{O}$, and cup product induces an isomorphism $\bigwedge^3 R^1\pi_*\mathbb{Z} \simeq R^3\pi_*\mathbb{Z} = \mathbb{V}$. Let $\mathbb{W} = \bigwedge^3_{\mathcal{O}} R^1\pi_*\mathbb{Z}$ (the third exterior product of $R^1\pi_*\mathbb{Z}$, viewed as a sheaf of $\mathcal{O}$-modules), a $\mathbb{Z}$-VHS with an $\mathcal{O}$-action. Lemma 4.6 shows that $\mathbb{W} \otimes \mathbb{Q} \simeq \mathbb{V}^{\mu_3} \otimes \mathbb{Q}$, so $\mathrm{J}(\mathbb{W})$ and $\mathrm{J}(\mathbb{V}^{\mu_3})$ are isogenous.

To analyze $\mathrm{J}(\mathbb{W})$, we analyze the $\mathbb{Z}$-VHS $\mathbb{W}(1)$ more closely. It has an $\mathcal{O}$-action by construction, Lemma 4.6 shows that it has constant rank 2, and Lemma 4.5 shows that it has type $(1,0)+(0,1)$. Since $\mathcal{O}$ has class number 1, there exists a unique $\mathbb{Z}$-Hodge structure with these properties, hence $\mathbb{W}_f(1) \simeq \mathrm{H}^1(E;\mathbb{Z})$ for every $f \in S(\mathbb{C})$, where E is the elliptic curve with Weierstrass equation $y^2 = x^3 + 1$.

Therefore $\mathrm{J}(\mathbb{W}) \to S(\mathbb{C})$ is an isotrivial family of elliptic curves: analytically locally on $S(\mathbb{C})$, it is isomorphic to $E(\mathbb{C}) \times S(\mathbb{C}) \to S(\mathbb{C})$. It follows that the sheaf of local isomorphisms between $\mathrm{J}(\mathbb{W})$ and $E(\mathbb{C}) \times S(\mathbb{C})$ is an $\mathrm{Aut}(E)$-torsor in the analytic topology on $S(\mathbb{C})$. Since $\mathrm{Aut}(E) \simeq \mu_6$ is finite, this torsor is the analytification of an étale μ_6-torsor on S.

To analyze étale μ_6-torsors on S, consider the following exact sequence induced by the Kummer exact sequence in étale cohomology:

$$\mathbb{G}_m(S) \xrightarrow{(-)^6} \mathbb{G}_m(S) \to \mathrm{H}^1_{\mathrm{et}}(S, \mu_6) \to \mathrm{Pic}(S)[6] \to 0.$$

The Picard group $\mathrm{Pic}(S)$ vanishes, being a quotient of $\mathrm{Pic}(\mathbb{A}^3_{\mathbb{C}})$, hence the outer term in the sequence vanishes. We claim that $\mathbb{G}_m(S) = \mathrm{H}^0(S, \mathcal{O}_S)^\times = \{c \cdot \mathrm{disc}^n \mid c \in \mathbb{C}^\times,\, n \in \mathbb{Z}\}$. Indeed, every $c \cdot \mathrm{disc}^n$ is clearly a unit in $\mathrm{H}^0(S, \mathcal{O}_S)$. Conversely, given a unit $f \in \mathrm{H}^0(S, \mathcal{O}_S)$, seen as a rational function on $\mathbb{A}^3_{\mathbb{C}}$, its divisor $\mathrm{div}(f)$ of zeros and poles must be supported on the zero locus of disc. Since $\mathrm{disc} \in \mathbb{C}[a,b,c]$ is irreducible (a general fact about discriminant polynomials), $\mathrm{div}(f) = n \cdot [\{\mathrm{disc} = 0\}]$ for some $n \in \mathbb{Z}$. Then $\mathrm{div}(f/\mathrm{disc}^n) = 0$ as a rational function on $\mathbb{A}^3_{\mathbb{C}}$, hence f/disc^n is a unit in $\mathbb{C}[a,b,c]$, hence $f/\mathrm{disc}^n \in \mathbb{C}^\times$, proving the claim. We conclude using the exact sequence that the group $\mathrm{H}^1_{\mathrm{et}}(S, \mu_6)$ classifying μ_6-torsors is generated by the image of disc.

