EXPLORING A (2+1)-DIMENSIONAL TOPOLOGICAL FIELD THEORY

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Abstract. We begin with the basic definitions of a (2+1)-dimensional Topological Quantum Field Theory (TFT) and the inverse limit construction. Then, given a group G, we define a (2+1)-d TFT Z for triangulated manifolds. Using the inverse limit construction we produce a (2+1)-d TFT Z' which is independent of triangulations (Dijkgraaf–Witten theory). In the remainder of the paper we examine the properties of Z', extending it to 2-manifolds with boundary and paying particular attention to the case of the trinion and sphere with four holes.

1. Terminology and basic notation

In this paper, by a *manifold of dimension n without boundary* is meant a Hausdorff space with a countable basis, s.t. each point has an open neighborhood homeomorphic to an open ball in $\mathbb{R}^n$. (For example the surface of a 3-dimensional ball is a 2-dimensional manifold without boundary.) The definition of a *manifold with boundary* is the same as for a manifold without boundary, except that now each point must have an open neighborhood homeomorphic to either an open ball $\{\mathbf{x} \mid |\mathbf{x}| < 1\}$ or to a half ball $\{\mathbf{x} \mid x_n \geq 0, |\mathbf{x}| < 1\}$. The set of those points with neighborhoods homeomorphic to a half ball is called the *boundary* of the manifold, denoted ∂M for a manifold M. (For example a 3-dimensional ball and a 2-dimensional disc are manifolds with boundary). When we say a manifold is *closed*, we will mean that it is without boundary. All the manifolds we will be dealing with can be assumed to be compact and orientable. We will be working with both connected and disconnected manifolds, but unless explicitly written manifolds are to be considered connected.

The terms *n-dimensional manifold* and *n-manifold* will be used interchangeably. Throughout this paper we will generally hold the convention that M denotes a 3-manifold and Σ a 2-manifold.

For a given triangulation T of some manifold M, we use T^n to denote the set of simplices in T of dimension n. So in particular $\left|T^0\right|$ is the number of vertices in T. Also ∂T will denote the triangulation of ∂M induced by T. Further notation and terminology will be introduced as we proceed.

2. Laying down the groundwork

2.1. **Definition of a (2+1)-d Topological Field Theory.** A (2+1)-d TFT is a theory which,

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(1) to a closed 2-dimensional manifold, Σ, assigns a complex vector space denoted by $Z(\Sigma)$; and

(2) To a 3-dimensional manifold, M, assigns a vector which resides in $Z(\partial M)$ satisfying certain constraints. Here the manifolds are all considered to be oriented. Note that the manifolds may carry additional information such as triangulations or a metric. Apart from certain naturality constraints, $Z(\Sigma)$ and $Z(M)$ are required to satisfy the following four axioms:

Axiom 1 (VACUUM): $Z(\emptyset) = \mathbb{C}$.
Axiom 2 (DUALITY): $Z(\Sigma^*) = Z(\Sigma)^*$
Axiom 3 (MULTIPLICATIVITY): $Z(\Sigma_1 \amalg \Sigma_2) = Z(\Sigma_1) \bigotimes Z(\Sigma_2)$
Axiom 4 (GLUING):

(1) $Z(M_1 \amalg M_2)$, $Z(M_1) \otimes Z(M_2)$ are equal as elements of $Z(\partial M_1) \otimes Z(\partial M_2)$

(2) If M is a manifold with $\partial M = \Sigma \amalg \tilde{\Sigma} \amalg \tilde{\Sigma}^*$, and $\cup_{\tilde{\Sigma}} M$ is the manfiold which results from M by identifying its boundary components $\tilde{\Sigma}, \tilde{\Sigma}^*$, then $Z(\cup_{\tilde{\Sigma}} M) = \circ_{Z(\tilde{\Sigma})} Z(M)$, where $\circ_{Z(\tilde{\Sigma})}$ notes the natural contraction map $Z(\Sigma) \otimes Z(\tilde{\Sigma}) \otimes Z(\tilde{\Sigma})^* \to Z(\Sigma)$

