RESEARCH STATEMENT

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My research interests lie broadly in algebraic geometry, though my expertise is more specifically in logarithmic algebraic geometry. Logarithmic algebraic geometry was introduced in the late eighties and early nineties by Kazuya Kato [Kat89], following ideas of Fontaine and Illusie, to address certain delicate questions in arithmetic geometry. Kato's ideas have been subsequently developed in the works of many authors into a deep and beautiful theory. The leitmotif of logarithmic geometry is to bring out a hidden smoothness of singular objects, called log smoothness, which allows one to formally treat singular objects as if they were smooth in the enlarged, logarithmic context. The techniques of logarithmic geometry are especially suited for the study of varieties arising from degenerations, and have thus found numerous applications in moduli theory. My papers can be roughly grouped into two categories: the study of logarithmic and tropical moduli spaces ([MW18],[AM16],[MW], [Mol16], [Wis14],[MR14],[GM]) and the study of foundations in logarithmic and toric geometry ([MT19],[GM15a], [GM15b], [Mol16], [Wis14]). Briefly, the theory of logarithmic moduli spaces is concerned with the construction of modular compactifications of moduli problems; tropical moduli spaces appear as the underlying combinatorial structure governing a logarithmic moduli problem. My work on log moduli problems has been on Picard schemes and abelian varieties, broken toric varieties, and stable logarithmic maps and log Gromov-Witten invariants. The foundational papers have often been motivated by the need to develop the theory necessary to study such problems. In this direction, I have studied toric stacks; weak semistability, that is, the geometric and combinatorial structure of the degenerations that appear in log moduli problems; properties of log regularity, the logarithmic version of classical regularity for applications in the theory of resolution of singularities; and how the ideas of log geometry can be also be applied in a different framework to study degenerate objects in differential geometry – in particular objects such as manifolds with corners.

1. Logarithmic Moduli Spaces

A general heuristic principle in log moduli theory is that, given a (generally non-proper) moduli problem parametrizing certain smooth objects, if one studies instead the corresponding log moduli problem, where all objects have been equipped with additional data of logarithmic structures, the resulting moduli space will be automatically compact; thus one obtains a modular compactification of the original space. Furthermore, the resulting moduli space tends to have good formal properties – for instance, deformation theory that behaves much like ordinary smooth deformation theory.

1.1. **Picard Functors.** The paper [MW18] can be seen as an incarnation of this principle. In [MW18], Jonathan Wise and I study the logarithmic Picard group of a log curve. More precisely, starting with a family of log curves $X \to S$, we consider the stack that to a log scheme $T \to S$ assigns the groupoid $B\mathbb{G}_m^{\log}(X/S)(T)$ of torsors under the associated group $M_{X_T}^{\mathrm{gp}}$ of the log structure M_{X_T} on $X_T = X \times_S T$. We define $\mathbf{LogPic}(X/S)(T)$ to be a certain substack of $B\mathbb{G}_m^{\log}(X/S)(T)$, consisting of torsors that satisfy a certain combinatorial condition we call *bounded monodromy*, which ensures that they deform algebraically. The stack $\mathbf{LogPic}$ has several remarkable properties: when the underlying family of $X \to S$ is smooth, it coincides with the classical stack $\mathrm{Pic}(X/S)$;

though a priori only a stack in the étale topology, it is also a stack in the log étale topology – roughly, this is the topology generated by étale maps, and taking blowups and root stacks along the loci where the log structure is concentrated.Though **LogPic** is not algebraic, it is "log algebraic" – it has a log smooth cover by a log scheme. Furthermore, **LogPic** fits the heuristic principle of log moduli theory: it is proper. In summary, we show:

Theorem 1. [MW18, Theorem A.] *The stack **LogPic**(X/S) is a logarithmically smooth, proper log algebraic stack, which is a commutative group object.*

In particular, given a family of nodal curves $X \to S$ which is smooth over some open subscheme $U \subset S$, **LogPic**(X/S) provides a log smooth, proper, group compactification of **LogPic**$(X_U/U) = \mathrm{Pic}(X_U/U)$. On the other hand, it is known that a smooth, group compactification of $\mathrm{Pic}(X_U/U)$ cannot exist in classical algebraic geometry. Indeed, **LogPic** is not algebraic, but merely log algebraic, so in a sense a genuinely new type of object. As such, it can be hard to understand completely. We therefore proceed to study its structure. First, we look at its underlying sheaf of isomorphism classes, which we denote by LogPic. We find that the structure of this sheaf fits in a context that has already been studied by Kato, Kajiwara and Nakayama:

Theorem 2. [MW18, Theorem B.] *The sheaf* LogPic(X/S) *is a log abelian variety in the sense of* [KKN08].

