LOG STABLE MAPS AS MODULI SPACES OF FLOW LINES

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Abstract. We enrich the Chow quotient $X /\!\!/_C T$ of a toric variety X by a one parameter subgroup T of its torus with the structure of a toric stack, from combinatorial data associated to the T action. We then show that this stack coincides with the stack $K_\Gamma(X)$ of stable log maps of Abramovich-Chen and Gross-Siebert, for appropriate data Γ determined by the T action.

1. Introduction

In Morse theory, one begins with compact manifold X and a Morse or Morse-Bott function $f : X \to \mathbb{R}$, that is, a function satisfying certain genericity assumption. The function f yields a vector field $-\nabla f$. The trajectories of $-\nabla f$ are called the flow lines of f. A now classical theorem states

Theorem 1. *The space of broken flow lines of f is a manifold with corners M. Furthermore, the homology of X can be reconstructed from the flow lines of f, that is, from M.*

The function f enters the picture only to define the vector field $-\nabla f$. This vector field is not arbitrary; the function satisfies certain non-degeneracy conditions which translate to certain properties of $-\nabla f$. Nevertheless, $-\nabla f$ equips X with an $\mathbb{R}^*$-action: $t \cdot x =$ flow x for time e^t along $-\nabla f$, and the orbits of this action are the flows of $-\nabla f$. Therefore, an appropriate space of orbits of this action is sufficient to reconstruct the homology of X. One cannot take the naive space of orbits because the homology of X requires the data of broken flow lines of $-\nabla f$, not simply the flows.

This point of view leads us to the following question in the algebraic world. Suppose X is a variety over $\mathbb{C}$ equipped with a $\mathbb{C}^*$ action: what is an appropriate quotient $X/\mathbb{C}^*$ that mimics the space of broken flow lines for this action? A possible candidate for this space is the Chow quotient $X /\!\!/_C \mathbb{C}^*$. Kapranov, Sturmfels and Zelevinsky gave a beautiful description of $X /\!\!/_C \mathbb{C}^*$ in their paper [KSZ91] in the case when X is a toric variety and $\mathbb{C}^*$ is a one-parameter subgroup T of the torus T' of X. More specifically, they showed that $X /\!\!/_C \mathbb{C}^*$ is a toric variety and described its fan in terms of natural combinatorial data arising from the fan of X and the subtorus T. In [CS12], Chen and Satriano showed that $X /\!\!/_C \mathbb{C}^*$ coincides with the coarse moduli space of an important stack coming from Gromov-Witten theory, the stack $K_\Gamma(X)$ of log stable maps of Abramovich-Chen [AC11] and Gross-Siebert [GS11b]. The variety X is given its natural log structure as a toric variety and the discrete data Γ necessary to specify the stack are determined by T.

In this paper, we endow $X /\!\!/_C \mathbb{C}^*$ with the structure of a toric stack, which we call $\mathcal{M}$. This is done by enriching the fan of $X /\!\!/_C C^*$ by natural combinatorial data arising from X and T, which are in fact present in the [KSZ91] picture. We then show that this stack is isomorphic to the actual stack $K_\Gamma(X)$. This is interesting to us because it gives very explicit descriptions of several stacks of log stable maps. A connection with Morse theory arises by geometrically realizing the fans of $X, \mathcal{M}$ in the category of positive log differentiable spaces [GM13], which produces a manifold with corners equipped with an $\mathbb{R}^*$ action and a manifold with corners $\mathcal{M}$ coinciding with the moduli space of flow lines for the $\mathbb{R}^*$ action.

2. Toric Stacks

We begin by outlining a procedure for constructing toric stacks, that is, stacks equipped with an action of a dense torus T.

Definition: A toric stack is the data (F, N_σ, N), where F is a fan in a lattice N; for each cone $\sigma \in F$, N_σ is a monoid, which is finitely generated and saturated in N_σ^{gp}; $N_\sigma^{\text{gp}} \subset N$ is a sublattice. This data is subject to the following compatibility condition: If $\tau < \sigma$ is a face, then $N_\tau = \tau \cap N_\sigma$. We further assume that for the maximal cones σ, $N_\sigma^{\text{gp}} \subset N$ has finite index.