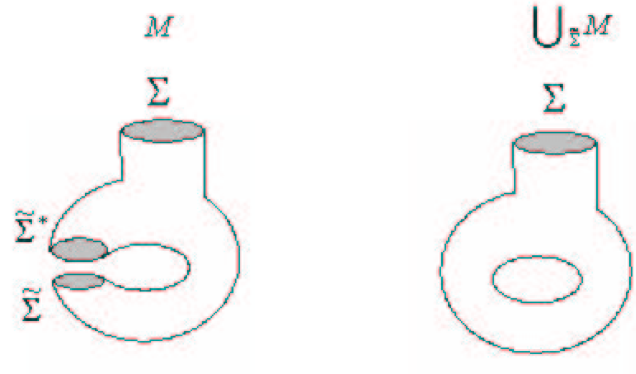

FIGURE 2.1. Gluing M's boundary components $\tilde{\Sigma}$,$\tilde{\Sigma}^*$, to get $\cup_{\tilde{\Sigma}} M$

Remarks and clarifications:

(i) The first axiom tells us that a closed 3-manifold is assigned a complex number by a (2+1)-d theory.

(ii) The second axiom states that switching orientation on 2-manifolds results in the dual vector space.

(iii) As a result of the multiplicativity and duality axioms if M is such that $\partial M = \Sigma_1^* \amalg \Sigma_2$ then $Z(M) \in Z(\partial M) = Z(\Sigma_1^* \amalg \Sigma_2) = Z(\Sigma_1)^* \otimes Z(\Sigma_2)$, which can be seen as a linear map $Z(\Sigma_1) \to Z(\Sigma_2)$.

(iv) If M_1, M_2 are such that $\partial M_1 = \Sigma_1^* \amalg \Sigma$, $\partial M_2 = \Sigma^* \amalg \Sigma_2$ then using axiom 4 we have $Z(M) = Z(M_2) \circ Z(M_1) : Z(\Sigma_1) \to Z(\Sigma_2)$, where M is the manifold obtained by gluing M_1, M_2 across their common boundary.

(v) If Σ is a 2-manifold then by remark (iii), a (2+1)-d TFT assigns to the cylinder $\Sigma \times I$, a linear map $i_\Sigma : Z(\Sigma) \to Z(\Sigma)$. Gluing two copies of this cylinder together along a common boundary component results in a cylinder homeomorphic to the original, and therefore by the gluing axiom, $i_\Sigma \circ i_\Sigma = i_\Sigma$, in other words i_Σ is an idempotent.

(vi) Axioms 3 and 4 are closely related, stating effects of gluing on manifolds of dimensions 2 and 3, respectively. (Note that in the world of (2+1)-d TFTs, the only 2-manifolds discussed are without boundary and so the only possible gluing operation on them is disjoint union.) Also Axiom 3 allows us to focus our definitions on connected manifolds, since the case of disconnected manifolds follows immediately using this axiom.

(vii) Sometimes a (2+1)-d TFT may also be extended to include assignments to 2-manifolds with boundary. This would give the notion of a *(2+1)-d TFT with corners* or of an *extended TFT (ETFT)*, see [1]. In this case there is usually added some special labeling on boundary components of a 2-manifold with boundary, and a vector space is only supplied to the manifold when thus labelled.

2.2. **The inverse limit construction.** As mentioned above a TFT may be defined for manifolds endowed with extra information. Not surprisingly it is often simpler to define a TFT when extra data is provided for the manifolds. Given a (2+1)-d TFT, Z, defined for manifolds with additional data, it is only natural to ask if it is possible to eliminate the dependency on this extra data. One would hope to retrieve a smaller theory Z' contained in Z, which is independent of any extra information, and is thus a topological theory in the pure sense. By "contained", we mean that $Z'(\Sigma) \subseteq Z(\Sigma, d)$ (or at least that $Z'(\Sigma)$ can be naturally embedded inside $Z(\Sigma, d)$) for any additional data d that Σ may be endowed with.

Under a natural assumption which will be given shortly, it turns out that not only does such a theory, Z', exist, but there is also a well defined process by which one can construct it. The following paragraphs give a concise survey of this process called the *inverse limit construction*. Later in this paper, after we have defined a (2+1)-d TFT for triangulated manifolds, we will make use of it to construct a TFT which is independent of triangulations. It is important to note that following the inverse limit construction guarantees us a new TFT. A detailed survey of this process can be found in [2].