To get a more concrete description, we study the tropicalization of **LogPic**. We begin by studying an analogous moduli problem on the tropicalization $\mathfrak{X}$ of $X \to S$. The tropicalization, roughly speaking, is the system of dual graphs of the fibers of $X \to S$, further marked to remember some information about the log structure; on $\mathfrak{X}$, we define sheaves of piecewise linear and linear functions, and define **TroPic**$(\mathfrak{X})$ to be the stack (in some appropriate topology) of bounded monodromy torsors for the sheaf of linear functions. The stack **TroPic** is a concrete combinatorial object, which in certain cases has already been studied: for instance, the $\mathbb{R}_{\geq 0}$ points of the underlying sheaf of isomorphism classes of the degree 0 part of **TroPic** are known in the tropical literature as the tropical Jacobian. We relate the combinatorial object TroPic to **LogPic**(X/S) by observing that **LogPic**(X/S) contains the multidegree 0 part $\mathrm{Pic}^{[0]}(X)$ of the Jacobian of X as a subgroup, and that we have:

Theorem 3. [MW18, Theorem C.] *There is an exact sequence of group stacks*

$$0 \longrightarrow \mathrm{Pic}^{[0]}(X) \longrightarrow \boldsymbol{LogPic}(X/S) \longrightarrow \boldsymbol{TroPic}(\mathfrak{X}) \longrightarrow 0$$

In other words, **TroPic**$(\mathfrak{X})$ is the tropicalization of **LogPic**(X/S), and the fibers of the tropicalization map are torsors under the semi-abelian variety $\mathrm{Pic}^{[0]}(X)$.

1.2. **Broken Toric Varieties.** The paper [AM16] also follows the same principle of using logarithmic geometry to construct a proper moduli space, but the space obtained is a more traditional object – an honest algebraic stack with a log structure. In [AM16], Kenneth Ascher and I study the moduli space of broken toric varieties in a toric variety V. Specifically, for a fixed toric variety V and a fixed subtorus H of its torus, we study the stack $\mathcal{AB}$, introduced in the work [AB06] of Alexeev and Brion, which assigns to a scheme S diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & V \\ {\scriptstyle\pi}\downarrow & & \\ S & & \end{array}$$

where π is flat and has reduced fibers, X is a broken toric variety with torus H – that is, a variety X such that each irreducible component of X is a toric variety with torus H – and f is H-equivariant. The stack $\mathcal{AB}$ is shown to be a proper Deligne-Mumford stack in [AB06], but it is not irreducible. In the logarithmic setting, we study instead the stack $\mathcal{K}$ which assigns over S diagrams the same data, but where now X and S are equipped with log structures, the morphism π is log smooth, and f is a log morphism. We show that the stack $\mathcal{K}$ is a proper, log smooth Deligne-Mumford stack. We show that the natural forgetful morphism $F : \mathcal{K} \to \mathcal{AB}$ is finite; since π is an isomorphism over the locus where X is smooth, we obtain

Theorem 4. [AM16, Theorem 5.12] *The morphism $\mathcal{K} \to \mathcal{AB}$ is the normalization of the closure of the main component of $\mathcal{AB}$.*

We thus obtain a fairly small modular compactification of the moduli space of smooth broken toric varieties in V. This result is analogous to Olsson's compactification of the moduli space of broken toric varieties, carried in [Ols08] – which is the special case of theorem 4 when V is some appropriate projective space. This is the main result of [AM16], but the methods are very different from those of [Ols08]. In [AM16], the stack $\mathcal{K}$ is studied by using the Chow quotient stack $[V /\!/_C H]$ of V by H. In my view the most important result of [AM16] is the following:

Theorem 5. [AM16, Proposition 5.11] *The stack $\mathcal{K}$ is isomorphic to the Chow quotient stack $[V /\!/_C H]$.*

The advantage of this description is that the Chow quotient stack is easy to understand: it is a toric stack, so it can be described by combinatorial data – see the section on toric stacks for details – and it comes with a universal family which is also a toric stack. So we get a complete, efficient, *global* combinatorial description of the stack. Essentially, the Chow quotient stack can be understood as the tropicalization of $\mathcal{K}$, and, as $\mathcal{K}$ is toric, its tropicalization determines it completely.

The Chow quotient stack was introduced in joint work with William Gillam, in an unpublished article [GM]. The Chow quotient stack is a certain toric stack, a stacky enrichment of the Chow quotient of [KSZ91]. In fact, in [GM] we study the Chow quotient stack $[V /\!/_C H]$ when $H = \mathbb{C}^*$ was a one parameter subgroup of V. In this case, the Chow quotient stack is related to the stack of stable logarithmic maps of Chen [Che10],Abramovich-Chen [AC11], and Gross-Siebert [GS11]. The main result of [GM] is:

Theorem 6. [GM, Theorem 2] *The stack $[V /\!/_C \mathbb{C}^*]$ is isomorphic to the moduli stack $K_\Gamma(V)$.*

Here, Γ denotes certain discrete data of the stable maps in question: the genus of the curves, the number of markings, and the type of the morphisms. In our case, the genus is 0, the markings are 2, and the type is determined from the one parameter subgroup. This result improves on a result of Qile Chen and Matthew Satriano, who showed the isomorphism of 6 on the level of coarse moduli spaces [CS12].