Let $\mathcal{E}_i \to S$ be the relative elliptic curve with equation $y^2 = x^3 + \mathrm{disc}(f)^i$. The previous paragraph shows that $\mathrm{J}(\mathbb{W})$ is isomorphic to $\mathcal{E}_i(\mathbb{C})$ for some $i \in \{0,1,2,3,4,5\}$. We show that $i = 1$, using our previous results on *bielliptic* Picard curves in [25]. Let $T \subset S$ be the closed subscheme where $b = 0$, parametrizing even quartic polynomials $f = x^4 + ax^2 + c$. If $f \in T(\mathbb{C})$, the μ_3-action on $\mathcal{C}_f$ extends to a μ_6-action. A calculation shows $\mathrm{disc}|_T = 16c(-a^2+4c)^2$. By applying the singular cohomology realization functor to [25, Theorem 5.1], the relative complex torus $\mathrm{J}((\mathbb{V}|_{T(\mathbb{C})})^{\mu_6}) \to T(\mathbb{C})$ is isogenous to $\mathcal{E}_1(\mathbb{C})|_{T(\mathbb{C})} \to T(\mathbb{C})$. Since $\mathrm{J}((\mathbb{V}|_{T(\mathbb{C})})^{\mu_6})$ is a subtorus of $\mathrm{J}((\mathbb{V}|_{T(\mathbb{C})})^{\mu_3})$ of the same dimension, they must be equal, hence $\mathrm{J}(\mathbb{W})|_{T(\mathbb{C})}$ is isogenous to $\mathcal{E}_1(\mathbb{C})|_{T(\mathbb{C})}$ over $T(\mathbb{C})$. On the other hand, let $i \in \{0,1,2,3,4,5\}$ be such that $\mathrm{J}(\mathbb{W}) \simeq \mathcal{E}_i(\mathbb{C})$. Then $\mathrm{J}(\mathbb{W})|_{T(\mathbb{C})} \simeq \mathcal{E}_i(\mathbb{C})|_{T(\mathbb{C})}$, hence $\mathcal{E}_1(\mathbb{C})|_{T(\mathbb{C})}$ is isogenous to $\mathcal{E}_i(\mathbb{C})|_{T(\mathbb{C})}$.

We show that the latter can happen only if $i = 1$. Indeed, let $\varphi\colon \mathcal{E}_1(\mathbb{C})|_{T(\mathbb{C})} \to \mathcal{E}_i(\mathbb{C})|_{T(\mathbb{C})}$ be an isogeny. Since the domain and target of φ are isotrivial relative elliptic curves with $\mathcal{O}$-multiplication, φ factors as $\psi \circ \gamma$, where γ is an endomorphism of $\mathcal{E}_1(\mathbb{C})|_{T(\mathbb{C})}$ and ψ an isomorphism. Therefore $\mathcal{E}_1(\mathbb{C})|_{T(\mathbb{C})}$ and $\mathcal{E}_i(\mathbb{C})|_{T(\mathbb{C})}$ are isomorphic. Since the monodromy representations of $\mathcal{E}_1(\mathbb{C})|_{T(\mathbb{C})}$ and

$\mathcal{E}_i(\mathbb{C})|_{T(\mathbb{C})}$ are non-isomorphic when $i \neq 1$, we conclude that $i = 1$ and that $\mathrm{J}(\mathbb{W})$ is isogenous to $\mathcal{E}_1(\mathbb{C})$ over $S(\mathbb{C})$.

In summary, we have shown that $\mathrm{J}(\mathbb{V}^{\mu_3})$, $\mathrm{J}(\mathbb{W})$ and $\mathcal{E}_1(\mathbb{C})$ are isogenous. Since $\mathcal{E}_1(\mathbb{C})$ is isomorphic to $\mathcal{E}(\mathbb{C})$, we conclude the proof. $\square$

Proposition 4.15. *The group of (algebraic) sections of $\mathcal{E} \to S$ is free of rank 1 over $\mathcal{O} = \mathbb{Z}[\omega]$, and contains the $\mathcal{O}$-span of $P := (a^2 + 12c, 72ac - 2a^3 - 27b^2)$ as a finite index subgroup.*

