TROPICALIZING THE MODULI SPACE OF BROKEN TORIC VARIETIES

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1. Introduction

Moduli spaces of broken toric varieties appeared in the work of Alexeev [Ale02]. Later on, Olsson constructed moduli spaces of broken toric varieties equipped with a logarithmic structure, and used them to compactify the main component of Alexeev's moduli spaces. Alexeev and Brion generalized the constructions of [Ale02] by constructing moduli spaces $AB(V)$ of morphisms $f : X \to V$ whose source X is a broken toric variety and whose target V is a classical toric variety. This is a generalization of the results of [Ale02], since moduli spaces of broken toric varieties can be recovered from the moduli space $AB(\mathbb{P}^r)$ of morphisms to some appropriate projective space. In [AM14], we constructed the logarithmic version of this moduli space and showed it compactifies the main component of the moduli space of [AB06], in analogy with Olsson's work. An interesting feature of this construction is that in [AM14], we worked entirely with the "N" side of toric geometry, i.e with fans in the cocharacter lattice N of the torus of V, whereas all of the constructions of [Ale02],[AB06],[Ols08] take place on the "M" side, i.e the authors work with polytopes in the character lattice M.

The goal of this paper is to bridge this gap. We briefly recall some results from Alexeev's and Olsson's work. In [Ale02], a broken toric variety is combinatorially described by subdividing a fixed polytope Q in the lattice M into the subpolytopes defined as the domains of linearity of some piecewise affine linear function on Q. In this paper, given the data of Q and the piecewise affine linear function f, we construct a fan in the lattice $N \oplus \mathbb{N}$ lying over $\mathbb{N}$, i.e a one parameter toric degeneration, whose general toric variety is the toric variety associated to Q and whose central fiber is the broken toric variety determined by the f. The fan we construct is actually rather simple: it is the cone over a polyhedral complex determined easily by f – this is definition 2.0.4 in the text. In fact, this polyhedral complex coincides with the domains of linearity of the Legendre transform of the function f. This process is self-dual, and functions f which give isomorphic (polarized) broken toric varieties give isomorphic toric degenerations, so we obtain a one to one correspondence between broken toric varieties and certain toric degenerations.

Next, we study this problem in families. We study the functor $\mathcal{F}_S$ on the category of cones which assigns to every cone σ all fans in $\sigma^{\mathrm{gp}} \times N$ with general fiber the toric variety associated to Q and central fiber the broken toric variety associated to a specific subdivision of Q, together with a polarization. The main result of this paper is that this functor is representable. This is theorem 4.0.31. In fact, the cone representing the functor is the cone of all piecewise affine linear upper convex functions on Q whose domain of linearity is the given subdivision. We then study the functor $\tilde{\mathcal{F}}_S$ consisting of isomorphism classes of $\mathcal{F}_S$, and show in 4.0.37 that it is representable by the cone of piecewise affine linear upper convex functions on Q with domain of linearity the subdivision S, up to linear functions. This result has two interpretations: it may either be interpreted as giving the tropicalization of a certain stratum in Alexeev's moduli space, or, as a "mirror" type result as it relates combinatorial data in the lattice M to combinatorial data in the lattice N.

Theorem 4.0.37 closely resembles certain results in Olsson's work: a central tool in [Ols08] is, given a subdivision S of the polytope Q, a construction of a monoid H_S and of a "standard" family

$$\mathcal{P}_S \to \operatorname{Spec} \mathbb{Z}[H_S]$$

with analogous properties on the general and central fiber; all logarithmic broken toric varieties in Olsson's work are locally pulled back from one of these standard families. Theorem 4.0.37 provides an interpretation of H_S, or rather, of its saturation H_S^{sat} as the dual of the monoid of affine linear functions on Q which induce precisely the subdivision S, up to linear functions.

2. The Legendre transform

In this section we fix two vector spaces, $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$, which we take dual to each other. We furthermore fix a full dimensional polyhedral complex Q in $M_{\mathbb{R}}$. In the applications we have in mind Q is a polytope, but this is not important for the discussion that follows: a polyhedral complex can mean any subset of a vector space which is the convex hull of a finite collection of affine linear subspaces. We furthermore use the notation $\overline{N}_{\mathbb{R}} = N_{\mathbb{R}} \oplus \mathbb{R}$ and $\overline{M}_{\mathbb{R}} = M_{\mathbb{R}} \oplus \mathbb{R}$ to save notation.

Let $f : Q \to \mathbb{R}$ be a piecewise affine linear upper convex function on Q. By this we mean that there is a decomposition of Q into polytopes $(\omega_i)_{i\in S}$, such that the maximal chambers ω_i are precisely the domains where the function f is affine linear – in other words, f on ω_i is given as $\langle v_i, \rangle + a_i$ for some $v \in N_{\mathbb{R}}$. Such a decomposition is called a paving in Alexeev's and Olsson's work.

The *affine* linearity is a bit inconvenient from the point of view of classical toric geometry, so we circumvent it using the following construction. We consider the cone $C(Q)$ over Q in $\overline{M}_{\mathbb{R}}$, i.e the closure of the set

$$\{t(x, 1) : t \in \mathbb{R}_{\geq 0}, x \in Q\}$$

The paving S on Q extends naturally into a paving of $C(Q)$, whose cells are $C(\omega_i)_{i\in S}$, and which we will also refer to as S. The function f gives rise to a function $F : C(Q) \to \mathbb{R}$ by declaring $F(x, 1) = f(x)$ and extending to the rest of $C(Q)$ by linearity. The function F contains essentially the same information as f but is now an honest linear function, albeit defined on a higher dimensional complex.

We can associate several "dual" objects to the function f. First, we can define the *Legendre transform* $\hat{f}$ of f. This is the function with domain

$$\hat{Q} = \{v \in N_{\mathbb{R}} : \sup_{x \in Q} \langle v, x \rangle - f(x) < \infty\}$$

and whose value at $v \in \hat{Q}$ is

$$\hat{f}(v) = \sup_{x \in Q} \langle v, x \rangle - f(x)$$

We note that since f is piecewise affine linear, and Q is a polyhedral complex, the extreme points of $\langle v, x \rangle - f(x)$ can only occur at the vertices of Q. Hence $\hat{f}$ is a supremum of a finite number of affine linear functions, hence affine linear itself. The domains of linearity of $\hat{f}$ define a dual paving $\hat{S}$ of $\hat{Q}$. The function $\hat{f}$ is also upper convex – this is a well known classical property of the Legendre transform. Finally, the transform is involutive, i.e $\hat{\hat{f}} = f$.

Remark 2.0.1. In our case of interest, when Q is a polytope, the domain of definition $\hat{Q}$ of $\hat{f}$ is all of $N_{\mathbb{R}}$. This is evident since then Q is compact.

We now want to understand the relationship between $\hat{f}$ and $\hat{F}$, where $F : C(Q) \to \mathbb{R}$ is the function on the cone over Q as above. The function $\hat{F}$ itself turns out to be uninteresting - it is always the 0 function. However, its domain of definition $\hat{C(Q)}$ is interesting.

Lemma 2.0.2. *The set* $\hat{C(Q)}$ *is the set*

$$\{(v, a) \in \overline{N}_{\mathbb{R}} : \langle v, x \rangle + a \leq f(x) \text{ for every } x \in X\} \tag{2.0.1}$$

In particular, it is lower convex, and we have a surjective projection $\hat{C(Q)} \to \hat{Q}$*. The function* $\hat{f}$ *can be recovered from* $\hat{C(Q)}$*: the upper envelope of* $\hat{C(Q)}$ *is the graph of* $-\hat{f}$*.*

Proof. We first prove that $\hat{C(Q)}$ has the description (2.0.1). The domain $\hat{C(Q)}$ is by definition the set of $(v, a) \in \overline{N}_{\mathbb{R}}$ such that

$$\sup_{(y,b) \in C(Q)} \langle v, y \rangle + ab - F(y, b) < \infty$$

Since $C(Q)$ is the cone over Q, there is a bijection between $(y, b) \in C(Q)$ and (bx, b) for $x \in Q$. So the supremum above can be simplified as

$$\sup_{b \in \mathbb{R}_+, x \in Q} b(\langle v, x \rangle + a - f(x))$$

This number is finite if, and only if, $\langle v, x \rangle + a - f(x) \leq 0$ for every x, as claimed. Note that if v is in $\hat{Q}$, the expression $\langle v, x \rangle - f(x)$ has $\hat{f}(v)$ as its least upper bound as x ranges in Q, and hence (v, a) is in $\hat{C(Q)}$ for precisely all a with $a \leq -\hat{f}(v)$. From this it follows that the projection $\hat{C(Q)} \to \hat{Q}$ is surjective and that the upper envelope of $\hat{C}$ is the graph of $-\hat{f}(v)$. Furthermore, since f is convex, so is $\hat{f}$, and thus $\hat{C(Q)}$ is convex.