An interesting deviation between the approach of [AM16] and the approach of [Ale02],[AB84],[Ols08] is that the combinatorial data in [AM16] – the Chow quotient stack – live in the chocharacter lattice N of the toric variety V, while the combinatorial data in [Ale02],[AB84],[Ols08] live in the character lattice M: the combinatorics of a broken toric variety are determined by a paving S of a polytope Q. A paving is the subdivision of Q into the subpolytopes where some upper convex affine piecewise linear function f is linear. In [MW], given such an f, we show how to construct a fan over $\mathbb{N}$ in $N \oplus \mathbb{N}$, whose central fiber is precisely the polarized broken toric variety associated to f. The fan is constructed from the Legendre transform $\hat{f}$ of f; conversely, by applying the Legendre transform again we show how certain fans in $N \oplus \mathbb{N}$ correspond to pavings in Q. We thus obtain a dictionary between data in M and data in N. This construction works in families, and we are led to consider the functor over the category of all cones defined by

$$\overline{\mathcal{F}}_S(\sigma) = \left\{ \begin{array}{c} (F,\phi) \subset N \oplus \sigma^{\mathrm{gp}} \ : p \text{ is proper, flat, with reduced fibers }, p^{-1}(0) = R_Q \\ \Big\downarrow p \\ \sigma \\ \phi \text{ upper convex }, \hat{\phi_v} \text{ defines the paving } S \text{ on } Q \text{ for an interior } v \in \sigma \end{array} \right\} / \sim$$

Here R_Q is the normal fan of the polytope Q and $\sim$ denotes that this data is considered up to isomorphism. The moduli problem $\overline{\mathcal{F}}_S$ can be regarded as a toric or tropical moduli space. We show

Theorem 7. [MW, Theorem 4.0.37] *The functor $\overline{\mathcal{F}}_S$ is representable.*

The monoid representing $\overline{\mathcal{F}}_S$ can be regarded as the tropicalization of the moduli space of polarized broken toric varieties. A different description of this monoid is that it is the dual of the monoid H_S appearing in Olsson's "standard construction" in [Ols08].

1.3. **Gromov-Witten Theory.** A special application of the techniques of logarithmic geometry in the construction of moduli spaces is in constructing Gromov-Witten invariants of singular spaces. In the project [MR14], E. Routis and I study the space of log stable maps of Bumsig Kim [Kim10]. This space is the logarithmic analogue of the space of relative stable maps of Jun Li [Li01] – a space inspired by the works of Li-Ruan [LR01] and Ionel-Parker [IP04] in symplectic geometry. In [Li01], Jun Li studies the Gromov-Witten theory of singular spaces X of a special form: $X = Y_1 \cup_D Y_2$ is a d-semistable, that is, nodal, union of two smooth varieties Y_1 and Y_2 along a common divisor D. Jun Li constructs a space of stable maps from nodal, marked curves to X, relative to the divisor D. The Gromov-Witten invariants constructed from Li's space, called relative Gromov-Witten invariants, are the first instance of Gromov-Witten invariants of singular targets. They are important even in ordinary Gromov-Witten theory, due to their deformation invariance: if X is the central fiber of a family of varieties with smooth total space and smooth general fiber, the Gromov-Witten invariants of the general fiber coincide with the relative Gromov-Witten invariants of the central fiber. Therefore, one may calculate the Gromov-Witten invariants of a smooth variety by degenerating it to a nodal singular variety with simpler components. The main calculational tools for these invariants are the degneration formula of [Li01] and the localization formula of Graber-Vakil [GV05].

The paper [Li01] is groundbreaking, but introduces one technical difficulty: the space of relative stable maps to X is a locally closed subset of the space of all maps to X, and hence the ordinary deformation/obstruction theory of maps cannot be used. The perfect obstruction theory of [Li01] is constructed by ad hoc methods and produces a virtual fundamental class which is hard to understand. This difficulty is overcome in the work [Kim10] of Bumsig Kim. In [Kim10], the author constructs the space $\mathcal{M}^{\mathrm{Kim}}(X)$ of logarithmic stable maps from logarithmically smooth curves into X. The advantage of the stack $\mathcal{M}^{\mathrm{Kim}}(X)$ is that it is an open substack of the space of all logarithmic maps into X, and hence its deformation theory is the one induced from the latter's. This deformation theory has been studied by Olsson [Ols05] and has formal properties almost identical to the deformation theory in ordinary Gromov-Witten theory – it is governed by the logarithmic cotangent complex.

On the other hand, it is proven in [AMW12] that $\mathcal{M}^{\mathrm{Kim}}(X)$ is related to $\mathcal{M}^{\mathrm{Li}}(X)$ via a finite map $\pi : \mathcal{M}^{\mathrm{Kim}}(X) \to \mathcal{M}^{\mathrm{Li}}(X)$ which preserves virtual fundamental classes: $\pi_*[\mathcal{M}^{\mathrm{Kim}}(X)]^{\mathrm{vir}} =$

$[\mathcal{M}^{\mathrm{Li}}(X)]^{\mathrm{vir}}$. Hence, relative Gromov-Witten invariants and log Gromov-Witten invariants coincide, and one may work with the technically easier space.

In our paper, we derive a localization formula for the space $\mathcal{M}^{\mathrm{Kim}}(X)$. We begin by studying the geometry of the stack. We show that if a torus $\mathbb{C}^*$ acts on X, then

Theorem 8. [MR14, Theorem 3.1] *The stack $\mathcal{M}^{\mathrm{Kim}}(X)$ is a global quotient stack of the form $[V/G]$, where G is a reductive group and V is a locally closed subset of a smooth projective W with an action of G extending that of V. The $\mathbb{C}^*$ action on $[V/G]$ descends from a $\mathbb{C}^* \times G$ action on W.*

In particular, it follows that $\mathcal{M}^{\mathrm{Kim}}(X)$ admits a closed embedding to a smooth Deligne-Mumford stack, and therefore the localization theorem of Graber-Pandharipande [GP99] applies to it. With this in mind, we analyze the deformation theory of $\mathcal{M}^{\mathrm{Kim}}(X)$. Our analysis follows closely the analysis of the deformation theory of the stack of stable maps in ordinary Gromov-Witten theory. We identify the fixed loci of the $\mathbb{C}^*$ action and compute the equivariant Euler classes of their normal bundles. We thus arrive at a localization formula:

Theorem 9. [MR14, Theorem 5.1] *There is a localization formula for $\mathcal{M}^{\mathrm{Kim}}(X)$. Under the pushforward π_*, this formula recovers the formula of Graber-Vakil.*

The reader interested in the precise form of the localization formula is referred to [MR14].