Proof. Given $s = (a, b, c) \in S(\mathbb{C})$, consider the closed subscheme $T = \{(at, bt, ct^2) : t \in \mathbb{A}^1_{\mathbb{C}}\} \cap S$ of S and the restriction $\pi : \mathcal{E}|_T \to T$. The variety $\mathcal{E}|_T$ is an open subscheme of an elliptic surface with Weierstrass equation $y^2 = x^3 + t^4 g(t)$, where $g(t) \in \mathbb{C}[t]$ has degree ≤ 2, using the formula (4.1). There exists a dense open $U \subset S$ such that for all $s \in U(\mathbb{C})$, $g(t)$ has two distinct nonzero roots. Let $s \in U(\mathbb{C})$. Using Tate's algorithm, we see that the elliptic surface has a singular fiber above $t = 0$ with Kodaira type IV*, singular fibers above the roots of $g(t)$ with Kodaira type II, and is smooth above $t = \infty$. The presence of a fiber of type II implies that $\mathcal{E}|_T \to T$ has no torsion sections [38, Lemma 7.8]. Moreover the Shioda–Tate formula [38, Theorem 6.3, Proposition 6.6 and §8.8] shows that the group of sections of $\mathcal{E}|_T \to T$ is free of rank 2. Since $\mathcal{E}|_T$ receives an $\mathcal{O}$-action, its group of sections is free of rank 1 over $\mathcal{O}$. We will now show that these facts can be used to prove the claims of the proposition by varying s in $U(\mathbb{C})$.

We first show that $\mathcal{E}(S)$ is torsion-free. Suppose $Q \in \mathcal{E}(S)$ is a torsion section. Then $N \cdot Q = 0$ for some $N \geq 1$. Hence $N \cdot Q|_T = 0$ for every $s \in U(\mathbb{C})$. Since the group of sections of $\mathcal{E}|_T \to T$ is torsion-free, $Q|_T = 0$ for every $s \in U(\mathbb{C})$. Hence $Q|_U = 0$. Since U is dense in S, we must have $Q = 0$, as desired.

The fact that P defines a section of $\mathcal{E} \to S$ can be verified by direct computation (see §4.1). Let $L = \mathbb{Z}\langle P, \omega \cdot P\rangle$ be the $\mathcal{O}$-span of P, which is a subgroup of the group of sections $\mathcal{E}(S)$ of $\mathcal{E} \to S$. The previous paragraph shows that L is free of rank 1 over $\mathcal{O}$. It remains to show that it has finite index in $\mathcal{E}(S)$. Suppose for the sake of contradiction that there exists a third section $Q \in \mathcal{E}(S)$ which is not in $L \otimes \mathbb{Q}$. Then the locus S_{dep} of $s \in S(\mathbb{C})$ where $P_s, \omega \cdot P_s$ and Q_s are linearly dependent is a countable union of closed proper algebraic subvarieties. For every $s \in U(\mathbb{C})$, the group of sections of $\mathcal{E}|_T \to T$ is free of rank 1 over $\mathcal{O}$. There exists a possibly smaller dense open $V \subset U$ such that if $s \in V(\mathbb{C})$ then P_s is nonzero, hence $\langle P_s, \omega \cdot P_s\rangle$ is a finite index subgroup of $\mathcal{E}(T)$. Therefore $Q_t, P_{1,t}, P_{2,t}$ are linearly dependent for all $s \in V(\mathbb{C})$. Since $V(\mathbb{C}) \setminus S_{\mathrm{dep}}$ is nonempty, we obtain a contradiction. $\square$

4.7. **Proof of the vanishing criterion in the Chow group.**

Proof of Theorem 4.2. We may assume (using Lemma 2.2) that $k = \mathbb{C}$. Recall that $3 \cdot \mathrm{AJ}(\kappa_{\mathcal{C}})$ defines a holomorphic section of $\mathrm{J}(\mathbb{V}^{\mu_3}) \to S(\mathbb{C})$. Choose an isogeny of complex tori $\mathrm{J}(\mathbb{V}^{\mu_3}) \to \mathcal{E}(\mathbb{C})$ using Proposition 4.14 and let σ be the image of $3 \cdot \mathrm{AJ}(\kappa_{\mathcal{C}})$ under this isogeny. This is a holomorphic section of $\mathcal{E}(\mathbb{C}) \to S(\mathbb{C})$.

We claim that σ is not a torsion section. If it were torsion, then $\mathrm{AJ}(\kappa_{\mathcal{C}})$ would be a torsion section of $\mathrm{J}(\mathbb{V}^{\mu_3})$, hence, by Proposition 4.12, κ_f would be torsion for every $f \in S(\mathbb{C})$. This is not the case, since κ_f is of infinite order if $f = x^4 + x^2 + 1$ by [25, Corollary 2.9]. We conclude that σ is not a torsion section.