□

In fact, we can also recover the paving $\hat{S}$ from $\hat{C(Q)}$. Given a polyhedral complex X, and subcomplexes X_1 and X_2 of X, we say that there is a break between X_1 and X_2 if no line segment in X can connect a point of $X_1 - (X_1 \cap X_2)$ with a point of $X_2 - (X_1 \cap X_2)$.

Since the breaks in the graph of $-\hat{f}$ happen precisely where the domains of linearity of $\hat{f}$ change, we see that the paving $\hat{S}$ can be read off from the breaks of the upper envelope of $\hat{C}$.

We now give another construction of a "dual" polyhedral complex associated to $f : Q \to \mathbb{R}$. Note that the cells of the paving S can be classified as *internal* and *external*: by an internal cell we mean a cell that intersects the interior of Q, and by an external cell a cell that does not intersect the interior of Q, i.e is a cell of the original polytope Q. For example, all maximal cells in S are internal and all faces of S that are also faces of Q are external.

On a maximal cell ω_i of S the function f is affine linear, so is given by a unique vector $(v_i, a_i) \in \overline{N}_{\mathbb{R}}$, i.e f is evaluation of (v_i, a_i) against $(x, 1)$ for $x \in \omega_i$. Whenever ω_1 and ω_2 intersect on a facet, the function f agrees on $\omega_1 \cap \omega_2$ with evaluation against vectors (v, a) on a unique line in $\overline{N}$, which passes through (v_1, a_1) and (v_2, a_2). Generally, let ω be an internal codimension k cell, and ω' the cells containing ω as a face. Then f takes the same value on ω as a k-plane of vectors in $\overline{N}$, which contains the corresponding cells for ω'. We let P_ω denote the convex hull of the vertices corresponding to the maximal cones containing ω.

For an external facet ω, the function $f|_\omega$ coincides again with evaluation against a line of vectors (v, a) again. This line contains the vector (v_i, a_i) for the *unique* maximal cell ω_i that contains the external facet ω. We let P_ω denote the half line that agrees with f on ω, begins at (v_i, ω_i), and decreases on ω_i. For a general external face ω, we take P_ω to be the convex hull of the half lines P'_ω for the external facets that contain ω. We summarize these considerations in the following definition:

Definition 2.0.3 (Dual Polyhedral Complex)**.** Let $\overline{P}$ be the polyhedral complex in $\overline{N}_{\mathbb{R}}$ with vertices $\overline{P}_{\omega_i} = (v_i, a_i)$ for the maximal cells ω_i of S, where (v_i, a_i) is the unique vector which agrees with f on ω_i; and with a k-cell $\overline{P}_\omega = ConvHull(\overline{P}_{\omega_i})$ for each *internal* codimension k cell ω in S, where ω_i range through the maximal cells of S that contain ω. Furthermore, for each external facet ω of S, $\overline{P}$ has as a cell the half-line $\overline{P}_\omega$ which agrees with f on ω, begins at $\overline{P}_{\omega_i}$ and decreases on ω_i for the unique maximal face $\omega \subset \omega_i$; and for each external face ω in general the convex hull $ConvHull(\overline{P}_{\omega'})$ where ω' ranges through the external facets of S containing ω.

Definition 2.0.4 (Dual Polyhedral Complex)**.** We also define a polyhedral complex P in $N_{\mathbb{R}}$ as the image of $\overline{P}$ in $N_{\mathbb{R}}$ under the natural projection $\overline{N}_{\mathbb{R}} \to N_{\mathbb{R}}$.

Lemma 2.0.5. *The complex $\overline{P}$ maps isomorphically onto its image under the projection $\overline{N}_{\mathbb{R}} \to N_{\mathbb{R}}$.*

Proof. We simply have to show that the projection $\overline{N}_{\mathbb{R}} \to N_{\mathbb{R}}$ is injective on $\overline{P}$. Suppose two vectors $(v_i, a_i) \in \overline{P}$ project to the same vector in $N_{\mathbb{R}}$, i.e $v_1 = v_2 = v$. By construction, there are two cells ω_1, ω_2 in Q such that $f(x) = \langle v, x\rangle + a_i$ on ω_i. Since f is upper convex, it follows that $\langle v, x\rangle + a_2 \leq \langle v, x\rangle + a_1$ on ω_1, and vice versa on ω_2. Hence, we must have $a_1 = a_2$, which means that $(v_1, a_1) = (v_2, a_2)$. □

Example 2.0.6. Let Q be the polytope in $\mathbb{R}^2$ with vertices $(1, 1), (1, -1), (-2, 1)$ (this polytope corresponds to $\mathbb{P}^2$ with the line bundle $\mathcal{O}(7)$). We take for f the function whose value at the three vertices is 1 and whose value at $(0,0)$ is 0. On the cell determined by $(0,0),(1,1),(1,-1)$, the function f agrees with the vector $(\alpha,\beta,\gamma) \in \overline{\mathbb{R}^2} = \mathbb{R}^3$ such that

$$\begin{aligned} 0\alpha + 0\beta + \gamma &= 0 \\ \alpha + \beta + \gamma &= 1 \\ \alpha - \beta + \gamma &= 1 \end{aligned}$$

Solving gives the vector $(\alpha,\beta,\gamma) = (1,0,0)$. For the two other maximal cells, determined by $(0,0),(1,-1),(-2,1)$ and $(0,0),(1,1),(-2,1)$ we compute similarily the vectors $(-2,-3,0)$ and $(0,1,0)$ respectively. For the face determined by $(0,0)$ and $(1,1)$ we attach the line segment between $(0,1,0)$ and $(1,0,0)$. For the face determined by $(0,0)$ and $(1,-1)$ we attach the line segment between $(0,1,0)$ and $(-2,-3,0)$ and for the face determined by $(0,0)$ and $(1,1)$ the line segment between $(1,0,0)$ and $(-2,-3,0)$. For the cell $(0,0)$ we attach the triangle with vertices $(1,0,0),(0,1,0),(-2,-3,0)$. This exhausts the internal cells of S.

The external cells of S are the three vertices of Q and the line segments between $(1,1),(1,-1),(-2,1)$. For the facet between $(1,1)$ and $(1,-1)$ we attach the half line of vectors (α,β,γ) such that

$$\begin{aligned} \alpha + \beta + \gamma &= 1 \\ \alpha - \beta + \gamma &= 1 \end{aligned}$$

i.e $\gamma = 1-\alpha$ which starts at $(1,0,0)$ and decreases on $(0,0),(1,1),(1,-1)$. This is the half line $(\alpha,0,1-\alpha)$, with $\alpha \geq 1$. Similarily, for the facet determined by $(1,1)$ and $(-2,1)$ we attach the half line (α,β,γ) which satisfies

$$\begin{aligned} \alpha + \beta + \gamma &= 1 \\ -2\alpha + \beta + \gamma &= 1 \end{aligned}$$

and decreases on $(0,0),(1,1),(-2,1)$, i.e the line $\alpha = 0, 3\beta+3\gamma = 3$, i.e $(0,\beta,1-\beta)$, $\beta \geq 1$. Finally, to the facet $(1,-1),(-2,1)$ we attach the half line $(2\gamma-2, 3\gamma-3, \gamma)$ for $\gamma \leq 0$.