2. Foundations

2.1. **Regularity.** The theory of logarithmic smoothness is central in log geometry, but an analogous theory for regularity does not exist. Work of Kato [Kat94] gave a definition of log regular schemes, but a relative notion for morphisms $X \to Y$ had been missing. In [MT19], Michael Temkin and I study this problem with applications in the theory of resolution of singularities in mind. Using Martin Olsson's formalism of the stack $\mathcal{L}og_X$ parametrizing log schemes over X, we define a morphism $X \to Y$ to be log regular if the associated morphism of stacks $\mathcal{L}og_X \to \mathcal{L}og_Y$ is regular. This definition has good formal properties (for example, it is easy to check that compositions and pullbacks of log regular morphisms are log regular), but it is unwieldy to work with, because the stack $\mathcal{L}og_X$ is itself unwieldy. To mitigate this, we first give an explicit description of the stack $\mathcal{L}og_X$. We do this by first describing its fan, which is a complicated combinatorial object, and then how to obtain $\mathcal{L}og_X$ through a general geometric realization functor. We then show

Theorem 10. [MT19, Theorem 4.3.1] *Suppose $\mathcal{P}$ is a property of representable morphisms of stacks which is stable under pullbacks and can be check étale locally on the source and flat locally on the target. Then $\mathcal{L}og_X \to \mathcal{L}og_Y$ has property $\mathcal{P}$ if and only if $X \to \mathcal{L}og_Y$ has property $\mathcal{P}$.*

As a consequence, we also obtain a chart criterion, analogous to Kato's chart criterion for log smooth morphisms.

Theorem 11. [MT19, Theorem 5.2.8] *A map $f : X \to Y$ is log regular if and only if, for any fppf chart $Q \to M_Y$, étale locally around any x in X, there exists an injective chart $Q \to P$ for the morphism f such the torsion part of the cokernel of $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ has order invertible in $\mathcal{O}_{X,\overline{x}}$, and such that the morphism $X \to Y \times_{\mathrm{Spec}\,\mathbb{Z}[Q]} \mathrm{Spec}\,\mathbb{Z}[P]$ is regular.*

2.2. **Toric Stacks.** The Chow quotient stack of [AM16],5, 6, as well as the stacks referred to in 13 are examples of toric stacks. There are many divergent theories of toric stacks in the literature – for examples in the works of [Iwa09], [BCS05], [FMN07] for Deligne-Mumford stacks and [GS11] for general Artin stacks. However, the toric stacks produced in the afforementioned papers are not general enough for our purposes. In [GM15b], together with W. Gillam, we develop a theory of toric stacks which is both simpler and more flexible and incorporates the stacks that appear in [AM16],[GM]. Toric stacks in our setting are described via the combinatorial information of a "KM" fan. A KM fan is the data of a usual toric fan F in a lattice N together with the additional data of a sublattice N_σ of N for each cone σ in F: in other words, a triple $(F, N, \{N_\sigma\}_{\sigma\in F})$. The only requirements are that each N_σ has finite index in the lattice spanned by $N \cap \sigma$, and that $N_\sigma \cap \tau = N_\tau \cap \tau$ when τ is a face of σ. The definition is thus very similar to the ordinary definition of a toric variety. This data has a geometric realization to a separated, normal Deligne-Mumford toric stack – details can be found in [GM15b]. We must mention here that, unbeknownst to us, the definition had been essentially given before by I.Tyomkin in [Tyo12], but there was no discussion of their properties. In [GM15b], we go on to develop the fundamental properties of toric stacks. These are essentially as expected: for instance, the coarse moduli space of the toric stack associated to $(F, N, \{N_\sigma\})$ is the toric variety associated to (F, N), the orbit-cone correspondence is still valid, and the data $(F, N, \{N_\sigma\})$ can still be recovered from the one parameter subgroups of the geometric realization. Our theory also has the advantage of being complete, when restricted to DM toric stacks. Specifically, we show:

Theorem 12. [GM15b, Theorem 3.10.7] *Every normal, separated Deligne-Mumford toric stack over a field of characteristic* 0 *is the geometric realization of a unique KM fan.*

We further show that the description of toric Artin stacks presented [GS11], contrary to what's stated, is not general enough to yield all toric Deligne-Mumford stacks, and a fortiori not all toric stacks. A description of all Artin toric stacks needs to combine their description with ours.