Next we claim that σ is the analytification of an algebraic section of $\mathcal{E} \to S$. Let $N \geq 1$ be an integer such that $N \cdot \kappa_f$ is algebraically trivial for every Picard curve $f \in S(\mathbb{C})$ (such an integer exists by Theorem 4.1). By the algebraicity of the Abel–Jacobi map for algebraically trivial cycles ([1, Theorem 1]), there exists a relative algebraic subtorus $\mathrm{J}_a(\mathbb{V}) \subset \mathrm{J}(\mathbb{V})$ with the following property: the section $\mathrm{AJ}(3N \cdot \kappa_{\mathcal{C}})$ lands in $\mathrm{J}_a(\mathbb{V})$ and the corresponding holomorphic map $S(\mathbb{C}) \to \mathrm{J}_a(\mathbb{V})$ is algebraic. On the other hand, $\mathrm{AJ}(3N \cdot \kappa_{\mathcal{C}})$ also lands in $\mathrm{J}(\mathbb{V}^{\mu_3})$, is not a torsion section by the previous paragraph, and $\mathrm{J}(\mathbb{V}^{\mu_3})$ has relative dimension 1 over $S(\mathbb{C})$. Therefore $\mathrm{J}(\mathbb{V}^{\mu_3})$ is the smallest relative subtorus of $\mathrm{J}(\mathbb{V})$ containing the image of $\mathrm{AJ}(3N\cdot\kappa_{\mathcal{C}})$. Hence $\mathrm{J}(\mathbb{V}^{\mu_3})\subset \mathrm{J}_a(\mathbb{V})$. We conclude that $\mathrm{AJ}(3N\cdot\kappa_{\mathcal{C}})\colon S(\mathbb{C})\to \mathrm{J}(\mathbb{V}^{\mu_3})$ is algebraic, so $\mathrm{AJ}(3\cdot\kappa_{\mathcal{C}})$ is algebraic, so σ is algebraic.

Since σ is an algebraic section of $\mathcal{E}\to S$, Proposition 4.15 shows that there exists an integer $M\geq 1$ and an element $\gamma\in\mathcal{O}$ such that $M\cdot\sigma=\gamma\cdot P$. Since σ is not a torsion section, $\gamma\neq 0$.

Putting everything together, we have for $f\in S(\mathbb{C})$: κ_f is torsion if and only if $\mathrm{AJ}^2_{J_f}(\kappa_f)$ torsion (by Proposition 4.12), if and only if $\mathrm{AJ}(\kappa_{\mathcal{C}})_f\in \mathrm{J}(\mathbb{V}^{\mu_3})_f$ torsion, if and only if $\sigma_f\in\mathcal{E}_f(\mathbb{C})$ torsion, if and only if P_f torsion. Tracing through the equivalences, the quotient of the torsion orders $\mathrm{ord}(\kappa_f)/\mathrm{ord}(P_f)$ (if defined) takes only finitely many values as f ranges in $S(\mathbb{C})$. □

4.8. **Families of Picard curves with torsion Ceresa cycle of arbitrarily large order.** Theorem 4.2 shows that there are infinitely many plane quartic curves over $\mathbb{Q}$ with torsion Ceresa cycle, since we may take $a=c=-12$. In fact, we can find explicit families of torsion Ceresa cycles of arbitrarily large order by analyzing the function $f\mapsto (I(f),J(f))$.

Let C_f be a Picard curve defined by the polynomial $f(x)=x^4+ax^2+bx+c$ over a field k with homogenization $F(x,z)=x^4+ax^2z^2+bxz^3+cz^4$. Associated to C_f is the genus 1 curve D_f with equation $y^2=F(x,z)$. Viewing D_f as an elliptic curve with origin $(1:1:0)$, it is isomorphic to the elliptic curve $y^2=x^3-I(f)x/3-J(f)/27$. The point $(1:-1:0)$ is sent to $(-2a/3,b)$ in this new model. Conversely, given a nonzero k-point (α,β) on the elliptic curve $E_{I,J}\colon y^2=x^3-Ix/3-J/27$, the quartic

$$f(x)=x^4-3\alpha x^2/2+\beta x+(I/12-3\alpha^2/16) \tag{4.3}$$

has the property that $(I(f),J(f))=(I,J)$. It follows that if $(I,J)\in k$ is such that $4J^3-J^2\neq 0$, then $f\mapsto(-2a/3,b)$ induces a bijection between the set of polynomials $f=x^4+ax^2+bx+c\in k[x]$ satisfying $(I(f),J(f))=(I,J)$, and the set of nonzero elements of $E_{I,J}(k)$.