The dual polyhedral complex considered in definition 2.0.3/2.0.4 is actually intimately connected with the Legendre transform.

Lemma 2.0.7. *The dual polyhedral complex $\overline{P}$ coincides with the graph of the function $-\hat{f}$. Therefore, the polyhedral complex P coincides with the domains of linearity of $\hat{f}$.*

Proof. Let (v,a) be a point in $\overline{P}$. For an element $x \in Q$, we note that by definition

$$f(x) = \langle v_i, x\rangle + a_i$$

for some vertex (v_i,a_i) of $\overline{P}$, since every $x \in Q$ belongs to at least one maximal cell ω_i of the paving S. Note also that

$$f(x) = \langle v, x\rangle + a$$

for any $x \in Q$ which belongs to the cell ω corresponding to (v,a). Note furthermore that $\langle v,x\rangle + a \leq f(x)$ for all $x \in Q$ by convexity, with equality precisely for $x \in \omega$. We thus have

$$\langle v, x\rangle - f(x) \leq -a$$

for all x, with equality on ω. Thus

$$\hat{f}(v) = \sup_{x \in Q} \langle v, x \rangle - f(x) = -a$$

which is the content of the lemma. □

We now note a few special cases of interest.

Lemma 2.0.8. *Let $f : Q \to \mathbb{R}$ be the 0 function. Then the dual polyhedral complex P is the normal fan of the polytope Q.*

Proof. In the case when $f = 0$, the Legendre transform is $\hat{f} = \sup_{x \in Q} \langle v, x \rangle$. This coincides with the supremum $\sup_{x_i \in Q} \langle v, x_i \rangle$, where x_i denote the extreme vertices of the polytope Q since $\langle v, \rangle$ is linear on Q. On the other hand, the maximal cones of the normal fan of Q are precisely the cones $\sigma_i = \{v \in N : \langle v, x - x_i \rangle \geq 0 \forall x \in Q\} = \{v \in N : \langle v, x_j - x_i \rangle \geq 0 \forall x \in Q\}$. A vector is in σ_i then if and only if $\hat{f}(v) = \langle v, x_i \rangle$, as desired. □

Corollary 2.0.9. *Let $f : Q \to \mathbb{R}$ be linear, given by evaluation against $\langle u, \rangle$, $u \in N_{\mathbb{R}}$. Then the dual polyhedral complex of f is the normal fan of Q translated by the vector u.*

Lemma 2.0.10. *If $f - g$ is constant, then the dual polyhedral complexes P_f, P_g are the same.*

Proof. If $f - g = c$, then $\hat{f} - \hat{g} = -c$, and thus the result follows from lemma 2.0.7. □

Combining corollary 2.0.9 with lemma 2.0.10 gives

Corollary 2.0.11. *If $f - g$ is affine linear, then P_f is the same as P_g up to translation.*

We now make the following interesting observation. Given a function f on Q, and a paving S as above, we may regard the paving S as a category with one object for each cell in S and a unique map $\omega \to \omega'$ whenever ω is a face of ω'. We furthermore know that for each maximal cell ω, the polyhedral complex P_{f_ω} associated to $f_\omega : \omega \to \mathbb{R}$ is the normal cone of ω, translated by the vector u which agrees with f_ω on ω.

Lemma 2.0.12. *We have $P = \varprojlim_{\omega \in S} P_{f_\omega}$, where the inverse limit is taken in the category of polyhedral complexes in $N_{\mathbb{R}}$, that is, agrees with intersections in $N_{\mathbb{R}}$.*

We have in particular that

Lemma 2.0.13. *The star of a vertex $v \in P$ is the same as the normal fan of the maximal cell ω corresponding to v.*

Proof. By definition, the star of v is the set

$$\{\mathbb{R}_{\geq 0}(P_f - v) : f < \omega\}$$

where $f < \omega$ is a face of ω in S. Look now at a neighborhood U of v which is so small that it contains no other vertex of P. On such a neighborhood we know that $\hat{f}$ coincides with $\widehat{f|_\omega}$. By corollary 2.0.8, on this neighborhood of v, the polytope P coincides with the polytope associated to $f|_\omega$, i.e the normal fan to ω translated by v. Thus the sets $(P_f \cap U) - v$ for a face f of ω and the cones of the normal fan of ω intersected with $(U - v)$ coincide. Dilating these cells by $\mathbb{R}_{\geq 0}$ yields the result. □

3. One Parameter Degenerations

We will now fix two dual lattices N, M with associated real vector spaces $N_{\mathbb{R}}, M_{\mathbb{R}}$, and a full dimensional polytope Q in M. In this section we will consider functions $f : Q \to \mathbb{R}$ which satisfy the following property: The function $F : C(\omega) \to \mathbb{R}$ determined by f on the cone over ω takes integer values on the integral points of $C(\omega)$ for every cell ω in the paving S defined by f. We will denote such a function by $f : Q \to \mathbb{Z}$ for simplicity. We observe:

Lemma 3.0.14. *If $F : C(\omega) \to \mathbb{R}$ takes integral values on the integral points of $C(\omega)$ for a maximal cell ω, then F actually takes integral values on all of $\overline{M}$. Hence the vertices of the dual polyhedral complex of f of definition* 2.0.3 *lie on the lattice $\overline{N} \subset \overline{N}_{\mathbb{R}}$.*

Proof. For a maximal cell ω the cone $C(\omega)$ is a full dimensional cone in $\overline{M}$. Therefore, a subset of the integral points of $C(\omega)$ forms a basis for M (see [Ful93] – this is the standard result necessary for resolution of singularities in toric geometry). Therefore, if f takes integral values on all integral points of $C(\omega)$ it takes integral values on all of $\overline{M}$. □

Question 3.0.15. We do not know if it would suffice to require that f takes integral values simply on the points of the cells ω of S – in other words, we do not know whether it is possible to choose a basis for $\overline{M}$ that lies in $\omega \oplus 1 \subset \overline{M}$. This is probably a standard result but eludes us at the moment.

Definition 3.0.16. Fix a polytope Q. We denote the set of functions $f : Q \to \mathbb{Z}$ by $Aff(Q)$. If we further fix a paving S, we denote by $Aff_S(Q)$ the set of functions $f \in Aff(Q)$ which define S as their associated paving.

Lemma 3.0.17. *The set $Aff(Q)$ is a monoid. The set $Aff_S(Q)$ is also a monoid, isomorphic to $\varprojlim_{\omega \in S} \operatorname{Hom}(C(\omega), \mathbb{Z})$.*

Remark 3.0.18. The function $f : Q \to \mathbb{Z}$ defines a paving on S, and a broken toric variety with a morphism to $\mathbb{P}^r$, for $r = card(Q \cap M) - 1$. The idea is simple: The broken toric variety is simply

$$\varinjlim_{\omega \in S} X(\omega)$$

where $X(\omega)$ is the toric variety associated to the polytope ω, and the morphism is the direct limit of the canonical morphisms $X(\omega) \to \mathbb{P}^{|\omega \cap M|-1}$ determined by each polytope ω. For details, see [Ale02].

Definition 3.0.19 (One Parameter Degeneration Associated to f)**.** Let $f : Q \to \mathbb{Z}$ be an upper convex piecewise affine linear function on Q. To f we associate the fan F in $\overline{N}$ which is the cone over the polyhedral complex P of f in N. The fan F has a canonical morphism to $\mathbb{N}$, namely the restriction of $\overline{N} \to \mathbb{Z}$ to F; and furthermore F comes equipped with a canonical upper convex function which sends (v, a) to $a\hat{f}(\frac{v}{a})$ – in other words, the function is the cone over the Legendre transform $\hat{f}$ on P –, which we will also denote by $\hat{f}$.
We abbreviate this data as $(F, \hat{f}) \to \mathbb{N}$ and call it the one parameter degeneration associated to f.

Remark 3.0.20 (Conventions)**.** In what follows, we will denote by $X(F)$ the toric variety associated to a fan F. Furthermore, we will use the same notation $\mathbb{P}^r$ for r-dimensional projective space and its fan. Context should make clear which of the two notions we are referring to.