2.3. **Weak Semistable Reduction.** The families $X \to S$ that commonly appear in log moduli problems are commonly (for instance in all the afforementioned examples) are demanded to be integral and saturated: a condition which I call "weak semistability" in [Mol16], weakening terminology introduced in [AK00]. To understand the significance of weak semistability in log moduli problems, one needs to keep in mind that fiber products of log schemes are ill-behaved. For instance, when working in the category of fine and saturated log schemes, the underlying scheme of the fiber product does not coincide with the fiber product of the underlying schemes. The importance of weak semistability in log moduli theory is two-fold: it implies that the underlying families are flat and have reduced fibers, and that pullbacks of weakly semistable family are well behaved. The main problem of the subject is: given a family $X \to S$ of toroidal embeddings (in the sense of [KKMSD73]), find an alteration $T \to S$, and a modification Y of $X \times_S T$ for which $Y \to T$ is weakly semistable. This problem was solved in a strong way – with Y and T both smooth – in [KKMSD73] when the base S has dimension one, and in [AK00] for higher dimensions, but only ensuring smoothness of T. The main result of [Mol16] is that for toroidal embeddings, weak semistable reduction can be done universally if we allow families $Y \to T$ to be Deligne-Mumford stacks rather than schemes. Specifically, I show

Theorem 13. [Mol16, Theorem 1.0.1] *Let $X \to S$ be a proper, surjective, log smooth morphism of toroidal embeddings.There is a commutative diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathcal{S} \arrow[r] & S \end{tikzcd}$$

where $\mathcal{X} \to \mathcal{S}$ is a weakly semistable morphism of toroidal Deligne-Mumford stacks, and any weakly semistable $Y \to T$ mapping to $X \to S$ factors uniquely through $\mathcal{X} \to \mathcal{S}$.

The result is proved by extending the formalism of [GM15b] to log smooth schemes, and describing the fans of $\mathcal{X}$ and $\mathcal{S}$ from the map of fans associated to $X \to S$. In particular, if X and S are toric, the stack $\mathcal{X}, \mathcal{S}$ are also toric.

2.4. **Minimal Log Structures.** A central technical problem in the study of traditional log moduli spaces such as those of [AM16],[MR14] is the problem of finding minimal log structures. The situation can be summarized as follows. One starts with a stack $\mathcal{M}$ over the category of schemes which one wishes to compactify, and considers instead the associated stack $\mathcal{M}^{\log}$ over the category of log schemes. To do geometry, one would like to show that $\mathcal{M}^{\log}$ is actually representable by a log stack $(\mathcal{K}, M_{\mathcal{K}})$ – a stack $\mathcal{K}$ defined over the category of schemes, equipped with a log structure. For instance, in the context of [AM16], the stack $\mathcal{K}$ alluded to in the theorem 4 is precisely the stack representing $\mathcal{AB}^{\log}$. In practice, showing representability of $\mathcal{M}^{\log}$ is equivalent to finding certain log structures for each family, called minimal log structures.

In [Wis14], it is shown that if one requires $X \to S$ to be weakly semistable, finding the minimal log structures is always possible. The description of the minimal log structure given in [Wis14] is, however, complicated. In Appendix C. in [Wis14] I derive a simple concrete formula for the characteristic monoid of the minimal log structure. Specifically, suppose a diagram of the form

$$\begin{tikzcd}
(X, M_X) \arrow[r] \arrow[d, "\pi"] & (V, M_V) \arrow[dl] \\
(S, M_S) &
\end{tikzcd}$$

where π is log smooth and weakly semistable, and S is a geometric point is given. Then, I show that the characteristic monoid of the minimal log structure on S (and consequently X) is unique and can be explicitly calculated as a quotient of two monoids that depend only on the following data: the log structure f^*M_V at the irreducible components of X and the irreducible components of the nodes where precisely two irreducible components intersect; and the two generization maps at such nodes. In the special case when the diagram is absolute, i.e when (V, M_V) is not considered as an (S, M_S) scheme we have:

Theorem 14. [Wis14, Theorem C.5] *The associated group of the characteristic monoid is the cokernel of*

$$\oplus_q P_q^{\mathrm{gp}} \longrightarrow \oplus_\eta P_\eta^{\mathrm{gp}} \oplus (\overline{M}_S)^{\mathrm{gp}}$$

where $\{\eta\}$ is the set of irreducible components of X, $\{q\}$ the set of irreducible components of nodes which are the intersection of two irreducible components, $P = f^*M_V$, and the morphism is determined by the generization maps $P_q \to P_\eta$. The characteristic monoid is the saturation of the image of $\oplus_\eta P_\eta \oplus (\overline{M}_S)$ in the group.

Thus, in this case we recover the same formula that appears in the study of stable logarithmic maps in [GS11]. Note however that here X has arbitrary dimension. The reason why this formula depends only on the values of the monoids P, M_S on the generic components and nodes of X regardless of dimension is due to the following theorem:

Theorem 15. [Wis14, Corollary C.1] *Let $(X, M_X) \to (S, M_S)$ be a log smooth morphism of logarithmic schemes, which is flat with reduced fibers, and with $S = \mathrm{Spec}\, k$ a geometric point. Then, étale locally, every irreducible component of a node in X is the intersection of precisely two irreducible components in X, and we have*

(1) $M_{X,\eta} \cong M_S$ *at the generic point of any irreducible component.*

(2) $M_{X,q} \cong M_S \oplus_{\mathbb{N}} \mathbb{N}^2$ *at the generic point of an irreducible component of a node.*

(3) $M_{X,p} \cong M_S \oplus \mathbb{N}$ *at the generic point of certain divisors which are entirely contained inside an irreducible component.*

The divisors in (3) are analogous to marked points of nodal curves. In other words, F.Kato's description of log smooth curves in [Kat00] holds in arbitrary dimension, at least over irreducible components, nodes, and marked points.