Using this description, it is not hard to cook up explicit 1-parameter families of torsion Ceresa cycles of arbitrarily large order, as in the following corollary of Theorem 4.2:

Corollary 4.16. *For every integer N, there exists a number field $k=k(N)$ such that there exists infinitely many $\bar{\mathbb{Q}}$-isomorphism classes Picard curves C_f defined over k with Ceresa cycle κ_f of order at least N in $\mathrm{CH}^2(J_{f,\bar{\mathbb{Q}}})$.*

Proof. Let M be an integer satisfying the conclusion of Theorem 4.2. Fix a nonzero $D\in\mathbb{Q}$ and let $(I,J)\in\bar{\mathbb{Q}}^2$ be a torsion point of order $M\cdot N$ on the elliptic curve $y^2=x^3+D$. Let k be the field of definition of (I,J). For $t\in k$, define $\lambda(t)=t^3-It/3-J/27$, $(\alpha_t,\beta_t)=(t\lambda(t),\lambda(t)^2)$ and the quartic $f_t(x)=x^4-3\alpha_tx^2/2+\beta_tx+(I/12-3\alpha_t^2/16)$ using Formula (4.3). Since (α_t,β_t) defines a point on the elliptic curve $E_{\lambda(t)^2I,\lambda(t)^3J}$, the quartic f has the property that $(I(f_t),J(f_t))=(\lambda(t)^2I,\lambda(t)^3J)$. If $t\in k$ is such that $\lambda(t)\neq 0$, then $(\lambda(t)^2I,\lambda(t)^3J)$ is an N-torsion point on the elliptic curve $y^2=x^3+\lambda_t^6D$. By Theorem 4.2, κ_{f_t} is torsion in $\mathrm{CH}^2(J_{f_t,\bar{k}})$ and its torsion order $\mathrm{ord}(\kappa_{f_t})$ is divisible by N. When varying over $t\in k$ such that $\lambda(t)\neq 0$, the Picard curves C_{f_t} cover infinitely many $\bar{\mathbb{Q}}$-isomorphism classes, using [12, Lemma 1.21(b)] and the fact that $\beta_t^2/\alpha_t^3=\lambda(t)/t^3$ is not a constant function of t. □

Conversely, for any number field k, the order of a torsion Ceresa cycle $\kappa(C_f)$, with C_f defined over k, is bounded, with the bound depending only on the degree of k over $\mathbb{Q}$:

Corollary 4.17. *For every $d\geq 1$, there exists $N=N(d)\geq 1$ such that for every Picard curve C_f over a number field k of degree d, the order of κ_f in $\mathrm{CH}_1(J_{f,\bar{k}})$ is either infinite or less than N.*

Proof. This follows from Theorem 4.2 and the uniform bound (depending only on d) on the order of a k-rational torsion point on an elliptic curve $y^2 = x^3 + D$ over any number field k of degree d. $\square$

5. Automorphism strata and Ceresa vanishing loci in genus 3

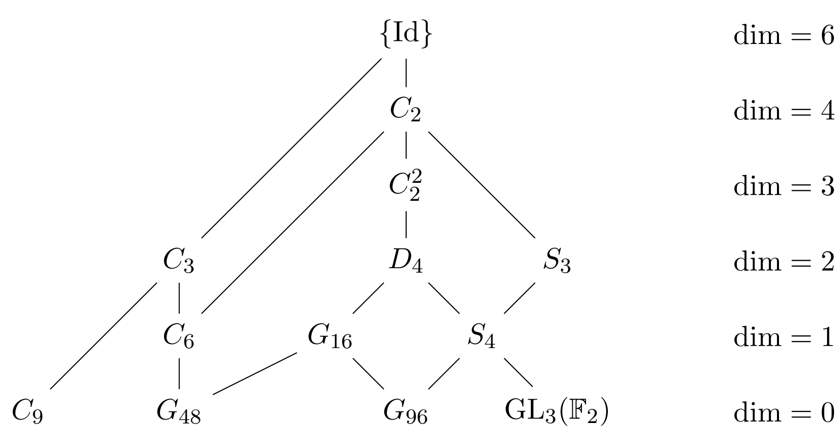
Fix $g \geq 3$ and let $\mathcal{M}_g$ be the (coarse) moduli space of genus-g curves, seen as a variety over $\mathbb{Q}$. Let $V_g^{\mathrm{rat}} \subset \mathcal{M}_g$ be the subset of curves $[C]$ for which $\kappa(C)$ vanishes in $\mathrm{CH}_1(\mathrm{Jac}(C))_{\mathbb{Q}}$, in the notation of §2.4. Since the vanishing of $\kappa(C)$ only depends on the geometric isomorphism class of C (by Lemma 2.2), this locus is well defined. Similarly define the locus $V_g^{\mathrm{alg}} \subset \mathcal{M}_g$ where $\bar{\kappa}(C) \in \mathrm{Gr}_1(\mathrm{Jac}(C))_{\mathbb{Q}}$ vanishes.