Theorem 3.0.21. *The data $(F, \hat{f}) \to \mathbb{N}$ defines a diagram*

$$\begin{array}{ccc} X(F) & \xrightarrow{\phi} & \mathbb{P}^r \\ {\scriptstyle p}\downarrow & & \\ \mathbb{A}^1 & & \end{array}$$

where $r = card(Q \cap M) - 1$. Furthermore, the map p is proper, flat, and has reduced fibers; the fiber $p^{-1}(z)$ for $z \in \mathbb{C}^*$ is isomorphic to the toric variety associated to Q, and ϕ restricts to the canonical map of this toric variety to $\mathbb{P}^r$; and the fiber $p^{-1}(0)$ is isomorphic to the broken toric variety determined by the paving S with its canonical map to $\mathbb{P}^r$.

Proof. We first show that the one parameter family $X(F) \to \mathbb{A}^1$ has the stated properties. Note that on the level of fans we have a morphism $p : F \to \mathbb{N}$, and $p^{-1}(n)$ is the dual polyhedral complex associated to the function nf. Hence, $p^{-1}(0) = R_Q$, the normal fan of Q by lemma 2.0.8. This is precisely the fan associated to the toric variety defined by the polytope Q. Thus the general fiber of the morphism $X(F) \to \mathbb{A}^1$ is the toric variety associated to Q, as claimed. We now look at $p^{-1}(1)$. By proposition 3.5 of [NS06], the geometric realization of this fiber is the direct limit

$$\varinjlim{}_{P_\omega} X_{P_\omega}$$

for P_ω the cells of P, where by X_{P_ω} we mean the toric variety with fan

$$\{R_{\geq 0}(P_f - P_\omega) : f < \omega\}$$

In particular, for a vertex v corresponding to a maximal face ω of S, we have the set

$$\{R_{\geq 0}(P_f - v) : f < \omega\}$$

which by lemma 2.0.13 is precisely the normal fan to ω. Hence, the geometric realization of $p^{-1}(0)$ has the same components as the broken toric variety associated to the paving S as constructed by Alexeev. Furthermore, the gluing relations for the two schemes are the same, and thus they are isomorphic.

The fact that p is proper follows from remark 2.0.1, since whenever Q is compact, the support of the dual polyhedral complex of any f on Q is all of $N_{\mathbb{R}}$. To prove the claim that p is flat and has reduced fibers, we make the following observation [M]: to show that a map $F \to G$ is flat and has reduced fibers, it suffices to show that

(1) Every time the interior of a cone $\sigma \in F$ maps into the interior of a cone $\tau \in G$, σ surjects onto τ. (Equidimensionality)

(2) Whenever σ surjects onto τ, we also have that the integral points of σ surject onto the integral points of τ. (Reduced Fibers)

In our case, condtion (1) is automatic, and condition (2) follows from the fact that the vertices of P are integral.

The only thing that remains to be seen is that F maps to $\mathbb{P}^r$. This is simple – the function $\hat{f}$ on F is upper convex, so by the results of [Ful93], we obtain a morphism

$$X(F) \to \mathbb{P}^{|P_{\hat{f}} \cap \overline{M}|}$$
$$z \mapsto [\chi^{-y}(z)]_{y \in P_{\hat{f}} \cap \overline{M}}$$

where $P_{\hat{f}}$ denotes the dual polytope $P_{\hat{f}} = \{y \in \overline{M}_{\mathbb{R}} : y \leq \hat{f}\}$ and χ^y denotes the character associated to y in $\mathbb{C}[\overline{M}]$ (Here we have changed the conventions of [Ful93] slightly, since we are working with upper convex rather than lower convex functions; we chose to do so to keep consistent with Alexeev's choice of upper convex functions and the literature on the Legendre transform). By lemma 2.0.2 this is precisely the set of vectors in $\overline{M}_{\mathbb{R}}$ that lie under the graph of $-(\hat{\hat{f}}) = -f$. But note that if for some z all sections $\chi^{(-x,f(x))}(z) = 0$, then all sections $\chi^{(-x,f(x)-a)}(z) = 0$ as well, for $a \in \mathbb{N}$. So the rational map

$$z \mapsto [\chi^{(-x,f(x))}(z)]_{x \in Q \cap M} \in \mathbb{P}^r$$

is in fact regular (in other words, it suffices to only consider characters of vectors that lie on the upper envelope here of $P_f \cap \overline{M}$ here). This gives the morphism to $\mathbb{P}^r$. It further agrees over $p^{-1}(t)$ with the morphism $X(Q) \to \mathbb{P}^r$ associated to the polytope Q and over 0 with the morphism associated to the paving S, and thus completes the proof. □

Remark 3.0.22. The morphism to $\mathbb{P}^r$ of theorem 3.0.18 in fact has the following pleasing interpretation. Note that on the level of fans, this morphism is given by

$$F \to \mathbb{N}^{r+1}$$
$$(v,a) \mapsto (a\hat{f}(\frac{v}{a}) - (\langle v, x\rangle - af(x)))_{x \in Q \cap M}$$

The upper convex function $\hat{f}$ on F corresponds to a divisor D; Consider the cone

$$C = \{(v,a,r) : (v,a) \leq rf(x)\}$$

This is the cone of all vectors in $\overline{N} \oplus \mathbb{Z}$ lying under the graph of $-\hat{f}$ on F. The sections of the line bundle $\mathcal{O}(D)$ correspond to morphisms

$$C \to F \times \mathbb{N}$$

over F. A basis for these sections is given by the points $x \in Q \cap M$, taking an $(l = (v,a), r) \in C$ to $rf(x) - l(x)$. Restricting to the upper envelope, i.e the points $(v, -r\hat{f}(\frac{v}{r}), r)$ gives precisely the morphism $(v,r) \mapsto r\hat{f}(\frac{v}{r}) + f(x) - \langle v, x\rangle$ - thus precisely the morphism of theorem 3.0.18.

Example 3.0.23. Let N and M be $\mathbb{Z}$, and fix Q the polytope between $-2, 2$ in M. We will consider two different functions, f and g, giving two different subdivisions. First, take f to be the function with values $f(-2) = 0, f(-1) = 0, f(0) = 0, f(1) = 1, f(2) = 2$. Note that on the maximal cell $[-2, 0]$ the function f is the 0 function, hence is represented by $(0, 0) \in \overline{N}$, and on the maximal cell it is given by the linear function with slope 1, hence represented by $(1, 0)$. Thus, the dual polyhedral complex P is the unique polyhedral complex in N whose support is all of $N_{\mathbb{R}}$ with vertices at 0 and 1. The function $\hat{f}(v)$ is the supremum of the three functions

$$-2v, 0, 2v - 2$$

hence coincides with $-2v$ on $[-\infty, 0]$, with 0 on $[0, 1]$, and with $2v-2$ on $[1, \infty]$. The fan F is the fan generated by the three cones $\sigma_{-2v} = \mathrm{Span}(-1, 0), (0, 1)$, $\sigma_0 = \mathrm{Span}(0, 1), (1, 1)$, $\sigma_{2v-2} = \mathrm{Span}(1, 1), (1, 0)$. The function $\hat{f}$ takes the form

$$(v, a) \mapsto \begin{cases} -2v & \text{if } v \leq 0 \\ 0 & \text{if } v \leq a \\ 2v - 2a & \text{if } v \geq a \end{cases}$$

(i.e $\hat{f}$ has domains of linearity precisely the 3 cones of F). The morphism to $\mathbb{P}^4$ sends (v, a) to the element

$$(4v - 2, 3v - 2, 2v - 2, v - 1) \in \mathbb{Z}^4$$

(Here $\mathbb{Z}^4$ is the character lattice of $\mathbb{P}^4$), or, in projective coordinates, it is the map

$$z \mapsto [\chi^{(2,0)}(z) : \chi^{(1,0)}(z) : \chi^{(0,0)}(z) : \chi^{(-1,1)}(z) : \chi^{(-2,2)}(z)]$$