2.5. **Logarithmic Differential Geometry.** The paper [GM15a] is rather long, so it is difficult to discuss all of its aspects in such short space. Many of the results are foundational results in the algebraic geometry of monoids. I choose to discuss here the results related to the original motivation for writing this paper.

The project can be seen as a response to the papers [KM11] and [Joy09] on manifolds with corners. Manifolds with corners do not form a good category. Firstly, even though a definition of manifolds with corners is more or less accepted, there is no consensus on what a morphism between manifolds with corners should be. Secondly, the category of manifolds with corners is not closed under inverse limits: for example, fiber products of transverse maps of manifolds with corners do not exist as manifolds with corners. In [GM15a], we propose that manifolds with corners should be studied in the enlarged category of differentiable spaces – a category which is roughly speaking built from the category of manifolds in the same way as the category of schemes is built out of the category of smooth varieties; for details see [Gil]. In fact, we enlarge the category even more by allowing differentiable spaces to carry log structures. This allows us to give a natural definition of a morphism of manifolds with corners – they are precisely the log morphisms. This notion coincides with the notion of b-maps of [KM11]. We may then characterize manifolds with corners in terms of their log structure:

Theorem 16. [GM15a, Theorem 6.7.8] *Manifolds with corners are positive log differentiable spaces which are log smooth with a free log structure.*

The payoff of working with differentiable spaces, which form a nicer category, is that many objects obtain singularities – for instance, the motivating example of fiber products of manifolds with corners. We thus discuss resolution of singularities in a rather general context. We start from any category of "log spaces", which can be for example the category of log schemes, log analytic spaces or log differentiable spaces. We discuss a notion of logarithmic smoothness for any log space and we describe a procedure to assign combinatorial data to a log space, which we call a monoidal space. This is analogous to the data of the fan of a toric variety. We discuss how operations on this data, analogous to subdivisions of fans, have a geometric realization that produces a new log space with a log smooth map to the original one. Following ideas in the theory of Kempf, Knudsen, Mumford and Saint-Donat [KKMSD73] in the theory of toroidal embeddings, of Kazuya Kato in [Kat94], of Niziól [Niz06], Wlodarczyk [Wło03] and Bierstone-Millman [BM97], we show how to resolve any monoidal space, so that the geometric realization of this resolution gives a space with a free log structure. In the example of positive differentiable spaces, it yields

Theorem 17. [GM15a, Theorem 10.4.1] *Any (fs) positive log differentiable space X admits a log smooth, surjective, locally projective map $X' \to X$ from a positive log differentiable space with a free log structure, which is an isomorphism over the locus where the log structure of X is free.*

In other words, X' is a resolution of singularities of X. If we begin with a log smooth space, the resulting resolved space will be log smooth and free - a manifold with corners. The theorem of Kottke and Melrose follows: The notion of b-transversality of manifolds with corners translates to our notion of log transversality; a fiber product of log transverse maps of positive log differentiable spaces is log smooth, and can thus be resolved to a manifold with corners.

3. Ongoing Projects

The stack $\mathbf{LogPic}(X/S)$ and associated sheaf LogPic that appear in [MW18], 1,2 are, as mentioned, not algebraic stacks with log structures, but rather honest analogues of stacks for the log-étale topos. The type of spaces that appear have remarkable properties and suggest many new possible research directions, but our understanding of their main properties, general structure, and their uses is still in its infancy and remains to be studied. In ongoing work, we have begun to undertake this study.

3.1. **Duality.** With M. Ulirsch and J. Wise, we study duality for $\mathbf{LogPic}$. Let $X \to S$ be a family of logarithmic curves. Following ideas introduced by Deligne in his study of duality for $\mathbf{Pic}$, we construct a pairing that takes two M_X^{gp} torsors and produces an M_S^{gp} torsor, that is, a pairing $\mathbf{LogPic}(X/S) \times \mathbf{LogPic}(X/S) \to \mathbf{LogPic}(S)$. More generally, we construct an Abel-Jacobi map $X \to \mathbf{LogPic}(X/S)$ and show that for any commutative 2-group stack $\mathcal{G}$ that satisfies a certain list of axioms, which I won't reproduce here, the following "universal coefficients" theorem holds:

Theorem 18. *If X is a proper, vertical logarithmic curve over S then every S-morphism $X \to \mathcal{G}$ factors over S through the Abel map $X \to \mathbf{LogPic}(X/S)$. We obtain a pairing*

$$\boldsymbol{LogPic}(X/S) \times \mathcal{G}(X/S) \to \mathcal{G}$$

that is perfect in the sense that

$$\mathcal{G}(X/S) \to \mathrm{Hom}(\boldsymbol{LogPic}(X/S), \mathcal{G})$$

is an equivalence.