Lemma 5.1. *The subsets $V_g^{\mathrm{rat}}, V_g^{\mathrm{alg}} \subset \mathcal{M}_g$ are countable unions of proper closed algebraic subvarieties.*

Proof. Let $\widetilde{\mathcal{M}}_g$ be the fine moduli space parametrizing genus-g curves C with full symplectic level-5 structure and a degree-1 divisor class $e \in \mathrm{CH}_0(C)$ such that $(2g-2)e$ is canonical. Considering the universal curve over it together with its degree-1 divisor, we can define a universal Ceresa cycle on the universal Jacobian over $\widetilde{\mathcal{M}}_g$; Lemma 2.3 then implies that the locus in $\widetilde{\mathcal{M}}_g$ where this Ceresa cycle vanishes (with $\mathbb{Q}$-coefficients) is a countable union of closed algebraic subvarieties. Since the forgetful map $\widetilde{\mathcal{M}}_g \to \mathcal{M}_g$ is proper, the same is true for the image of this locus, which is exactly V_g^{rat}. The proof for V_g^{alg} is identical. $\square$

These vanishing loci have the following basic properties: $V_g^{\mathrm{rat}} \subset V_g^{\mathrm{alg}} \subset \mathcal{M}_g$; the hyperelliptic locus is contained in V_g^{rat}; and $V_g^{\mathrm{alg}} \neq \mathcal{M}_g$ by Ceresa's famous result [13]. It would be interesting to obtain further information about the components of V_g^{rat} and V_g^{alg}. We end our paper by determining the automorphism group strata in $\mathcal{M}_3$ that are contained in V_3^{rat} or V_3^{alg}.

So let $g = 3$ and consider the open subscheme $\mathcal{M}_3^{\mathrm{nh}} \subset \mathcal{M}_3$ of non-hyperelliptic curves. There is a stratification $\mathcal{M}_3^{\mathrm{nh}} = \sqcup X_G$ into locally closed subvarieties such that a non-hyperelliptic curve C over $\mathbb{C}$ belongs to X_G if and only if $\mathrm{Aut}(C) \simeq G$. It turns out that X_G is irreducible and the closure of X_G is a union of other strata. We refer to [28, §2.2] and references therein for a complete description of the loci X_G and the closure relations between them. We reproduce here a diagram capturing these closure relations:



For $n \in \{16, 48, 96\}$, the symbol G_n the group of order n and GAP label $(16, 13), (48, 33)$, and $(96, 64)$ respectively [20]. See [28, Table 2] for models for a generic plane quartic in X_G. We make explicit the strata that are relevant for us: $\overline{X_{C_3}}$ is the locus of Picard curves studied in §4; $\overline{X_{C_6}}$ is the locus of *bielliptic* Picard curves studied in [26]; and the zero-dimensional strata each consist of a single automorphism-maximal curve with equation

$$\begin{cases} y^3z = x^4 + xz^3 & \text{if } G = C_9, \\ y^3z = x^4 + z^4 & \text{if } G = G_{48}, \\ x^4 + y^4 + z^4 = 0 & \text{if } G = G_{96}, \\ x^3y + y^3z + z^3x & \text{if } G = \mathrm{GL}_3(\mathbb{F}_2). \end{cases}$$

If C is a non-hyperelliptic genus 3 curve over a field k, we say C is a generic curve for X_G if the classifying map $\mathrm{Spec}(k) \to \mathcal{M}_3$ maps to the generic point of X_G. If X is an integral variety over $\mathbb{C}$, we say a property hold for a very general $x \in X(\mathbb{C})$ if it holds true outside a countable union of proper closed subvarieties of X.

Lemma 5.2. *For a group G in the above diagram, the following are equivalent:*

(1) $\kappa(C) \neq 0$ for some generic curve C for X_G.
(2) $\kappa(C) \neq 0$ for a very general C in $X_G(\mathbb{C})$.
(3) $\kappa(C) \neq 0$ for some C in X_G.
(4) $X_G \not\subset V_3^{\mathrm{rat}}$.
(5) $\overline{X_G} \not\subset V_3^{\mathrm{rat}}$.

Moreover, the analogous equivalences hold for $\bar{\kappa}(C) \in \mathrm{Gr}_1(J)_{\mathbb{Q}}$ and V_3^{alg}.