Next, consider the function $g : Q \to \mathbb{Z}$ defined by $g(-2) = 0, g(-1) = 0, g(0) = 0, g(1) = 0, g(2) = 1$. The paving associated to g is different here – its maximal cells are $[-2, 1]$ and $[1, 2]$. The dual polyhedral complex $\overline{P} \subset \overline{N}$ of g consists has as vertices the vectors $(0, 0), (1, -1)$; hence the polyhedral complex P of g is the same as of f: It is the unique polyhedral complex in $\mathbb{R}$ with support $\mathbb{R}$ and vertices at $0, 1$. However, the function $\hat{g}$ is different. It is given by

$$\sup\,(-2v, v, 2v - 1)$$

and hence coinicides with $-2v$ on $[-\infty, 0]$, with v on $[0, 1]$ and with $2v - 1$ on $[1, \infty]$. The fan G associated to g is the same fan F as of f then, with three cones $\sigma_{-2v} = \mathrm{Span}(-1, 0), (0, 1)$, $\sigma_v = \mathrm{Span}(0, 1), (1, 1)$, and $\sigma_{2v-1} = \mathrm{Span}(1, 1), (1, 0)$. The function $\hat{g}$ on G is given by

$$(v, a) \mapsto \begin{cases} -2v & \text{if } v \leq 0 \\ v & \text{if } v \leq a \\ 2v - a & \text{if } v \geq a \end{cases}$$

This gives a different map to projective space than f: the map sends (v, a) now to the element

$$(4v - 1, 3v - 1, 2v - 1, v - 1) \in \mathbb{Z}^4$$

or, in projective coordinates, $z \in X(G)$ goes to

$$[\chi^{(2,0)}(z) : \chi^{(1,0)}(z) : \chi^{(0,0)}(z) : \chi^{(-1,0)}(z) : \chi^{(-2,1)}(z)]$$

As one can see in the example above, even though two functions f and g may give isomorphic one parameter degenerations, the projective embedding associated to them is in general different. The precise result is as follows.

Definition 3.0.24. We consider two families $(F, \hat{f}) \to \mathbb{N}$ and $(G, \hat{g}) \to \mathbb{N}$ isomorphic if there is an isomorphism

$$\begin{array}{ccc} \overline{N} & \xrightarrow{m} & \overline{N} \\ \downarrow & & \downarrow \\ \mathbb{N} & \longrightarrow & \mathbb{N} \end{array}$$

which takes the fan F to the fan G; fixes the fiber over $0 \in \mathbb{N}$; and with the property that the morphisms from F to $\mathbb{P}^r$ defined by $\hat{f}$ and by $\hat{g} \circ m$ coincide.

Lemma 3.0.25. *The degeneration $(F, \hat{f}) \to \mathbb{N}$ associated to f is isomorphic to the degeneration $(G, \hat{g}) \to \mathbb{N}$ associated to g if and only if $f - g$ is affine linear on Q.*

Proof. Suppose $(F, \hat{f}) \to \mathbb{N}$ is isomorphic to $(G, \hat{g}) \to \mathbb{N}$. Note that the isomorphism m of definition 3.0.24 must fix $N \times 0$ and $0 \times \mathbb{N}$, and thus is represented by a matrix of the form

$$\begin{pmatrix} Id & u \\ 0 & 1 \end{pmatrix}$$

for a vector $u \in N$, or explicitly, has the form $m(v, a) = (v + au, a)$. Therefore, $\hat{g} \circ m(v, 1) = \hat{g}(v + u, 1) = \hat{g}(v + u) = \sup_{x \in Q} \langle v + u, x \rangle - g(x)$, which is the Legendre transform of $h = g - \langle u, \rangle$. In order for $\hat{f}$ and $\hat{g} \circ m$ to give the same embedding to $\mathbb{P}^r$ we need that all sections

$$\begin{aligned} \hat{h}(v) + g(x) - \langle v, x \rangle - \langle u, x \rangle &= \hat{f}(v) + f(x) - \langle v, x \rangle \\ \hat{h}(v) - \hat{f}(v) &= f(x) - g(x) + \langle u, x \rangle \end{aligned}$$

Since the left hand side of the equation does not depend on x and the right hand side does not depend on v, it follows that both expressions are constant. Hence $f(x) - g(x) + \langle u, x \rangle = c$ for all $x \in Q \cap M$. Since the values of $f - g + \langle u, \rangle$ are determined by their values on the points $Q \cap M$, we get that $f - g = \langle -u, \rangle = +c$ is affine linear. This proves the necessity of the statement of the lemma.

To prove the sufficiency, suppose $g(x) = f(x) + \langle u, x \rangle + c$ for all $x \in Q$. Then $\hat{g}(v) = \sup_{x \in Q} \langle v, x \rangle - \langle u, x \rangle - f(x) - c = \hat{f}(v - u) - c$, i.e $\hat{f}(v) = \hat{g}(v + u) + c$. Thus the polytope of definition 2.0.4 P_f associated with f and the polytope P_g associated with g are related by $P_f + u = P_g$. Thus the map $m : \overline{N} \to \overline{N}$ sending (v, a) to $(v + au, a)$ gives an isomorphism from F to G, and the sections $(v, 1) \mapsto (\hat{f}(v) + f(x) - \langle v, x \rangle)$ and $(\hat{g}(v + u) + g(x) - \langle v, x \rangle - \langle u, x \rangle) = (\hat{f}(v) - c + f(x) + \langle u, x \rangle + c - \langle v, x \rangle - \langle u, x \rangle) = (\hat{f}(v) + f(x) - \langle v, x \rangle)$ are equal. □

4. Moduli

We are now ready to study how toric varieties degenerate to broken toric varieties in families. As before, we fix a polytope Q in the lattice M, and let $r = card(Q \cap M) - 1$. We define first the following functor:

Definition 4.0.26. Let $\mathcal{F}$ be the functor Cones $\to$ Sets^{op} which to a cone σ assigns

$$\mathcal{F}(\sigma) = \left\{ \begin{array}{ccc} F \subset N \oplus \sigma^{\mathrm{gp}} & \longrightarrow & \mathbb{P}^r \\ \downarrow p & & \\ \sigma & & \end{array} : p \text{ is proper, flat, with reduced fibers }, p^{-1}(0) = R_Q \right\}$$

Remark 4.0.27. It will be convenient to replace the map to $\mathbb{P}^r$ with an upper convex function $\phi : F \to \mathbb{R}$, linear on each cone of F. We will do so in what follows, and talk about the equivalent functor

$$\mathcal{F}(\sigma) = \left\{ \begin{array}{c} (F,\phi) \subset N \oplus \sigma^{\mathrm{gp}} \;:\; p \text{ is proper, flat, with reduced fibers }, p^{-1}(0) = R_Q \\ \downarrow p \\ \sigma \\ \phi \text{ upper convex} \end{array} \right\}$$

Before we proceed, we need to understand why $\mathcal{F}$ is a functor. The key ingredient here is that p has reduced fibers.

Lemma 4.0.28. *Suppose*

$$\begin{array}{c} F \\ \downarrow p \\ \sigma \end{array}$$

is proper, flat, and has reduced fibers. Then, for a map $\tau \to \sigma$ of cones, the geometric realization of the diagram

$$\begin{array}{ccc} F_\tau = F \times_\sigma \tau & \longrightarrow & F \\ \downarrow p_\tau & & \downarrow p \\ \tau & \longrightarrow & \sigma \end{array}$$

is cartesian in the category of schemes, and p_τ is also proper, flat, with reduced fibers.