3.2. **Néron Models.** In joint work, D. Holmes, G. Orrechia, T. Poirét, J. Wise and I study the relationship of $\mathrm{LogPic}(X/S)$ with Néron models. Let $X \to S$ be a family of nodal curves over a regular base S, with smooth fiber over a dense open subscheme $U \subset S$. In a collection of papers, the first three authors study the existence of a Néron model $N_S \to S$ for $\mathrm{Pic}^0(X_U) \to U$, and show that existence of such a model is equivalent to a certain relation among the smoothing parameters of the nodes of X, which they call alignment; if we endow $X \to S$ with log structures, this condition is related to a condition on the tropicalization of X, which we call log alignment. We show that $\mathrm{LogPic}(X/S)$ "is" the Néron model in the category LogSch/S: in other words, let S be log regular, with $U \subset S$ the locus where the log structure is trivial. Let $T \to S$ be log smooth, and $V = T \times_S U$. Suppose a map $V \to \mathrm{LogPic}(X/S)$ is given. Then, there exists a unique extension to a map $T \to \mathrm{LogPic}(X/S)$. Consider the functor $\mathrm{strLogPic}(X/S)$ defined on Sch/S, which, to a log scheme $T \to S$ assigns $\mathrm{LogPic}(X/S)(T)$, where T has been given the log structure pulled back from S, and its tropicalization $\mathrm{strTroPic}(X/S)$. A consequence of the Néron mapping property for $\mathrm{LogPic}(X/S)$ is that $\mathrm{strLogPic}^0(X/S)$ *is* the Néron model for $\mathrm{Pic}^0(X_U/U)$ provided that it is representable and separated. We show that when the log structure on S is locally free, $\mathrm{strLogPic}(X/S)$ is always representable. Combining the two observations, our main result is the following:

Theorem 19. *Let $X \to S$ be a log curve and assume the log structure on S is locally free. Consider the following conditions:*

(1) *X/S is strictly aligned*
(2) *X/S is log aligned*
(3) *The geometric fibres of* $\mathrm{strTroPic}^0_{X/S}$ *are finite;*
(4) $\mathrm{strLogPic}^0_{X/S} \to S$ *is separated*

Then (2) $\iff$ (3) $\implies$ (4)*; moreover* (4) $\implies$ (2) $\iff$ (1) *if S is log regular.*

Hence, for a log regular S with locally free log structure a Néron model for $\mathrm{Pic}^0(X_U/U)$ exists if and only if any of the equivalent conditions is satisfied. In particular, we recover the results of Holmes, Orrechia and Poirét.

3.3. **Models of the Log Jacobian.** The sheaf $\mathrm{LogPic}(X/S)$ may not be algebraic, but it's failure to be algebraic can nonetheless be described explicitly through the tropical Jacobian $\mathrm{TroJac}(X/S) = \mathrm{TroPic}^0(X/S)$. Essentially, lack of algebraicity of LogPic is equivalent to lack of a polyhedral structure on the tropical Jacobian. Subdivisions of the tropical Jacobian that give it a polyhedral structure produce "models" of LogPic, that is, schemes Y with a proper log étale map to LogPic. In joint work with M. Melo, M. Ulirsch, F. Viviani and J. Wise we provide a method for constructing subdivisions of the tropical Jacobian. The approach is inspired by Caporaso's compactification of the universal Jacobian.

3.4. **McKay Correspondence.** One of the phenomena that we have come to recognize in recent years is that essential invariants in log geometry are invariants under log étale maps – for instance, $\mathbf{LogPic}(X/S)$ does not change under log étale maps $X' \to X$, and neither do log Gromov-Witten invariants or Chow groups for log schemes. As log étale maps are generated by étale maps, log blowups, and extracting roots along the log structure, this suggests that various invariants of a log scheme could be viewed as invariants on the (perhaps non-algebraic) spaces $X^{\mathrm{val}} = \varprojlim_{X' \to X \text{ log blowup}} X'$ or $\sqrt[\infty]{X} = \varprojlim_{X' \to X \text{ root}} X'$. In [SST18], the authors study the derived categories of such spaces. More precisely, starting from a special kind of degeneration $X \to S$, they show that the derived categories of $\sqrt[\infty]{X}$ and $(X_\infty)^{\mathrm{val}} := (X \times_S \sqrt[\infty]{S})^{\mathrm{val}}$ are canonically isomorphic, the isomorphism provided by a certain limit of Fourier-Mukai transforms. For [SST18], the degeneration $X \to S$ must have a very special form – essentially, the base S must be one dimensional, the morphism must be weakly semistable, and the total space X must be smooth. Furthermore, the proof is rather complicated. In joint work with G. Liu, we show that if one adopts the formalism of [GM15b],[Mol16], and adapts the main result of [KKMSD73], one can give a simpler proof of the result while also generalizing it: for any simplicial log scheme X, we show how to assign a stack X_∞ to it, for which there is an isomorphism between the derived categories of X_∞^{val} and $\sqrt[\infty]{X}$.

4. Future Work

Log algebraic spaces and log algebraic stacks can be used to compactify several moduli spaces in a natural way, while retaining much of the original structure of the moduli space being compactified. This raises hope that they can be used in a fruitful way to study the original, non-compact spaces. Here I outline some of the results we hope we will be able to approach using the methods already developed above, and some of the foundations we will need to develop to do so effectively.