Let Z be a (2+1)-d TFT for manifolds endowed with additional data. Let $D(M)$ be the set of extra data allowed on the 3 (or 2)-manifold M. Although we won't go into the details of what we allow as additional data, it is important that data on M induces data on ∂M, while data on two manifolds containing a common boundary inducing identical data on that boundary, can be glued into valid data on the glued manifold. In order for the construction to work we first assume $Z(M, d_1) = Z(M, d_2)$, for any two datum $d_1, d_2 \in D(M)$ which agree on ∂M. In other words we are assuming that M is independent of any data which is not on the boundary. In particular, for closed manifolds, this condition states that $Z(M)$ is a true topological invariant, independent of additional data. It follows that for a manifold M it is enough to specify the data on the boundary.

Fix a 2-manifold Σ.

Definition 2.1. The category C_Σ is defined to have set of objects

$$OBJ(C_\Sigma) = \{d | d \in D(\Sigma)\}$$

while the set of morphisms from d to d' is defined to be cylinders $\Sigma \times [0,1]$ endowed with additional data which restricts on $\Sigma \times \{0\}$ to d and on $\Sigma \times \{1\}$ to d'. That is,

$$MORPH(C_\Sigma)(d_1, d_2) = \left\{ d \in D(\Sigma \times [0,1]) \;\middle|\; d|_{\Sigma \times \{0\}} = d_1, d|_{\Sigma \times \{1\}} = d_2 \right\}$$

(The morphisms can be thought of as 3-cylinders with boundary $\Sigma^* \amalg \Sigma$, where the boundary components are endowed with some extra data.)

Let V^1 denote the category, which has as objects vector spaces, and linear transformations as morphisms. Then Z induces a $F : C_\Sigma \to V^1$, as follows.

Definition 2.2. The functor $F : C_\Sigma \to V^1$ is defined on objects by

$$F(d) = Z(\Sigma, d)$$

and on morphisms by

$$F(d) = Z(\Sigma \times [0,1], d)$$

Now we begin defining our new (2+1)-d TFT Z'.

Definition 2.3. $Z'(\Sigma) = \left\{ \begin{array}{l} a\,choice\, v(d) \in Z(\Sigma, d)\, for\, every\, datum\, d\, on\, \Sigma\ s.t. \\ \forall \phi \in MORPH(C_\Sigma)(d_1, d_2)\ ,\ F(\phi)\,(v(d_1)) = v(d_2) \end{array} \right\}$

Firstly note that, $Z'(\Sigma)$ is independent of any specific triangulation U as required. Secondly, if we assume that there exists a morphism between every two objects in C_Σ, as will be the case with the theory we later define, then for any fixed datum d_0 the choice $v(d_0)$ determines all other choices. So we have:

Corollary 2.4. $Z'(\Sigma) \cong \{v \in Z(\Sigma, d_0) \mid F(\phi)(v) = v, \forall \phi \in MORPH(C_\Sigma)(d_0, d_0)\}$

Since the theory Z was assumed to be independent of inner data, we have that for all cylinders ϕ, $F(\phi)$ depends only on the source and target of ϕ. In other words if we use the following notation:

Notation 2.5. $F_{d_0,d_0} := F(\phi)$, *where* $\phi \in MORPH(C_\Sigma)(d_0, d_0)$

We have:

Corollary 2.6. $Z'(\Sigma) \cong \ker(F_{d_0,d_0} - Id) \subseteq Z(\Sigma, d_0)$ *for any possible data* d_0 *on* Σ.

To define Z' on a 3-manifold M, with boundary Σ, pick some additional data d and define:

Definition 2.7. $Z'(M) = F_{\partial d, d_0}(Z(M, d)) \in Z'(\Sigma)$

Note that this definition is independent of d. It is easiest to understand this by thinking of the manifold M with data d which then has the cylinder $F_{\partial d, d_0}$ glued on to it, d is lost in the resulting manifold which is just M with boundary data d_0. The vector assigned to M is simply the image of the vector $Z(M, d)$, under the linear transformation that Z assigned to the cylinder $\Sigma \times I$, whose boundary is endowed with data $\partial d \cup d_0$. Recall that d_0 was arbitrarily chosen. Had we chosen any other boundary data d' instead of d_0 we would (by the above section) have an isomorphic vector space in which $F_{d_0,d'}(F_{\partial d, d_0}(Z(M, d)) = F_{\partial d, d'}(Z(M, d)) \in Z'(\Sigma)$.