Proof. The properness of p_τ is automatic, since proper morphisms are stable under pullback. The same is true for flatness. The interesting point is showing that the diagram is cartesian in the category of schemes, and that p_τ has reduced fibers. Both properties are local on F, so we may replace F by a single maximal cone C. Note that since p is flat with reduced fibers (i.e *saturated* in the terminology of [Ts]), the pushout $C^\vee \oplus_{\sigma^\vee} \tau^\vee$ of the dual monoids is saturated in its associated group $M \oplus (\tau^\vee)^{\mathrm{gp}}$. Furthermore, $C_\tau = C \times_\sigma \tau$ is saturated in $N \oplus \tau^{\mathrm{gp}}$. We claim that it follows that $C_\tau^\vee = C^\vee \oplus_{\sigma^\vee} \tau^\vee$. This is because the double dual of a cone in a lattice coincides with the sharpening of the saturation of the cone: that is, with the cone obtained by taking all elements of the lattice such that some multiple of the element is in the cone, and then taking the quotient by all the units in that cone. Therefore, taking the dual of $C_\tau^\vee$ recovers C_τ, while taking the dual of $C^\vee \oplus_{\sigma^\vee} \tau^\vee$ gives $C \times_\sigma \tau = C_\tau$ by the universal property of the pushout. Therefore, the two saturated, sharp monoids $C_\tau^\vee$ and $C^\vee \oplus_{\sigma^\vee} \tau^\vee$ have the same dual, i.e are isomorphic. On the other hand, the spectrum of the algebra $\mathbb{C}[C^\vee \oplus_{\sigma^\vee} \tau^\vee]$ coincides with the fiber product $X(C) \times_{X(\sigma)} X(\tau)$ in the category of schemes, since the functor $M \to \mathbb{C}[M]$ is left adjoint and thus commutes with direct limits. Hence, the fiber product $X(C) \times_{X(\sigma)} X(\tau)$ coincides with $X(C_\tau)$, as claimed. The fact that p_τ has reduced fibers follows from [T]. □

Therefore, $\mathcal{F}$ indeed defines a functor: given the morphism $i : \tau \to \sigma$, and the family (F, ϕ) over σ, we assign the family $(F_\tau, \phi \circ j)$ over τ, where $j : F_\tau \to F$ denotes the natural projection from the fiber product to one of its components.

We next make the following crucial obervation. Given a family $(F, \phi) \to \sigma$ over a cone σ, we may study for each $v \in \sigma$ the restriction ϕ_v of ϕ to the fiber F_v of F over v. The fact that $F \to \sigma$ is flat with reduced fibers implies the following:

Lemma 4.0.29. *The Legendre transform of each ϕ_v is a function $\hat{\phi_v} : Q \to \mathbb{Z}$; furthermore the domains of linearity of $\hat{\phi_v}$ are constant on the interior of each cone in σ.*

In other words, the function $\hat{\phi_v}$ all define the same paving in the interior of each cone of σ.

Proof. Let v be an interior point of σ. Let us further denote by $\phi_{\mathbb{N}v}$ the restriction of ϕ to fiber over the ray $\mathbb{N}v$, and by

$$P = \{(x, z) \in M_{\mathbb{R}} \times \Sigma : (x, z) \leq \phi\}$$
$$P_{\mathbb{N}v} = \{(x, b) \in \overline{M}_{\mathbb{R}} : (x, b) \leq \phi_{\mathbb{N}v}\}$$

the polytopes associated to ϕ, $\phi_{\mathbb{N}v}$, where $\Sigma := (\sigma^{gp})^\vee$ fir brevity. We claim that under the projection $M_{\mathbb{R}} \times \Sigma \to \overline{M}_{\mathbb{R}}$, the polytope P maps into the polytope $P_{\mathbb{N}v}$, and the upper envelope of P maps isomorphically to the upper envelope of $P_{\mathbb{N}v}$. It is clear that P maps into $P_{\mathbb{N}v}$, since if a vector (x, z) is smaller than ϕ on F, its restriction to $F_{\mathbb{N}v}$ is smaller than the restriction $\phi_{\mathbb{N}v}$ of ϕ to $F_{\mathbb{N}v}$. What is interesting is showing that the upper envelopes of the two polytopes are isomorphic. We note that since the projection $F \to \sigma$ is proper and flat, for any v in the interior of σ there is a bijection between the cones of $F_{\mathbb{N}v}$ and of F which preserves codimension. Since the function ϕ is linear on F, it defines a polyhedral complex in $M_{\mathbb{R}} \times \Sigma$ as in definition 2.0.3, with vertices the vectors $(x(\tau), z(\tau))$ which agree with ϕ on τ for the maximal cones $\tau \in F$. This polyhedral complex agrees with the upper envelope of P by lemmas 2.0.2 and 2.0.7. On the other hand, $P_{\mathbb{N}v}$ coincides with the upper envelope of the polyhedral complex of $\phi_{\mathbb{N}v}$, namely the polyhedral complex whose vertices agree with $\phi_{\mathbb{N}v}$ on the maximal cones of $F_{\mathbb{N}v}$. Since $\phi_{\mathbb{N}v}$ is the restriction of ϕ to $F_{\mathbb{N}v}$, the vector which agrees with ϕ on a maximal cone $\tau \in F$ projects to the vector which agrees wtih $\phi_{\mathbb{N}v}$ on the fiber $\tau_{\mathbb{N}v}$ of τ over $\mathbb{N}v$. Thus the vertices of P map into the vertices of $P_{\mathbb{N}v}$. We claim that this map on vertices is bijective. To see this, suppose τ, τ' are two adjacent maximal cones in F. Since ϕ is upper convex, the vector $(x(\tau), z(\tau))$ which agrees with ϕ on τ has values which are strictly larger than $(x(\tau'), z(\tau'))$ on the interior of τ. Hence, the values of $(x(\tau), z(\tau))$ on the restriction of the interior of $\tau_{\mathbb{N}v}$ are larger than the values of $(x(\tau'), z(\tau'))$ – in other words, the vectors $(x(\tau), z(\tau))$ and $(x(\tau'), z(\tau'))$ project to different vectors of $\overline{M}_{\mathbb{R}}$. Therefore the map from the vertices of P to the vertices of $P_{\mathbb{N}v}$ is injective. Since there is at most as many vertices in $P_{\mathbb{N}v}$ as in P – there cannot be more vertices in $P_{\mathbb{N}v}$ than the number of cones for which ϕ takes distinct values, and this is less than or equal to the number of cones of F for which ϕ takes distinct values –, it follows that the map from the vertices of P to the vertices of $P_{\mathbb{N}v}$ is a bijection. Thus, the projection $M_{\mathbb{R}} \times \Sigma \to \overline{M}_{\mathbb{R}}$ gives an isomorphism between the upper envelopes of the two polytopes, which are the convex hulls of the vertices.

The lemma now follows: The domains of linearity of $\hat{\phi_v}$ are determined by the upper envelope of $P_{\mathbb{N}v}$ by lemma 2.0.7. Since these are determined by the polytope P for any interior vector v, they are independent of the interior vector v, as desired. To show that ϕ_v does not change in the interior of a face $\rho\langle\sigma$ we may repeat the same argument with σ replaced by ρ. Note however that for a $v \in \rho$ the domains of linearity of $\hat{\phi}_v$ may not be the same as the domains of linearity of $\hat{\phi_{v'}}$for an interior v'. This is because the function ϕ may become equal on two maximal cones of the fiber F_ρ over ρ. However, even in this case the domains of linearity of $\hat{\phi}_v$ are unions of the domains of linearity of $\hat{\phi_{v'}}$ – a domain of linearity of $\hat{\phi_v}$ is the union of the domains of linearity of $\hat{\phi_{v'}}$ corresponding to the cones of F for which ϕ restricted to F_ρ becomes equal. In other words, the paving defined by ϕ_v for a face ρ of σ is subordinate to the paving defined by $\phi_{v'}$ for an interior point of σ. □

Lemma 4.0.29 leads us naturally to the following definition.

Definition 4.0.30. Let S be a paving of the polytope Q which arises from an upper convex function $f : Q \to \mathbb{Z}$. We define a functor

$$\mathcal{F}_S(\sigma) = \left\{ \begin{array}{c} (F,\phi) \subset N \oplus \sigma^{\mathrm{gp}} \ : p \text{ is proper, flat, with reduced fibers }, p^{-1}(0) = R_Q \\ \Big\downarrow p \\ \sigma \\ \phi \text{ upper convex }, \hat{\phi}_v \text{ defines the paving } S \text{ on } Q \text{ for an interior } v \in \sigma \end{array} \right\}$$

Theorem 4.0.31. *The functor $\mathcal{F}_S$ is representable, and can be represented by the monoid $Aff_S(Q)$.*

Proof. Let $(F,\phi) \to \sigma$ be a family. We denote by ϕ_v the restriction of ϕ to the fiber F_v of F over v as above. We then obtain a morphism $\sigma \to Aff_S(Q)$ by taking v to $\hat{\phi}_v$.Note that since the domains of linearity of $\hat{\phi}_v$ and $\hat{\phi}_{v'}$ are either equal or one is subordinate to the other, we have that the Legendre transform of $\phi_v + \phi_{v'}$ coincides with $\hat{\phi_v} + \hat{\phi_{v'}}$, so the morphism of $\sigma \to Aff_S(Q)$ is actually a morphism of monoids.