Lemma 1. *There is a geometric realization of the data (F, N_σ, N) to a toric stack.*

Proof: Each maximal cone σ gives rise to a stack as follows: The inclusion $N_\sigma^{\text{gp}} \to N$ induces a map of tori $T(N_\sigma^{\text{gp}}) \to T(N)$, with kernel a finite subgroup K. We define $\mathcal{X}_\sigma = [X_\sigma/K]$. Here X_σ is the toric variety associated to the cone σ inside the lattice N_σ^{gp}. The various $\mathcal{X}_\sigma$ glue due to the compatibility condition, to give a stack $\mathcal{X}(F, N_\sigma, N)$. This is the geometric realization.

The stack $\mathcal{X}(F, N_\sigma, N)$, is in fact a log stack: a chart for the log structure over the affine piece $\mathcal{X}_\sigma$ is given by the dual monoid $M_\sigma = \text{Hom}(N_\sigma, N)$ corresponding to σ.

Remark: There is a more general definition of a toric stack but the one given above is sufficient for our purposes.

Lemma 2. *The coarse moduli space of $\mathcal{X}(F, N_\sigma, N)$ is the toric variety $X(F)$ associated to the fan F.*

Proof: By lemma 2.2.2 of the paper [AV02] of Abramovich-Vistoli, it suffices to check the lemma for the etale affine chart $\mathcal{X}_\sigma$ of a single maximal cone. The result then follows from theorem 6.3 of [GS11a].

Remark: Our terminology 'toric stack' here potentially conflicts with the literature. A rather complete theory of toric stacks has been developed over the years by various authors. We have looked at works of Borisov-Chen-Smith [BCS05], Fantechi-Mann-Nironi [FMN07], and Satriano-Geraschenko [GS11a]. Our definition is similar in spirit to the definition given in [BCS05], but is more general. The theory in [GS11a] is the newest and encompases the results of [BCS05] and [FMN07] papers. It would not be surprising if our definition of a toric stack, which is seemingly different from than the one given in [GS11a], is in fact included in the results of [GS11a] as well. However, we do not pursue this here.

3. The Stack of Broken Flows

We begin by recalling the construction of the Chow quotient $X /\!\!/_C C^*$ given in [KSZ91]. For proofs we refer the reader to the original paper. Fix a toric variety X, defined by a fan F inside a lattice N. We consider a one parameter subgroup $\mathbb{C}^* = T$ of the torus T' of X. The torus T corresponds to a primitive vector v in N. Denote by L the lattice spanned by v, and by $p : N \to Q = N/L$ the projection onto the quotient.

Let G be the projection of the fan F along p. This is by definition the fan obtained by projecting each cone in F individually and taking their common refinement into a fan. More explicitly, G can be constructed as follows. For each vector ψ in $N_{\mathbb{R}}$, let $\mathcal{N}(\psi)$ be the collection of cones

$$\{\sigma \in F : (\psi + L) \cap \sigma \neq \emptyset\}$$

Two vectors ψ and ψ' are equivalent if $\mathcal{N}(\psi) = \mathcal{N}(\psi')$. The closure of each equivalence class of vectors forms an L invariant cone; the projections of these cones form a fan, which is the fan G. In [KSZ91] it is proven that $X /\!\!/_C \mathbb{C}^* = X(G)$, the toric variety associated to G.

Recall that the Chow quotient is a subspace of the space of algebraic cycles of X. A topology on this space is discussed in the paper [KSZ91]. Thus, every point of $X /\!\!/_C \mathbb{C}^*$ corresponds to an algebraic cycle in X. In this case, this embedding is very explicit. Let κ be a cone in G, and let $e_\kappa \in X(G)$ be the associated special point of κ. Let $\mathcal{N}_0(\kappa)$ be the collection of cones

$$\left\{\sigma \in F : p_{\mathbb{R}}^{-1}(\psi) \cap \mathrm{int}(\sigma) = \text{one point}\right\}$$

