around the positive vertices contain the topological vertex of Aganagic, Klemm, Marino and Vafa. In this setting the GromovWitten invariants in all genus are determined by a tropical gluing formula.

## The Logarithmic Picard Group

Sam Molcho
(joint work with Jonathan Wise)

Let $X \to S$ be a family of nodal curves with smooth fibers over some open subscheme $U \subset S$. The Jacobian $\mathrm{Pic}^0(X \times_S U/U)$ is an abelian variety over $U$, but in general there is no way to construct an abelian variety over all of $S$ whose restriction to $U$ agrees with $\mathrm{Pic}^0(X \times_S U/U)$. One must choose to either sacrifice properness and work with the generalized Jacobian, which is only a semiabelian scheme, or, as in [1] and [2], to compactify the generalized Jacobian in some way and sacrifice the group structure in the process.

Following ideas of Kato and Illusie, we describe a solution to this problem in the setting of logarithmic geometry: given a family of logarithmically smooth curves $\pi : X \to S$, we construct the log Picard group $\mathrm{LogPic}(X/S)$ over $S$. Let $\mathbb{G}_m^{\log}$ denote the functor on log schemes defined by $\mathbb{G}_m^{\log}(Y) = \Gamma(Y, M_Y^{\mathrm{gp}})$. We then define $\mathbf{LogPic}(X/S)$ to be the stack $(\pi_* B\mathbb{G}_m^{\log})^\dagger$ and $\mathrm{LogPic}(X/S)$ its sheaf of isomorphism classes. The $\dagger$ indicates that we are only taking $\mathbb{G}_m^{\log}$-torsors that satisfy a certain combinatorial condition on the dual graph of $X$, which we call bounded monodromy. We refer to the bounded monodromy torsors as log line bundles. The degree 0 log line bundles $\mathbf{LogPic}^0(X/S)$ form a proper group stack, which coincides with the usual stack $\mathbf{Pic}^0$ on the locus of $S$ where the log structure is trivial, i.e. over the smooth locus of $X \to S$. For instance, applying this construction to the universal curve $\overline{C}_{g,n} \to \overline{M}_{g,n}$ provides a compactification $\mathrm{LogPic}^0(\overline{C}_{g,n}/\overline{M}_{g,n})$ of the universal Jacobian $\mathrm{Pic}^0(C_{g,n}/M_{g,n})$. On the other hand, $\mathbf{LogPic}$ is not representable by an algebraic stack with a logarithmic structure. It is only a log algebraic stack – the analogue of an algebraic stack over the category of log schemes. It nevertheless has rich structure that allows one to study it; the log Jacobian $\mathrm{LogPic}^0$ is, for instance, a log abelian variety in the sense of [3].

As a first step, $\mathbf{LogPic}(X/S)$, a priori a stack in the étale topology, is, in fact, invariant under logarithmic blowups and under extracting roots of the log structure. It is thus a stack in the full log étale topology. A consequence of this observation is that $\mathbf{LogPic}(X/S)$ has a cover by the stacks $\mathbf{Pic}(Y)$ as $Y$ ranges over all log blowups of $X$, and that log line bundles on $X$ can thus be understood as actual line bundles on semistable models of $X$, up to some elaborate equivalence relation.

Additional structure of the log Picard group is revealed by studying its tropicalization. To a log curve $X \to S$, one associates its tropicalization $\mathfrak{X}$. This is a collection of dual graphs, which have edge lengths valued in the characteristic monoid of $M_S$, and are compatible with generization. The tropical curve $\mathfrak{X}$ carries a sheaf

$\mathcal{L}$ of linear functions, which allows us to define the tropical Picard group $\mathrm{TroPic}(\mathfrak{X})$ as the sheaf of isomorphism classes of the stack $\mathbf{TroPic}(\mathfrak{X})$: this is the stack of tropical line bundles on $\mathfrak{X}$, that is, the $\mathcal{L}$-torsors on $\mathfrak{X}$ which, again, have bounded monodromy. The stack $\mathbf{TroPic}(\mathfrak{X})$ is a combinatorial object that can be explicitly calculated. More importantly for our purposes, it coincides with the tropicalization of $\mathbf{LogPic}(X/S)$. Furthermore, $\mathbf{LogPic}(X/S)$ (resp. $\mathrm{LogPic}(X/S)$) contains the stack of multidegree 0 line bundles $\mathbf{Pic}^{[0]}(X)$ (resp. the generalized Jacobian) as a subgroup, and there are exact sequences of group stacks and sheaves respectively:

$$0 \to \mathbf{Pic}^{[0]}(X/S) \to \mathbf{LogPic}(X/S) \to \mathbf{TroPic}(\mathfrak{X}) \to 0$$
$$0 \to \mathrm{Pic}^{[0]}(X/S) \to \mathrm{LogPic}(X/S) \to \mathrm{TroPic}(\mathfrak{X}) \to 0$$

In particular, the failure of representability of $\mathrm{LogPic}(X/S)$ by a scheme with a logarithmic structure is entirely due to the failure of representability of $\mathrm{TroPic}(\mathfrak{X})$ by a polyhedral complex. The tropical Picard group, though not a polyhedral complex itself, has subdivisions which are polyhedral complexes. By pulling back subdivisions of $\mathrm{TroPic}(\mathfrak{X})$ under the tropicalization map, one obtains log blowups of $\mathrm{LogPic}(X)$, which in fact are representable by schemes. Restricting to the log Jacobian, one obtains by this procedure toroidal compactifications of the generalized Jacobian.

## References


[1] Oda, Seshadri, *Compactifications of the generalized Jacobian variety.*, Trans. Amer. Math. Soc. 253 (1979), 1–90.

[2] Caporaso, *A compactification of the universal Picard variety over the moduli space of stable curves.*, J. Amer. Math. Soc. 7 (1994), no. 3, 589–660.

[3] Kajiwara, Kato, Nakayama, *Logarithmic Abelian Varieties, Part II*, Nagoya Mathematical Journal **189** (2008), 63–136.


## Universal coefficients for logarithmic curves


Jonathan Wise

(joint work with Samouil Molcho, Martin Ulirsch)


Suppose that $X$ is a smooth, proper algebraic curve over a base $S$. Then there is a canonical symmetric biextension of its Jacobian, which we notate as a bilinear pairing:

$$\mathrm{Pic}^0(X/S) \times \mathrm{Pic}^0(X/S) \to \mathrm{B}\mathbf{G}_{m,S} \tag{1}$$

If $X$ is a flat, proper curve over $S$ with nodal fibers then the Jacobian will fail to extend to an abelian variety if the dual graph of a fiber of $X$ is not a tree. Furthermore, the Jacobian contains an algebraic torus in this case, and a theorem of Grothendieck says that there is no nontrivial biextension between an algebraic