This completes the construction of the inverse limit theory — a smaller theory, completely independent of additional data.

2.3. **A standard triangulation for cylinders.** We will shortly be introducing a (2+1)-d TFT Z for triangulated manifolds. Since we will use the inverse limit construction, it must hold that the result of Z for a 3-manifold will depend only on the boundary triangulations. For a cylinder, $\Sigma \times I$, this means that $Z(\Sigma \times I)$ will depend on triangulations only up to the triangulations on Σ^*, Σ. For reasons that will become clear later on, we need to define a standard triangulation for cylinders $\Sigma \times I$ where Σ^*, Σ are both triangulated by the same triangulation U. Luckily, there exists a very simple triangulation for such cylinders. The standard triangulation which we will use employs the prism triangulation shown in figure 2.2 as a basic building block.

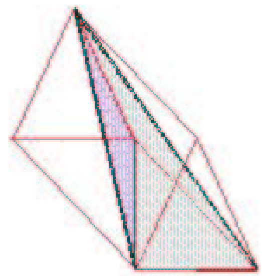

FIGURE 2.2. Basic building block for a standard triangulation

To obtain a standard triangulation for a general cylinder, first divide it into triangular cylinders and triangulate them as in figure 2.2. Now as long as the diagonals on adjacent faces match, there is no problem. On all faces where the diagonals from the two adjacent triangular cylinders do not match, insert a new tetrahedron whose edges are the four edges of the rectangular face and the two crossing diagonals, slightly bowed. This is illustrated in figure 2.3 .

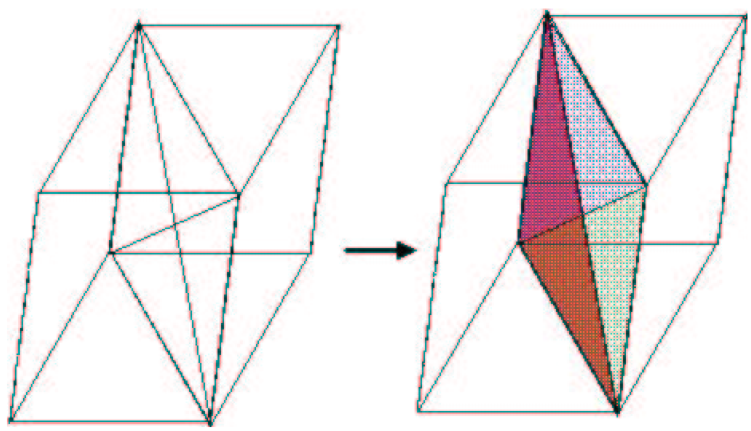

FIGURE 2.3. Insertion of a new tetrahedron

Notice that we have not inserted any inner vertices throughout this process. Figure 2.4 gives an example of a possible standard triangulation of a cylinder (omitting inner edges).

Note that there may be many standard triangulations for a single cylinder. When we say a cylinder is triangulated by a standard triangulation, we simply mean it is triangulated by some triangulation that can be produced by the process described above. The benefits of using standard triangulations will gradually become clear as we proceed.

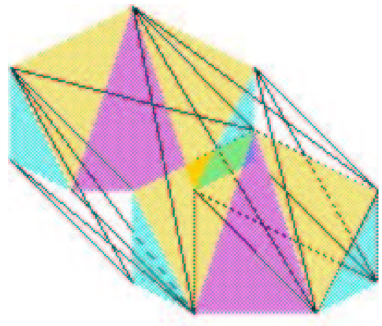

FIGURE 2.4. An example of a standard triangulation (omitting inner edges)

3. Introducing the theory Z

3.1. Aiming for an invariant. We shortly begin to define a (2+1)-d Topological Field Theory Z (henceforth TFT or simply theory), for triangulated manifolds. Obviously a theory that deals with triangulated manifolds does not define a topological invariant in the pure sense, since extra data, namely the triangulation, is taken into account. As mentioned above we will later use the inverse construction theory to derive a smaller theory, which is independent of triangulations.

Let G be a finite group with identity element e. The following is a topological invariant of a manifold M.

Definition 3.1. $Z(M) = \frac{|Hom(\pi_1(M) \to G)|}{|G|}$ where $\pi_1(M)$ denotes the fundamental group of M.