- the preimage of an interior point $\psi \in \kappa$ should intersect the interior of σ at precisely one point. For each σ in $\mathcal{N}_0(\kappa)$ we let

$$c(\sigma, L) = [\sigma \cap N + L : Cone(\sigma + L_{\mathbb{R}}) \cap N]$$

By $Cone(Set)$ we mean the $\mathbb{R}_{\geq 0}$-cone spanned by the elements of the set S. The algebraic cycle corresponding to e_κ is then

$$\sum_{\sigma \in \mathcal{N}_0(\kappa)} c(\sigma, L)\mathrm{closure}(Te_\sigma)$$

The cycle over a general point of $X(G)$ is obtained from this description by use of the torus T'/T of $X(G)$; specifically, any x lies in the orbit of a unique e_κ, that is, is unambiguously expressed as $c \cdot e_\kappa$. The cycle corresponding to x is then the same sum above with e_σ multiplied by c', where c' is any lift of c.

Having recalled the construction of $X /\!\!/_C \mathbb{C}^*$ we are now ready to define the corresponding stack structure.

Consider the fan G described above; for each cone $\kappa \in G$, let Q_κ be the intersection of the projection of the lattices determined by the cones σ with e_σ in the fiber of e_κ:

$$Q_\kappa = \bigcap_{\sigma \in \mathcal{N}_0(\kappa)} p(\sigma \cap N) \subset p(N) = Q$$

Lemma 3. *The data (G, Q_κ, Q) determines a toric stack.*

Proof: The only thing we need to show is the compatibility condition: if $\lambda \subset \kappa$ is a face, then $Q_\lambda = \lambda \cap Q_\kappa$. Suppose σ is a cone in F mapping isomorphically to κ. Since p is an isomorphism on σ, there is a unique face τ of σ mapping isomorphically to λ. As $\tau \subset \sigma$ and τ maps to λ, we have $p(\tau \cap N) \subset p(\sigma \cap N) \cap \lambda$. On the other hand, $p^{-1}(p(\sigma \cap N) \cap \lambda) \cap \sigma \subset \tau \cap N$. Therefore, $p(\sigma \subset N) \cap \lambda \subset p(\tau \cap N)$. Taking intersections over $\mathcal{N}_0(\kappa)$ and $\mathcal{N}_0(\lambda)$ yields the result.

The geometric realization $\mathcal{X}(G, Q_\kappa, Q)$ will be the stack $\mathcal{M}$ referred to in the introduction. It is a DM-stack whose underlying coarse moduli space is the stack $X /\!\!/_C \mathbb{C}^*$. As we have a modular interpretation of $\mathcal{M}$ in mind, as the moduli stack of broken flows of the $\mathbb{C}^*$ action in X, we ought to construct its universal family

$$\mathcal{U} \to \mathcal{M}$$

The description of $\mathcal{U}$ is very simple: it is the minimal modification of the fan F of X into a toric stack $(F', L_{\sigma'}, L)$ that maps to both X and $\mathcal{M}$. The following lemmas make this precise.

Lemma 4. *Fix a diagram of morphisms of lattices*

$$\begin{array}{ccc} N & \xrightarrow{id} & N \\ {\scriptstyle p}\downarrow & & \\ Q & & \end{array}$$

and two fans F in N and G in Q. Let $\mathcal{C}$ be the category of fans F' in N that map to both F and G under the given map of lattices. Morphisms are maps of fans $F'' \to F'$ that commute with the maps to F and G. Then $\mathcal{C}$ has a terminal object.