4.1. **Double Ramification Cycles.** Fix a vector of integers $a = (a_1, \cdots, a_n)$ that sum up to 0. The vector a determines an Abel-Jacobi section $\sigma : M_{g,n} \to J_{g,n}$, where $M_{g,n}$ is the moduli space of genus g, n-marked smooth curves and $J_{g,n} \to M_{g,n}$ is the Jacobian of the universal curve. The double ramification cycle D_a can be defined as $\sigma^*(0)$, with 0 here denoting the zero section of $J_{g,n}$.The Abel-Jacobi section does not extend to a section over the Deligne-Mumford compactification $\overline{M}_{g,n}$. The reason, essentially, is that the Jacobian $J_{g,n}$ does not extend to an abelian variety over $\overline{M}_{g,n}$. Nevertheless, an extension of the double ramification has been given by Li and Graber-Vakil [Li01],[GV05] using techniques of Gromov-Witten theory, and by David Holmes [Hol19] by essentially resolving the indeterminancies of the rational map $\overline{M}_{g,n} \to \mathrm{Pic}^{[0]}(\overline{C}_{g,n})$ – here $\mathrm{Pic}^{[0]}(\overline{C}_{g,n})$ is the semi-abelian variety consisting of multi-degree 0 line bundles on the universal curve $\overline{C}_{g,n}$.

In the logarithmic world, the Jacobian $J_{g,n}$ does extend over $\overline{M}_{g,n}$ to the log Jacobian $\mathrm{LogPic}^0(\overline{C}_{g,n})$, and the Abel-Jacobi section extends to a section $\sigma : \overline{M}_{g,n} \to \mathrm{LogPic}^0(\overline{C}_{g,n})$ as well. One may then hope to use the extended Abel-Jacobi section to give a straightforward definition of the double ramification cycle. This approach carries its own set of difficulties – a major one being that we do not yet have a good understanding of the Chow groups of a log algebraic space such as LogPic. At the moment, the best way we understand Chow rings of a log scheme X is as a colimit $\lim_{X' \to X \text{ log blowup}} A(X')$ – this is an analogue of the bChow group defined similarly as a colimit along *all* blowups of X, but is much more manageable as there are far fewer log blowups of X. The proposed definition thus only produces a class in the log Chow ring of $\overline{M}_{g,n}$. Nevertheless, there is some evidence that this approach is useful. In [HPS19], it is seen that the double ramification cycle satisfies a certain multiplicativity relation, but this holds only in the bChow ring of $\overline{M}_{g,n}$ rather than the ordinary Chow ring; furthermore, on the locus of compact type curves, where the Abel-Jacobi section does extend, this relation is a consequence of the group structure of $\mathrm{Pic}^{[0]}(\overline{C}_{g,n})$. We expect that the multiplicativity relation will hold in the log Chow ring of $\overline{M}_{g,n}$ as well, and that it will be a consequence of the group structure of LogPic. Moreover, on the locus of compact type curves, Grushevsky and Zakharov show that there is a relationship between the double ramification cycle and the theta divisor [GZ14]; our work in duality produces an analogue of the theta divisor on LogPic, and we expect the analogous relation to hold in the log Chow ring of $\overline{M}_{g,n}$ as well.

4.2. **Other types of Invariants.** In some sense, the definition of log Gromov-Witten invariants is a happy accident: the space of stable log maps $K_\Gamma(X)$ is well behaved, but depends on X in a deep way, whereas the log Gromov-Witten invariants less so – the invariants do not change under logarithmic blowups, while $K_\Gamma(X)$ does. Nevertheless, the fact that $K_\Gamma(X)$ is easy to understand with techniques of traditional algebraic geometry is perhaps the reason there has been progress on the theory of log Gromov-Witten invariants. On the other hand, the log Hilbert scheme of a log scheme X is not as well behaved: it is not a scheme with a log structure, but behaves rather like the log Chow groups of X or $\mathrm{LogPic}(X)$. As we are beginning to get more comfortable with such spaces, we hope that we will be able to extract invariants in a natural way. A long-standing challenge for us has been to define analogues of other curve counting invariants – for example Donaldson-Thomas invariants – and study their relationship with log Gromov-Witten invariants.

4.3. **Log Abelian Varieties.** The first definition of a log abelian variety is presented in [KKN08]. However, the definition the authors give is rather complicated – essentially, log abelian varieties in [KKN08] are defined through log analogues of 1-motives, which have a particular form dictated by the theory of degenerations of abelian varieties described by Mumford and Faltings-Chai. Jonathan Wise and I propose an intrinsic definition: a log abelian variety is a log smooth, proper log algebraic space that is a group. A rough argument indicates that this definition is equivalent to the one given by [KKN08], but we are still far from a proof. We hope that such a definition will clarify the nature of log abelian varieties and can help simplify arguments involving degenerations of abelian varieties.

4.4. **Tropical Spaces.** Currently, a theory of tropical spaces sufficient for the purposes of log geometry is lacking. Essentially, what we understand fairly well is tropicalizations of log smooth schemes and of families of curves over simple bases. Tropicalization is a crucial tool in our understanding of log geometry, however, in general we don't even have a notion of a category of tropical spaces. In our study of duality for **LogPic**, we need to consider a version of tropicalization for families of curves over more general bases. This set of examples, together with the method of tropicalization carried out in [MW18], suggests a candidate for such a category. On the other hand basic notions geometric notions such as flatness, degree etc. remain unclear. The relationship of these kind of spaces with more traditional links between logarithmic geometry and tropicalization, such as Berkovich spaces, are still not understood, and neither is the relationship with other approaches to develop a general theory of tropical geometry (for instance that of Giansiracusa-Giansiracusa [GG16] or Mikhalkin-Rau).

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