Two examples are instructive. For the 3-sphere, S^3, the fundamental group is trivial and so $Z(S^3) = 1/|G|$. For $M = S^2 \times S^1$, the fundamental group is the infinite cyclic group and so a homomorphism from it to G is specified by the (arbitrary) image of a generator, so that $Z(S^2 \times S^1) = 1$.

We aim to build a theory which assigns this invariant to closed 3-manifolds. Thus it is clear that although the theory Z will in general be dependent on triangulations, it will not depend on the triangulation of closed 3-mainolds.

3.2. Legal labelings. Before we begin defining Z we must gain some understanding of triangulations on 2-dimensional manifolds. Let Σ be a 2-manifold triangulated by U. In particular we will be interested in functions $\alpha : Edges(U) \to G$, where edges of a triangulation are considered to be directed. Keeping in mind the invariant given in definition 3.1 it is clear that we aim to relate our theory with homomorphisms from the fundamental group of the manifold to G. It would thus seem wise if we were to restrict our interest to those functions which:

a. assign inverse elements to a fixed edge with opposite direction;

b. assign e to triangles. By this we mean that the multiplication of the elements assigned to 3 edges, which when traversed make up a triangle's perimeter, equals e. (The multiplication is taken in the order of the traversal.)

We will later see why these two conditions are sufficient, but in the meanwhile it may be a good idea to hold in mind that the inside of a triangle is always simply connected, so that the closed loop associated with the triangles perimeter is homological to a point, and is therefore equal to the identity element of the fundamental group. Throughout this paper when talking of a function $\alpha : Edges(U) \to G$ we implicitly assume that α observes condition (a). To specify that α observes condition (b) for a triangle Δ we write $\alpha_{\Delta=e}$. If the triangle Δ is not specified, we take $\alpha_{\Delta=e}$ to mean α assigns e to every trinagle in its domain. We will often use the term **legal labeling** to denote a function which observes the above conditions.

Notice, however, that a legal labeling σ will do more than assign e to every triangle. Indeed it will assign e around every contractible closed path of the triangulation. This is true since any such closed path can be broken up into triangles, where all edges that are not part of the closed path will cancel out (as shown in 3.1).

It is crucial to understand that this does not mean that every closed loop of the manifold will be assigned e. The difference is illustrated in figure 3.1. Follow the triangles in the left figure to see that the closed loop marked by thick arrows must be assigned e by any legal triangulation[1].

The right figure presents a simple triangulation of the 2-dimenional torus, (edges with the same arrow style are glued in the direction indicated by the arrow). Since, in the given triangulation of T^2, all four vertices are one and the same, every edge is a closed loop on T^2, but these loops most certainly do not have to be labeled by e in order for the labeling to be legal!

FIGURE 3.1. **Left:** A closed loop of triangulation edges will be assigned e by any legal triangulation. **Right:** Closed loops on T^2 may be assigned elements other than e by legal labelings.

It should be noted that there is a difference between the notion of a point on the manifold and that of a vertex of the triangulation. A point on the manifold may be represented by several different vertices in the triangulation of that manifold, which when glued become the same point on the manifold. When we talk about a path on the triangulation we are referring to the vertices and edges it traverses and

[1]Traverse the square *abce*, as follows: *abcacea* (the edge *ac* cancels out). This is just like traversing two triangles which gives e, so now we have that the paths *abc*, *aec* are equal, using the same trick on the quadrangle *ecdf* we get *ce*=*cdfe*, and obviously *dfe*=*de* so *abcde*=*abce*, which shows us that the closed loop is indeed assigned e. This process can be followed for any closed loop, by dividing it into quadrangles.

not the points. Until the end of this section we will use the notation $\tilde{a}$ to denote a point on the manifold, whereas a will denote a vertex corresponding to $\tilde{a}$.

Since, as noted above, every closed path on the triangulation must be assigned e by a legal labelng, the label on any path (given by multiplying the labels on the edges) depends only on its end vertices (two paths with different end **vertices** but the same end **points** may result with different labels). So each legal edge labeling can be seen as a multiplicative labeling of paths (as opposed to edges), that complies with conditions (a) and (b) stated above. It follows that each legal path labeling induces a homomorphism $\rho : \pi_1(M) \to G$ (it is a multiplicative map which depends on paths only up to their end vertices, so homotopic loops are assigned the same value).