Proof: The terminal object is simply the collection of cones $p^{-1}(\kappa) \cap \sigma$, where κ ranges through all cones in G and σ through all cones in F. To see that these form a fan, we only need to check that the intersection of two cones is in the collection and is a face of each. The first statement follows because $p^{-1}(\kappa_1)\cap\sigma_1\cap p^{-1}(\kappa_2)\cap\sigma_2 = p^{-1}(\kappa_1\cap\kappa_2)\cap(\sigma_1\cap\sigma_2)$. The second statement follows because if $x+y \in p^{-1}(\kappa_1)\cap\sigma_1$ is in $p^{-1}(\kappa_1\cap\kappa_2)\cap(\sigma_1\cap\sigma_2)$, then $p(x+y) = p(x)+p(y)$ is in $\kappa_1\cap\kappa_2$; hence $p(x)\in\kappa_1\cap\kappa_2$ and $p(y)\in\kappa_1\cap\kappa_2$; and $x+y\in\sigma_1\cap\sigma_2$, hence $x\in\sigma_1\cap\sigma_2$ and $y\in\sigma_1\cap\sigma_2$. Thus $x, y \in p^{-1}(\kappa_1\cap\kappa_2)\cap(\sigma_1\cap\sigma_2)$.

This is not quite enough to determine $\mathcal{U}$ in our situation, since $\mathcal{M}$ is a toric stack. However, the lemma above can be modified to incorporate this case; given a cone $p^{-1}(\kappa)\cap\sigma$, we define $N_{p^{-1}(\kappa)\cap\sigma} = \sigma\cap p^{-1}Q_\kappa$. Writing σ' for a cone in F', we have the lemma:

Lemma 5. *The collection $(F', N_{\sigma'}, N)$ is a toric stack. It is the minimal toric stack that maps to both X and $\mathcal{M}$.*

Proof: Again, the only thing that needs to be checked is the compatibility condition. The proof is similar to the proof of lemma 3.

This completes the construction of $\mathcal{U}\to\mathcal{M}$.

4. Log Stable Maps

In this section we recall the necessary facts about log stable maps and describe the correspondence $\mathcal{M}\leftrightarrow K_\Gamma(X)$. Let X be a log smooth fs log scheme. Recall the definition of a log curve, as in [Kat00].

Definition: A log curve C over a log scheme S is an integral log smooth morphism $C\to S$ whose geometric fibers are reduced curves.

In practice, this means the following. First, the fibers are forced to be nodal curves. Next,suppose the base $S = \text{Spec } A$ for A a strictly Henselian local ring; every point of S posseses an etale neigbhorhood of this sort. Let $\sigma : Q\to A$ be a chart for the log structure on S. Then, the log structure around a geometric point $p\in C$ has a chart of the following form:

- p is a smooth point, $\mathcal{O}_C\cong A[z]$ around p, and a chart is given by $\sigma : Q\to A\to A[z]$
- p is a smooth point, and a chart is given by $Q\oplus\mathbb{N}\to A[z]$, $(q,a)\mapsto\sigma(q)z^a$
- p is a node, $\mathcal{O}_C\cong A[z,w]/zw$ around p, and a chart is given by $Q\oplus_{\mathbb{N}}\mathbb{N}^2\to A[z,w]/zw$, $(q,(a,b))\mapsto\sigma(q)z^aw^b$. Here $\mathbb{N}\to\mathbb{N}^2$ is the diagonal map and $\mathbb{N}\to Q$ is determined by an integer $1\mapsto\rho_q$ corresponding to the node, which we consider part of the data.

The points p in the second situation may be regarded as the marked points of C.

Definition A log stable map to X over a log scheme S is a diagram of log schemes

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \downarrow & & \\ S & & \end{array}$$

where $C \to S$ is a log curve. Stability refers to the requirement that the group of automorphisms of f should be finite. An automorphism of f is an automorphism of log schemes of C over S commuting with f.

Over a geometric point $S = \operatorname{Spec} k$, a log stable map defines a type $(C, u_p \in \operatorname{Hom}(P_p, \mathbb{N}), u_q \in \operatorname{Hom}(P_q, \mathbb{Z}))$, where p, q range through the marked and nodal points of C, and P_x denotes the sharpened monoid $f^*\bar{M}_{X,x}$, as follows. We consider the category of diagrams

$$\begin{array}{ccc} \bar{M}_C & \xleftarrow{\phi} & f^*\bar{M}_X \\ \pi \uparrow & & \\ \bar{M}_S = Q & & \end{array}$$

of the characteristic log structures. The map u_p is determined by the map $P_p \to \bar{M}_{C,p} \cong Q \oplus \mathbb{N}$ by projection, $u_p = pr_2 \circ \phi_p$. The map u_q is a little more subtle. Given a node q, let η_1, η_2 denote the generic points of the two irreducible components containing it. Denote by $\chi_{q,\eta_i} : P_q \to P_{\eta_i}$ the two generization maps. These induce a commutative diagram

$$\begin{array}{ccc} \bar{M}_{C,\eta_i} = Q & \longleftarrow & P_{\eta_i} \\ \uparrow & & \uparrow \\ \bar{M}_{C,q} = Q \oplus_{\mathbb{N}} \mathbb{N}^2 & \longleftarrow & P_q \end{array}$$