Proof. Follows immediately from Lemma 5.1. □

We end with the proof of Theorem D, which we restate for convenience:

Proposition 5.3. *Let G be a group in the diagram. Then*

(1) $X_G \subset V_3^{\mathrm{alg}}$ if and only if $G = C_3, C_6, C_9, G_{48}$.
(2) $X_G \subset V_3^{\mathrm{rat}}$ if and only if $G = C_9, G_{48}$;

Proof. (1) The equivalence between (4) and (5) of Lemma 5.2 implies that if $X_H \subset \overline{X_G}$ and $X_G \subset V_3^{\mathrm{alg}}$, then $X_H \subset V_3^{\mathrm{rat}}$; we will use this observation in the remainder of the proof. Our analysis of Picard curves (Theorem C) shows that $X_{C_3} \subset V_3^{\mathrm{rat}}$, so $X_G \subset V_3^{\mathrm{rat}}$ for $G = C_6, C_9$ and G_{48} as well. On the other hand, the Ceresa cycle of the Fermat quartic and Klein quartic are known to be of infinite order in the Griffiths group; see [10, Theorem (4.1)] for the former and [23, §4] for the latter. By our observation, this means that $X_G \not\subset V_3^{\mathrm{rat}}$ for every stratum whose closure contains one of these curves. Since every stratum not contained in $\overline{X_{C_3}}$ has this property, we conclude the proof.

(2) Since $V_3^{\mathrm{rat}} \subset V_3^{\mathrm{alg}}$, Part (1) implies that $X_G \subset V_3^{\mathrm{rat}}$ only if $G = C_3, C_6, C_9$ or G_{48}. The criterion of Theorem A applies to the curves in X_{C_9} and $X_{G_{48}}$ (see Example 3.1), so V_3^{rat} contains these strata. On the other hand, there exist curves in X_{C_6} with nonvanishing $\kappa(C)$, by [25, Corollary 1.2]. So $X_{C_6} \not\subset V_3^{\mathrm{rat}}$ and $X_{C_3} \not\subset V_3^{\mathrm{rat}}$.

□

Remark 5.4. Not every irreducible component of V_3^{rat} is of the form $\overline{X_G}$ for some G. The proof of Corollary 4.16 shows that $V_3^{\mathrm{rat}} \cap \overline{X_{C_3}}$ is a union of countably many (open, possibly singular) rational curves.

References

[1] J. D. Achter, S. Casalaina-Martin, and C. Vial. Normal functions for algebraically trivial cycles are algebraic for arithmetic reasons. *Forum Math. Sigma*, 7:Paper No. e36, 22, 2019.

[2] J. D. Achter, S. Casalaina-Martin, and C. Vial. Parameter spaces for algebraic equivalence. *Int. Math. Res. Not. IMRN*, (6):1863–1893, 2019.

[3] Y. André. Motifs de dimension finie (d'après S.-I. Kimura, P. O'Sullivan…). Number 299, pages Exp. No. 929, viii, 115–145. 2005. Séminaire Bourbaki. Vol. 2003/2004.

[4] A. Beauville. Sur l'anneau de Chow d'une variété abélienne. *Math. Ann.*, 273(4):647–651, 1986.

[5] A. Beauville. Algebraic cycles on Jacobian varieties. *Compos. Math.*, 140(3):683–688, 2004.

[6] A. Beauville. A non-hyperelliptic curve with torsion Ceresa class. *C. R. Math. Acad. Sci. Paris*, 359:871–872, 2021.

[7] A. Beauville and C. Schoen. A non-hyperelliptic curve with torsion Ceresa cycle modulo algebraic equivalence. *Int. Math. Res. Not. IMRN*, (5):3671–3675, 2023.

[8] M. Bhargava, N. Elkies, and A. Shnidman. The average size of the 3-isogeny Selmer groups of elliptic curves $y^2 = x^3 + k$. *J. Lond. Math. Soc. (2)*, 101(1):299–327, 2020.

[9] M. Bhargava and A. Shankar. Binary quartic forms having bounded invariants, and the boundedness of the average rank of elliptic curves. *Ann. of Math. (2)*, 181(1):191–242, 2015.

[10] S. Bloch. Algebraic cycles and values of L-functions. *J. Reine Angew. Math.*, 350:94–108, 1984.

[11] S. Bloch. *Lectures on algebraic cycles*, volume 16 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, second edition, 2010.

[12] I. I. Bouw, A. Koutsianas, J. Sijsling, and S. Wewers. Conductor and discriminant of Picard curves. *J. Lond. Math. Soc. (2)*, 102(1):368–404, 2020.

[13] G. Ceresa. C is not algebraically equivalent to C^- in its Jacobian. *Ann. of Math. (2)*, 117(2):285–291, 1983.

[14] F. Charles. On the zero locus of normal functions and the étale Abel-Jacobi map. *Int. Math. Res. Not. IMRN*, (12):2283–2304, 2010.