Now, conversely, for any homomorphism $\rho : \pi_1(M, *) \to G$ and triangulation T, we can construct a legal labeling σ of the paths of T which assigns closed loops the same values as ρ. Here $*$ is an arbitrary basepoint. For any point $\tilde{a}$ that does not correspond to *, arbitrarily choose a single vertex, a, corresponding to $\tilde{a}$. Now choose some path γ_a on the triangulation edges from * to a, and let $\sigma(\gamma_a) = g_a \in G$. For any other vertex a' which also corresponds to the point $\tilde{a}$, the path $\gamma_{a,a'}$ between them is a closed loop on the manifold and therefore $\sigma(\gamma_{a,a'})$ can be derived from ρ as follows: Let ζ be the loop on the manifold which first traverses the path γ_a then traverses the loop $\gamma_{a,a'}$ and finally returns to * via the path γ_a^{-1}, and let $x = \rho(\zeta)$. Define $\sigma(\gamma_{a,a'}) := \sigma(\gamma_a) x \sigma(\gamma_a)^{-1} = g_a x g_a^{-1}$. Keeping in mind that we want σ to be a multiplicative map we define $\sigma(\gamma_{a'}) := \sigma(\gamma_{a,a'})\sigma(\gamma_a) = g_a x g_a^{-1} g_a = g_a x$, and more generally for any two vertices a,b define $\sigma(\gamma_{a,b}) := \sigma(\gamma_b \gamma_a^{-1}) = \sigma(\gamma_b)\sigma(\gamma_a^{-1})$.

σ is now well defined for all edges, and it is clear that it yields e around any triangle. Furthermore, a path on the triangulation corresponding to a closed loop ζ on the manifold based at vertex (point) *, will be assigned the value $\rho(\zeta)$. This is because σ is a multiplicative map and if $*'$ is a vertex corresponding to the same point as *, then by definition $\sigma(\gamma_{*,*'}) = exe^{-1} = x$ where x is exactly $\rho(\zeta)$.

We started with $\rho \in Hom(\pi_1(M, *) \to G)$ and arbitrarily chose for each point $a \neq *$ an element $g_a \in G$. Thus we had a total freedom of $|G|^{|T^0|-1}$ ($|T^0|$ is the number of different vertices, all vertices corresponding to the same point are counted as one vertex). Since any multiplicative legal path labeling can be constructed in this manner (using the homomorphism it induces) we have just proved the following:

Theorem 3.2. $\frac{\#(\alpha_{\Delta=e} : Edges(T) \to G)}{|G|^{|T^0|-1}} = |Hom(\pi_1(M) \to G)|$

3.3. **Defining** Z. For a closed 2-dimensional manifold Σ, triangulated by U we define:

Definition 3.3. $Z(\Sigma, U) = \langle e_\alpha \mid \alpha_{\Delta=e} : Edges(U) \to G \rangle$

This means that Z assigns to triangulated 2-manifolds a vector space spanned by all legal labelings on its triangulation's edges. (This does not in any way mean that all the vectors in the space represent legal labelings, rather that they represent some weighted combination of legal labelings.)

For a 3-manifold M triangulated by T define the vector $Z(M, T) \in Z(\partial M, \partial T)$ as follows:

Definition 3.4. $$Z(M,T)_\alpha = \sum_{\substack{\sigma : Edg(T) \to G \\ s.t.\ \sigma |_{\partial T} = \alpha}} \frac{1}{|G|^{|T^0|-1/2|\partial T^0|}} \prod_{\Delta \in T} \delta_{\Delta=e}$$

where the subscript α denotes the component as indexed by the natural basis given in definition 3.3, and $\delta_{\Delta=e}$ is defined to be 1 if the multiplication of the elements assigned to the 3 edges of the triangle Δ equals e and zero otherwise. Multiplying $\delta_{\Delta=e}$ over each triangle assures that we count only legal labelings σ.

This definition means that Z assigns to a triangulated 3-manifold, a vector that has each of its basis components factored by the number of ways in which the legal labeling given by the basis component can be legally extended, normalized by a constant $\frac{1}{|G|^{|T^0|-1/2|\partial T^0|}}$.