The monoid $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ is identified with the submonoid $\{(u_1, u_2) : u_2 - u_1 \in \rho_q \mathbb{Z}\}$ of $Q \times Q$ via the map $(m, (a, b)) \mapsto (m + \rho_q a, m + \rho_q b)$; in other words, the multiple of ρ_q corresponding to the point $(m, (a, b))$ is $b - a$. The generization diagram above induces a natural map $P_q \to Q \times Q$ factoring through the map $P_q \to Q \oplus_{\mathbb{N}} \mathbb{N}^2 \to Q \times Q$, and the homomorphism u_q then assigns to $m \in Q$ the multiple of ρ_q determined from the difference of the two factors in $Q \times Q$: $(\phi \circ \chi_{\eta_2} - \phi \circ \chi_{\eta_1})(m) = \rho_q u_q(m)$.

A major insight of the paper [GS11b], which is also apparent in the papers [Che10],[AC11] in slightly different language, is that given the data of the curve C and the type (u_p, u_q), there is a log structure Q^a on the base on the base $\operatorname{Spec} k$, called the basic log structure, with the given type. This is minimal in the sense that any log curve $(C, N_C) \to (\operatorname{Spec} k, M)$ with type $C, (u_p, u_q)$ is obtained by pullback from $(C, M_C) \to (\operatorname{Spec} k, Q^a)$.

The log structure admits a chart Q, which is constructed as follows.

Consider the map

$$a_q : P_q \to (\prod_\eta P_\eta \times \prod_q \mathbb{N})^{\mathrm{gp}} = \prod_\eta (P_\eta)^{\mathrm{gp}} \times \prod_q \mathbb{Z}$$
$$a_q(m) = (\cdots, \chi_{\eta_1}(m), \cdots, -\chi_{\eta_2}(m), \cdots, u_q(m), \cdots)$$

The indices run over the generic points η of irreducible components of C and the nodes q of C; the map a_q maps via the two generizations to the components P_{η_i} corresponding to the two irreducible components of C containing q, and via u_q to the copy of $\mathbb{Z}$ corresponding to q. All the other elements in the product are 0. We consider the saturated subgroup R generated by the images of the maps a_q, take the quotient, and consider the image of $\prod_\eta P_\eta \times \prod_q \mathbb{N}$ in the quotient. The saturation of this monoid is Q. It is useful to know its dual monoid $Hom(Q, \mathbb{N})$, which is perhaps simpler to describe:

$$\mathrm{Hom}(Q, \mathbb{N}) = \left\{ (V_\eta, e_q) \subset \prod Hom(P_\eta, \mathbb{N}) \times \prod_q \mathbb{N} : V_{\eta_2} - V_{\eta_1} = e_q u_q \right\}$$

We saw how a log stable map determines a type over each geometric point on the base. The log stable map is called basic if the log structure over each geometric point coincides with the basic log structure associated to the given type. The stack of basic log stable maps is denoted by $K_\Gamma(X)$, where Γ is the discrete data of the marked points, the genus of C, and the homology class of the image. The basicness condition essentially means that $K_\Gamma(X)$ is a log stack; a basic stable log map over S is determined by a map $S \to K_\Gamma(X)$; and a general stable log map by a log map $(S, N) \to K_\Gamma(X)$. In the papers [Che10],[AC11],[GS11b], it is shown that $K_\Gamma(X)$ is a proper DM stack.