For the converse, suppose a morphism $s : \sigma \to Aff_S(Q)$ is given. We construct the family over $(F,\phi) \to \sigma$ by mimicking definition 2.0.3:

Fix a cell $\omega \in S$. We consider the following set of cones in $\overline{N} \times \sigma^{\mathrm{gp}}$:

$$\overline{\sigma}_\omega = \{(v,a,u) : (v,a) = s(u) \text{ on } C(\omega), (v,a) \le s(u) \text{ on } C(\omega') \text{ for } \omega' \ne \omega\}$$

Note this is a cone, since s is a homomorphism. Furthermore, the $\overline{\sigma}_\omega$ fit together into a fan $\overline{F}$, since if ω contains ω', $\overline{\sigma}_\omega$ is a face of $\overline{\sigma}_{\omega'}$. The desired fan in $N \times \sigma^{\mathrm{gp}}$ is the image of $\overline{F}$ in $N \times \sigma^{\mathrm{gp}}$ under the natural map $\overline{N} \to N$.

Lemma 4.0.32. *The morphism $\overline{F} \to F$ is bijective.*

Proof. The proof is the same as that of lemma 2.0.5. □

The fan F comes equipped with a canonical function ϕ, defined by $\phi(v,u) = -a$, where a is the unique $a \in \mathbb{R}$ such that $(v,a,u) \in \overline{F}$. This makes sense by the lemma. We have

Lemma 4.0.33. *The function ϕ is upper convex and linear on each cone of F. In fact, ϕ coincides with $(v,u) \mapsto \hat{s(u)}(v)$.*

Proof. The fact that ϕ is linear on each cone follows from the fact that each $\overline{\sigma}_\omega$ is a cone. We next show that $\phi(v,u) = \hat{s(u)}(v)$. We have

$$\hat{s(u)}(v) = \sup_{x \in Q} \langle v, x \rangle - s(u)(x)$$

By construction, we know that a (v,u) is in F if and only if $s(u) = \langle v, x \rangle + a$ on a cell ω of the paving S, and $s(u) \geq \langle v, x \rangle + a$ on every other cell ω' of S. Thus, the expression $\langle v, x \rangle - s(u)(x) = -a$ on ω, and is $\leq -a$ on every other cell of S. Thus, the supremum of $\langle v, x \rangle - s(u)(x)$ occurs for $x \in \omega$ and equals $-a$, i.e $\phi(v,u)$.
This establishes that ϕ is upper convex over a fixed $u \in \sigma$. To establish convexity in general, let (u,v) and (u',v') be two vectors in F. By construction, there exist two constants a, a' and two faces ω, ω' of S such that

$$s(u)(x) = \langle v, x \rangle + a \text{ on } \omega, s(u)(x) \geq \langle v, x \rangle + a \text{ for } x \notin \omega$$
$$s(u')(x) = \langle v', x \rangle + a' \text{ on } \omega', s(u')(x) \geq \langle v', x \rangle + a' \text{ for } x \notin \omega'$$

Furthermore, by definition of ϕ, the constants a, a' are precisely $-\phi(v,u), -\phi(v',u')$. Consider now a vector $t(v,u) + (1-t)(v',u')$. This lies in some cone $\sigma_{\omega''}$ for some cell ω'' in S. Thus, there exists an a'' such that

$$\begin{aligned} ts(u) + (1-t)s(u') = s(tu + (1-t)u')(x) = \\ = \langle tv + (1-t)v', x \rangle + a'' = t\langle v, x \rangle + (1-t)\langle v', x \rangle + a'' \text{ on } \omega'' \end{aligned}$$

Since $\langle v, x \rangle + a$ and $\langle v', x \rangle + a'$ are less than or equal to $s(u), s(u')$ on ω'', it follows that $t(-\phi(v,u)) + (1-t)(-\phi(v',u')) = ta + (1-t)a' \leq a'' = -\phi(t(v,u) + (1-t)(v',u'))$, which establishes that ϕ is upper convex in general.

□

Lemma 4.0.34. *The restriction of the family $(F, \phi) \to \sigma$ to a ray $\mathbb{N}u \subset \sigma$ coincides with the one parameter degeneration $(F_u, \hat{s(u)}) \to \mathbb{N}$ associated to $s(u)$ of definition* 3.0.19*.*

Proof. We now fix a $u \in \sigma$. By lemma 4.0.33 it follows that ϕ restricted to $\mathbb{N}u$ equals $\hat{s(nu)}$ over nu, which agrees with definition 3.0.19. So what remains to be verified is that the fiber $F_{\mathbb{N}u}$ coincides with the one defined in 3.0.19. Note that by definition, the fiber over a fixed u is the projection to N of the sets

$$\overline{\sigma}_\omega = \{(v,a) : (v,a) = s(u) \text{ on } \omega, (v,a) \leq s(u) \text{ for } \omega \neq \omega'\}$$

This is precisely the polyhedral complex of definition 2.0.3, i.e the projection is precisely the polyhedral complex $P_{s(u)}$ of definition 2.0.4. So the fiber $F_{\mathbb{N}u}$ is the cone over $P_{s(u)}$, precisely as in definition 3.0.19. □

To conclude, every family $(F, \phi) \to \sigma$ gives us a function $\sigma \to Aff_S(Q)$ by sending u to $\hat{\phi}_u$, and a function $s : \sigma \to Aff_S(Q)$ gives us a family $(F, \phi) \to \sigma$ for which the function ϕ_u is precisely $\hat{s(u)}$. Since the Legendre transform is involutive, these operations are inverse to each other. This completes the proof. □

Corollary 4.0.35. *The functor $\mathcal{F}$ is representable.*

Proof. This is because $\mathcal{F}$ is the union of the $\mathcal{F}_S$ where S ranges through all pavings of Q, in virtue of lemma 4.0.29. □

We further obtain the following auxiliary result, which is perhaps more interesting. Define a functor

Definition 4.0.36. Let S be a paving of the polytope Q which arises from an upper convex function $f : Q \to \mathbb{Z}$. We define a functor

$$\tilde{\mathcal{F}}_S(\sigma) = \left\{ \begin{array}{c} (F,\phi) \subset N \oplus \sigma^{\mathrm{gp}} \ : p \text{ is proper, flat, with reduced fibers }, p^{-1}(0) = R_Q \\ \big\downarrow p \\ \sigma \\ \phi \text{ upper convex }, \hat{\phi_v} \text{ defines the paving } S \text{ on } Q \text{ for an interior } v \in \sigma \end{array} \right\} / \sim$$

where two families (F, ϕ) and (F', ϕ') over σ are considered equivalent if there is an isomorphism

$$\begin{array}{ccc} N \times \sigma^{\mathrm{gp}} & \xrightarrow{m} & N \times \sigma^{\mathrm{gp}} \\ \downarrow & & \downarrow \\ \sigma^{\mathrm{gp}} & \xrightarrow[=]{} & \sigma^{\mathrm{gp}} \end{array}$$

which takes F to F' and so that the projective embedding defined by ϕ and $\phi' \circ m$ is the same.