[15] A. Collino and G. P. Pirola. The Griffiths infinitesimal invariant for a curve in its Jacobian. *Duke Math. J.*, 78(1):59–88, 1995.

[16] P. Deligne. *Hodge Cycles on Abelian Varieties*, pages 9–100. Springer Berlin Heidelberg, Berlin, Heidelberg, 1982.

[17] C. Deninger and J. Murre. Motivic decomposition of abelian schemes and the Fourier transform. *J. Reine Angew. Math.*, 422:201–219, 1991.

[18] L. Fu, R. Laterveer, and C. Vial. Multiplicative Chow-Künneth decompositions and varieties of cohomological K3 type. *Ann. Mat. Pura Appl. (4)*, 200(5):2085–2126, 2021.

[19] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[20] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.13.0*, 2024.

[21] S. Kallel and D. Sjerve. On the group of automorphisms of cyclic covers of the Riemann sphere. *Math. Proc. Cambridge Philos. Soc.*, 138(2):267–287, 2005.

[22] E. Kani and M. Rosen. Idempotent relations and factors of Jacobians. *Math. Ann.*, 284(2):307–327, 1989.

[23] K.-i. Kimura. On modified diagonal cycles in the triple products of Fermat quotients. *Math. Z.*, 235(4):727–746, 2000.

[24] S.-I. Kimura. Chow groups are finite dimensional, in some sense. *Math. Ann.*, 331(1):173–201, 2005.

[25] J. Laga and A. Shnidman. Ceresa cycles of bielliptic Picard curves. Arxiv preprint, available at https://arxiv.org/abs/2312.12965v1, 2023+.

[26] J. Laga and A. Shnidman. The geometry and arithmetic of bielliptic Picard curves. Arxiv preprint, available at https://arxiv.org/abs/2308.15297v2, 2023+.

[27] D. T.-B. G. Lilienfeldt and A. Shnidman. Experiments with Ceresa classes of cyclic Fermat quotients. *Proc. Amer. Math. Soc.*, 151(3):931–947, 2023.

[28] D. Lombardo, E. Lorenzo García, C. Ritzenthaler, and J. Sijsling. Decomposing Jacobians via Galois covers. *Exp. Math.*, 32(1):218–240, 2023.

[29] B. Moonen. Special subvarieties arising from families of cyclic covers of the projective line. *Doc. Math.*, 15:793–819, 2010.

[30] B. J. J. Moonen and Y. G. Zarhin. Weil classes on abelian varieties. *J. Reine Angew. Math.*, 496:83–92, 1998.

[31] B. J. J. Moonen and Y. G. Zarhin. Hodge classes on abelian varieties of low dimension. *Math. Ann.*, 315(4):711–733, 1999.

[32] J. P. Murre. Applications of algebraic K-theory to the theory of algebraic cycles. In *Algebraic geometry, Sitges (Barcelona), 1983*, volume 1124 of *Lecture Notes in Math.*, pages 216–261. Springer, Berlin, 1985.

[33] J. P. Murre, J. Nagel, and C. A. M. Peters. *Lectures on the theory of pure motives*, volume 61 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2013.

[34] C. Qiu and W. Zhang. Vanishing results in Chow groups for the modified diagonal cycles. Preprint, available at https://arxiv.org/abs/2209.09736v3, 2022.

[35] J. J. Ramón Marí. On the Hodge conjecture for products of certain surfaces. *Collect. Math.*, 59(1):1–26, 2008.
[36] C. Schoen. Addendum to: "Hodge classes on self-products of a variety with an automorphism" [Compositio Math. **65** (1988), no. 1, 3–32; MR0930145 (89c:14013)]. *Compositio Math.*, 114(3):329–336, 1998.
[37] A. J. Scholl. Classical motives. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 163–187. Amer. Math. Soc., Providence, RI, 1994.
[38] M. Schütt and T. Shioda. Elliptic surfaces. In *Algebraic geometry in East Asia—Seoul 2008*, volume 60 of *Adv. Stud. Pure Math.*, pages 51–160. Math. Soc. Japan, Tokyo, 2010.
[39] T. Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2018.
[40] C. Voisin. *Hodge theory and complex algebraic geometry. II*, volume 77 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps.
[41] S.-W. Zhang. Gross-Schoen cycles and dualising sheaves. *Invent. Math.*, 179(1):1–73, 2010.

Department of Pure Mathematics and Mathematical Statistics, Wilberforce Road, Cambridge, CB3 0WB, UK

Email address: jeflaga@hotmail.com

Einstein Institute of Mathematics, Hebrew University of Jerusalem, Israel

Email address: ari.shnidman@gmail.com