5. The Correspondence

In this section we describe the correspondence $K_\Gamma(X) \leftrightarrow \mathcal{M}$ between log stable maps and broken flows. We fix some notation. Let F, N be the fan and lattice of X, $L \cong \mathbb{Z}$ the sublattice corresponding to the one parameter subgroup T, $Q = N/L$ the quotient lattice, (G, Q_κ, Q) the toric-fan of $\mathcal{M}$,and (F', N'_σ, N) the toric fan of $\mathcal{U}$, as in section 3. Let us also denote the coarse moduli space $X /\!\!/_C T$ of $\mathcal{M}$ by M to simplify notation, and the coarse moduli space of the universal family $\mathcal{U}$ by U. The closure of T in X is a $\mathbb{P}^1$, which defines a homology class β of X; this $\mathbb{P}^1$ also intersects X along the torus invariant divisor $X - T'$, and hence specifies a contact order with the divisor at 0 and ∞ - these contact orders in fact corresponds to two vectors in the lattice N. These vectors are the primitive vector v along L defining T, and its negative $-v$, respectively. The stack of log stable maps $K_\Gamma(X)$ we consider will be log stable maps from genus 0 two marked curves with image the homology class β and the prescribed contact orders v,$-v$ along the divisor.

Lemma 6. *The morphism $\mathcal{U} \to \mathcal{M}$ has reduced fibers.*

Proof: The statement is local, so it suffices to check over the etale base change $X'_\sigma \to X_\tau$ of $\mathcal{U} \to \mathcal{M}$ for a cone τ in G and a cone $\sigma' \in F$ mapping to τ. The map $X'_\sigma \to X_\tau$ has reduced fibers if and only if $p(\sigma' \cap N) = \tau \cap Q$ - see for example lemma 5.2 in the paper of Abramovich-Karu [AK00]. In other words, we need $p(N'_\sigma) = N_\tau$. This holds by construction.

In fact, the morphism $\mathcal{U} \to \mathcal{M}$ is the minimal modification of $U \to M$ with this property. This statement can be interpreted in two ways. The first interpretation is that the monoids N'_σ, Q_τ are the maximal submonoids of $N \cap \sigma$, $Q \cap \tau$ for which the associated morphism of toric stacks has reduced fibers - every other choice of such monoids yields submonoids of N'_σ, Q_τ. This is clear. The other, more formal formulation is the following.

Lemma 7. *Consider a diagram of lattices*

$$\begin{array}{ccc} K & \xrightarrow{i} & N \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ P & \xrightarrow[j]{} & Q \end{array}$$

and a map

$$\begin{array}{ccc} Z & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & M \end{array}$$

of toric stacks in these lattices. Suppose $Z \to Y$ has proper and reduced fibers. Then $Z \to Y$ factors uniquely through $\mathcal{U} \to \mathcal{M}$.

Proof: Suppose v is a lattice point in the monoid P_κ corresponding to a cone κ in P. By the criterion given in the proof of lemma 6, there is a lattice point w_λ mapping to v under q in every monoid K_λ corresponding to a cone λ in P mapping to κ. Suppose v maps to $p(v) \in \tau$ in Q, and let σ denote all the cones mapping to τ isomorphically. By properness, all of $p^{-1}(\kappa)$ is a union of cones in the fan of X, hence, for each σ, there is at least a cone λ with $i(\lambda) \subset \sigma$. Thus $p(v) = q(i(w_\lambda))$ is in Q_τ, and hence also $i(u)$ is in N_σ for any u in K_λ with $i(\lambda) = \sigma$. The uniqueness part of the statement follows because the lattices Q_τ, N_σ map injectively into Q, N.

Observe that we in fact do not need to assume that Z maps to U, but simply to X. Then, by lemma 4, a map $Z \to U$ is automatic.