An isomorphism m as in definition 4.0.36 corresponds to a morphism $m : \sigma \to N$. To say that ϕ and $\phi' \circ m$ provide the same embedding to projective space is to say that for each $u \in \sigma$, all sections

$$\phi_u(v) + \hat{\phi_u}(v) - \langle v, x \rangle = \phi'_u(v + m(u)) + \hat{\phi'_u}(x) - \langle v + m(u), x \rangle$$

coincide. In other words, by lemma 3.0.25, we need that $\hat{\phi_u} - \phi'\hat{(u)} = \langle -m(u), \rangle + c_u$ for a constant c_u (or, equivalently, $\phi - \phi'$ is affine linear). Therefore, if we denote by $Aff_0(Q)$ the affine piecewise linear function with respect to the paving obtained from the zero function – i.e simply the affline linear functions on Q, we obtain:

Theorem 4.0.37. *The functor $\tilde{\mathcal{F}}_S$ is representable by the monoid $Aff_S(Q)/Aff_0(Q)$.*

Proof. This is straightforward: A family $(F, \phi) \to \sigma$ is equivalent to a morphism $s : \sigma \to Aff_S(Q)$, by theorem 4.0.31. An isomorphic family gives a different morphism $s' : \sigma \to Aff_S(Q)$, but such that for each $u \in \sigma$, the functions $s(u)$ and $s'(u)$ differ by an affine linear function. So an isomorphism class $(F, \phi) \to \sigma$ gives an equivalence class of a function in $Aff_S(Q)/Aff_0(Q)$. Conversely, given a function $s : Aff_S(Q)/Aff_0(Q)$, we build an isomorphism class $(F, \phi) \to \sigma$ by choosing an explicit representative of the image function. For instance, we may choose a representative uniformly for every cone σ as follows: we fix a maximal cell ω of S, and we look at functions $f \in Aff_S(Q)$ which are 0 on ω.

Lemma 4.0.38. *There is an isomorphism*

$$Aff_S(Q)/Aff_0(Q) \to \{f \in Aff_S(Q) : f = 0 \text{ on } \omega\}$$

Proof. Given a function f in $Aff_S(Q)$, we know that f is affine linear on ω, say equal to a unique vector $(u, a) \in \overline{N}$. We send f to $f - (u, a)$, which vanishes on ω. If f is affine linear, then $f = (u, a)$ everywhere, so the map just defined factors through $Aff_S(Q)/Aff_0(Q)$.

We obtain an inverse map by sending a function f which is 0 on ω to its equivalence class in $Aff_S(Q)/Aff_0(Q)$. $\square$

This lemma concludes the proof: given $s : \sigma \to Aff_S(Q)/Aff_0(Q)$, we simply build the family $(F, \phi) \to \sigma$ whose fiber over $u \in \sigma$ corresponds to the unique representative of $Aff_S(Q)$ vanishing on ω. $\square$

Finally, we remark between the connection of $Aff_S(Q)/Aff_0(Q)$, i.e the functor $\overline{\mathcal{F}}_S$ and Olsson's construction of his monoid "H_S". We briefly recall the definition for the reader's convenience. We fix $Q \in M_{\mathbb{R}}$ and a paving S as above. There is a set map

$$\rho : C(Q)^{\text{gp}} \to \varinjlim{}_{\omega \in S} C(\omega)^{\text{gp}}$$

which simply sends an element in $C(Q)$ to its image in $C(\omega)$ (Warning: this map is not a homomorphism). We then consider the monoid H_S generated by elements

$$H_S = \langle \rho(p_1) + \rho(p_2) - \rho(p_1 + p_2) : p_i \in C(\omega_i) \rangle$$

Lemma 4.0.39. *The monoid $Aff_S(Q)/Aff_0(Q)$ coincides with the* $\operatorname{Hom}(H_S, \mathbb{N})$.

Proof. Note that the quotient $\varprojlim{}_{\omega \in S} \operatorname{Hom}(C(\omega)^{\text{gp}}, \mathbb{Z}) / \operatorname{Hom}(C(Q)^{\text{gp}}, \mathbb{Z})$ is the set of all functions on Q, which are affine linear on each cell of S and whose value on ω and ω' coincides on $\omega \cap \omega'$ – in other words, the continous affine piecewise linear functions on Q whose domains of linearity are the cells of S – up to linear functions. Thus $Aff_S(Q)/Aff_0(Q)$ is a submonoid of $\varprojlim{}_{\omega \in S} \operatorname{Hom}(C(\omega)^{\text{gp}}, \mathbb{Z}) / \operatorname{Hom}(C(Q)^{\text{gp}}, \mathbb{Z})$. For a function $f \in Aff_S(Q)/Aff_0(Q)$, we obtain a homomorphism $h(f) : \varinjlim C(\omega)^{\text{gp}} \to \mathbb{Z}$, by simply taking an element x to its value $f(x)$. This homomorphism is actually non-negative on H_S: if f is upper convex, then $f(\rho(p_1)) + f(\rho(p_2)) \geq f(\rho(p_1 + p_2))$ for all $p_i \in \omega_i$. Conversely, given a homomorphism $h : H_S \to \mathbb{N}$, we build a function on Q by choosing a "base" maximal cell ω of S, and defining

$$f(p) = 0 \text{ for } p \in C(\omega)$$
$$f(q) := h(\rho(q) + \rho(p) - \rho(p + q))$$

where we have *chosen* a $p \in C(\omega)$ such that $p + q$ is in $C(\omega)$ as well. This is always possible by scaling p to be very large, but it is not clear at the moment that f is well defined. Suppose however p' is another choice such that $p' + q$ is also in $C(\omega)$. Then

$$h(\rho(q) + \rho(p) - \rho(p + q) - \rho(q) - \rho(p') - \rho(p' + q)) = h(\rho(q) + \rho(q) - \rho(p + q) - \rho(p' + q)) = 0$$

Furthermore, the function f just defined is linear on each cell of S. For suppose q, q' belong to the same cell ω'. Choose a p such that $p + q \in C(\omega)$, a p' such that $p' + q' \in C(\omega)$. Then for $p + p'$ we have $p + p' + q + q' \in C(\omega)$, and

$$\begin{aligned}
&f(q) + f(q') - f(q + q') = \\
&= h(\rho(q) + \rho(p) - \rho(p + q) + \rho(q') + \rho(p') - \rho(p' + q') - \rho(q + q') - \rho(p + p') + \rho(p + p' + q + q')) = \\
&= h(\rho(q) + \rho(q') - \rho(q + q')) + h(\rho(p) + \rho(p') - \rho(p + p')) - h(\rho(p + q) + \rho(p' + q') - \rho(p + p' + q + q')) = \\
&= 0 + 0 + 0 = 0
\end{aligned}$$

since each term in the parentheses in the third line is a sum of elements of the same cell. Thus, starting with $h : H_S \to \mathbb{N}$ we obtain a function $f(h) \in Aff_S(Q)$, hence in $Aff_S(Q)/Aff_0(Q)$. Note that $f(h)$ vanishes on ω by definition. We also observe that $f(h)$ is upper convex: The same calculation as above is still valid, except in the third row the term $h(\rho(q)+\rho(q')-\rho(q+q'))$ does not vanish if q and q' do not belong in the same cell, but instead is positive. This precisely means that $f(h(q))+f(h(q'))-f(h(q+q'))\geq 0$, which is to say that $f(h)$ is upper convex.

We finally note that the two processes $h\mapsto f(h)$, and $f\mapsto h(f)$ are inverse to each other: reversing the steps in the calculation above yields $h(f(h))=h$; conversely, if f is a function in $Aff_S(Q)/Aff_0(Q)$, i.e a function on Q, affine linear on each cell of S, and which vanishes on ω (c.f. lemma 4.0.38), we have

$$f(h(f))(x)=h(f)(\rho(x)+\rho(p)-\rho(x+p))=f(x)+f(p)-f(x+p)=f(x).$$

This shows that $\mathrm{Hom}(H_S,\mathbb{N})$ and the monoid $\{f\in Aff_S(Q): f=0 \text{ on } \omega\}$ are isomorphic, which concludes the proof in virtue of lemma 4.0.38. □

Corollary 4.0.40. *The "standard family"* $\mathrm{Proj}\,\mathbb{Z}[P_S]\to\mathrm{Spec}\,\mathbb{Z}[H_S]$ *constructed by Olsson corresponds to the geometric realization of the universal morphism* $\mathcal{U}\to Aff_S(Q)/Aff_0(Q)$ *given by the functor* $\tilde{\mathcal{F}}_S$*.*

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