Since the universal log stable curve $\mathfrak{C}_\Gamma(X) \to K_\Gamma(X)$ is proper and has reduced fibers, and its coarse moduli space is $U \to M$, we obtain

Corollary 1. *There exists a Cartesian diagram*

$$\begin{array}{ccc} \mathfrak{C}_\Gamma(X) & \longrightarrow & \mathcal{U} \\ \big\downarrow & & \big\downarrow \\ K_\Gamma(X) & \longrightarrow & \mathcal{M} \end{array}$$

To show that the horizontal arrows are isomorphisms, we need maps in the opposite direction as well. To get such maps, we need to study the maps of the toric data $(F', N'_\sigma, N) \to (G, Q_\tau, Q)$ more carefully. For every cone τ in G, there are precisely two kinds of cones σ in F' that map to τ surjectively. The first kind has $\dim(\tau) = \dim(\sigma)$ and maps isomorphically, and satisfies $N_\sigma \cong Q_\tau$ under p, by construction. The second has $\dim(\sigma) = \dim(\tau) + 1$. This second kind also breaks into two types:

Lemma 8. *Suppose σ is a cone in F' with $p(\sigma) = \tau$ and $\dim(\sigma) = \dim(\tau) + 1$. Then either*
(i) σ has precisely two faces τ_1, τ_2 mapping isomorphically to τ, or
(ii)σ has precisely one face mapping isomorphically to τ.

Proof: First note that σ has at least one such face. This follows by a dimension estimate. We prove this by induction on the dimension n of the cone τ. If $n = 1$, $\dim(\sigma) = 2$, with σ spanned by two linearly independent vectors v_1, v_2. If neither $p(v_1)$ nor $p(v_2)$ span the vector space spanned by τ, we must have $p(v_1) = p(v_2) = 0$. It follows that v_1 and v_2 are both in the kerner $\mathbb{R}v$ of p, which is a contradiction, since the v_i are linearly independent. Here v is the vector determining the one parameter subgroup T. In general, take $n+1 = \dim(\sigma)$ linearly independent vectors $v_1, \cdots, v_{n+1}$. Consider the vector space H spanned by the images of the first n vectors $p(v_1), \cdots, p(v_n)$. If its dimension is n, the face generated by $v_1, \cdots, v_n$ maps surjectively onto τ and there is nothing to prove. If not, the dimension of H is $n-1$, as the kernel of p is $\mathbb{R}v$ and is one dimensional. Therefore, the vectors $v_1, \cdots, v_n$ lie on the n-dimensional space $H \oplus \mathbb{R}v$. By induction, $n-1$ of them map surjectively onto H, say $v_1, \cdots v_{n-1}$. Consider the face generated by $v_1, \cdots, v_{n-1}, v_{n+1}$. If the image of this face had dimension $n-1$ as well, v_{n+1} would be in $H \oplus \mathbb{R}v$ as well. Since $v_1, \cdots, v_n$ span $H \oplus \mathbb{R}v$, we would get a linear relation between $v_1, \cdots, v_{n+1}$, a contradiction. This proves that there is at least one face mapping surjectively onto τ.

On the other hand, there cannot be more than two such faces. Suppose for instance there were three, and let p_1, p_2, p_3 be three points mapping to the same point in τ. By convexity, the line segment through p_1, p_3, which is parallel to v, has to lie entirely in σ and pass through p_2. On the other hand, by the definition of a face, this line segment can intersect a face only at its extreme points, a contradiction.

Geometrically, the cone $\tau \in G$ corresponds to a special point e_τ in $\mathcal{M}$, and the cones σ mapping surjectively to τ correspond to the fiber of e_τ in $\mathcal{U}$, which is a nodal curve. The cones σ with $\sigma \cong \tau$ correspond to the irreducible components of the curve. The cones σ of higher dimension correspond to the nodes of the curve if they are of type

(i) and to the marked points of the curve if they are of type (ii). In fact, more can be said.

Lemma 9. *(i) Suppose σ is a cone of F' mapping surjectively to τ with $\dim(\sigma) = \dim(\tau) + 1$ and two faces τ_1, τ_2 isomorphic to τ. Then*

$$N_\sigma \cong Q_\tau \times_{\mathbb{N}} \mathbb{N}^2$$

The map $Q_\tau \to \mathbb{N}$ is the map ρ_σ sending $w \in Q_\tau$ to $|p^{-1}(w)| - 1$, the number of preimages of w minus one.
(ii) Suppose σ is a cone of F' mapping surjectively to τ with $\dim(\sigma) = \dim(\tau) + 1$ and precisely one face isomorphic to τ. Then

$$N_\sigma \cong Q_\tau \times \mathbb{N}$$

Proof: (i) Over a point w in Q_τ, let w_1, w_2 be the unique preimages in N_{τ_1}, N_{τ_2}. Every point in σ in the fiber of w, which is a two dimensional cone, can be written as

$$\frac{\alpha}{\rho_\sigma(w)} w_1 + \frac{\beta}{\rho_\sigma(w)} w_2$$

with $\alpha + \beta = \rho_\sigma(w)$. Since every point in N_σ is in the fiber of some point in Q_τ, an isomorphism $Q_\tau \times_{\mathbb{N}} \mathbb{N}^2 \to N_\sigma$ is given by

$$(v, \alpha, \beta) \mapsto \frac{\alpha}{\rho_\sigma(v)} v_1 + \frac{\beta}{\rho_\sigma(v)} v_2$$

(ii) This case happens only when v or $-v$ is in σ. We assume without loss of generality that $v \in \sigma$. Given a point $w \in Q_\tau$, its fiber in N_σ is then precisely $w' + nv$ for the unique lift w' of w in the face of σ mapping isomorphically to τ.

Considering the dual lattices $\mathrm{Hom}(\cdot, \mathbb{N})$, which we denote by P_τ, M_σ for brevity, we obtain that one of the three holds:A. $M_\sigma \cong P_\tau$, in the case $\dim(\sigma) = \dim(\tau)$ corresponding to an irreducible component on the fiber. B. $M_\sigma \cong P_\tau \oplus_{\mathbb{N}} \mathbb{N}^2$, corresponding to the case of a node; the map $\mathbb{N} \to P_\tau$ here is the dual of the map ρ_q in the proof of lemma 9. C. $M_\sigma \cong P_\tau \oplus \mathbb{N}$, corresponding to a marked point. We have the following lemma on monoids:

Lemma 10. *For any integral monoid P, the map $P \to P \oplus_{\mathbb{N}} \mathbb{N}^2$ is integral, for any map $\rho : \mathbb{N} \to P, 1 \mapsto p$.*

Proof: We use the description of section 4,

$$P \oplus_{\mathbb{N}} \mathbb{N}^2 = \{(a, b) \in P \times P : b - a \in p\mathbb{Z}\}$$

Suppose $(x, x) + (s_1, s_2) = (y, y) + (t_1, t_2)$. Then, either $s_2 - s_1$ or $s_1 - s_2$ is in P, as a multiple of p. Assume $s_2 - s_1 = np, n \geq 0$ without loss of generality. We then have $t_2 - t_1 = s_2 - s_1$ is also in P. Therefore,

$$(x,x)+(s_1,s_2)=(x,x)+(s_1,s_1)+(0,s_2-s_1)=(y,y)+(t_1,t_1)+(0,t_2-t_1)$$

which proves integrality.

Since $P \to P, P \to P \oplus \mathbb{N}$ are trivially integral, we obtain

Corollary 2. *The map $\mathcal{U} \to \mathcal{M}$ is an integral map of log stacks; in particular, it is flat.*

Since $\mathcal{U} \to \mathcal{M}$ is an integral, proper map of log stacks with reduced fibers, we obtain

Corollary 3. *$\mathcal{U} \to \mathcal{M}$ is a log curve.*

Therefore, by the universal property of the stack $K_\Gamma(X)$, we obtain a cartesian diagram

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathfrak{C}_\Gamma(X) \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & K_\Gamma(X) \end{array}$$

Theorem 2. *The moduli stack $K_\Gamma(X)$ of log stable maps is isomorphic to the moduli stack $\mathcal{M}$ of broken flows of T.*

Proof: The compositions of the maps $K_\Gamma(X) \to \mathcal{M} \to K_\Gamma(X)$ and $\mathcal{M} \to K_\Gamma(X) \to \mathcal{M}$ are the identity by the universal properties of $\mathcal{M}$ and $K_\Gamma(X)$.

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