Before we begin to check the definitions given above, we present two quick examples which calculate the dimensions of the vector spaces that Z assigns to the 2-sphere S^2 and the 2-torus T^2 triangulated by U_1, U_2 respectively [See figure 3.2].

FIGURE 3.2. **Left:** a triangulation U_1 of S^2; **Right:** a triangulation U_2 of T^2. (Two edges ending with the same arrow shape are glued together.)

Example 3.5. $Z(S^2, U_1)$

Notice that there are only 3 distinct edges in U_1, and any two of these can be freely labeled. Two edge labels uniquely determine the third in a legal labeling. We have full freedom on two edges and so there are $|G|^2$ legal labelings and therefore we have $\dim(Z(S^2, U_1)) = |G|^2$.

Example 3.6. $Z(\mathrm{T}^2, U_2)$

T^2 is slightly more tricky. Although U_2 also consists of only 3 distinct edges, and the third one is determined by the first two, there is an extra constraint. If we label the first two edges g,h it must hold that $gh = hg$. In other words there is complete freedom only on one edge, while the label assigned to the second edge must commute with the label assigned to the first edge. So each label g on the first edge contributes a total of $\frac{|G|}{|[g]|}$ labelings (where $[g]$ denotes the conjugacy class of g in G). Let k denote the number of conjugacy classes in G, we have that $\dim Z(\mathrm{T}^2, U_2) = \sum\limits_{g \in G} \frac{|G|}{|[g]|} = |G|\, k$.

3.4. **Checking that Z is a TFT.** Given the above definition of Z we should immediately have two concerns:

i) We would like to check that Z as we defined it obeys the gluing property that every TFT should observe.

ii) We need to check that this definition restricts to the invariant given in definition 3.1 when applied to closed 3-manifolds.

i) Let M_1, M_2 be two 3-manifolds with a common boundary component Σ triangulated by T_1, T_2 respectively, such that T_1, T_2 agree on Σ. We need to check that summing over all possible labelings α on $T_1 \cap T_2$ gives $Z(M,T)$, where M is the manifold given by gluing M_1 and M_2 along their common boundary Σ, and $T = T_1 \cup T_2$. For simplicity reasons we assume that Σ is the only boundary component, so that M is a closed manifold[2]. The following equations give a proof.

Proof.

$$\begin{aligned}
&\sum_{\alpha_{\Delta=e}:Edg(T_1 \cap T_2) \to G} Z(M_1,T_1)_\alpha Z(M_2,T_2)_\alpha \\
&= \sum_{\substack{\sigma_1 : Edg(T_1) \to G \\ \sigma_2 : Edg(T_2) \to G \\ \sigma_1 |_{\partial T_1} = \sigma_2 |_{\partial T_2} = \alpha}} \frac{1}{|G|^{|T_1^0| + |T_2^0| - 1/2|U^0| - 1/2|U^0|}} \prod_{\Delta \in T_1 \cup T_2} \delta_{\Delta=e} \\
&= \sum_{\sigma:Edg(T) \to G} \frac{1}{|G|^{|T^0|}} \prod_{\Delta \in T} \delta_{\Delta=e} \\
&= Z(M,T)
\end{aligned}$$

□

ii) We need to show that our general definition for a 3-manifold, M, coincides with the invariant given in definition 3.1 when M is closed. Since M has no boundary, Z must assign to M a complex number. Using theorem 3.2 it is clear that:

$$Z(M,T) = \sum_{\sigma:Edg(T) \to G} |G|^{-|T^0|} \prod_{\Delta \in T} \delta_{\Delta=e} = \frac{|Hom : \pi_1(M) \to G|}{|G|}$$

which is what we wanted to show.

3.5. **2-manifolds with boundary.** A (2+1)-d TFT usually does not deal with 2-manifolds which are not closed, but there is no reason why it can not be extended to deal with such manifolds. However, one must exercise extra caution when extending the theory. Namely, one may be tempted to give an arbitrary definition without checking that it indeed restricts to the existing definition.

We will stay out of trouble by simply not defining the notion of a (2+1)-d TFT with corners and similarly not asserting that the definition we give does indeed satisfy the conditions for such. The purpose of the following definition is to serve as impetus for definitions that will be introduced in Z', the theory we obtain using the inverse limit construction. Needless to say, any use of these future definitions will have to be carefully checked, and their correctness proved.

For a closed 2-dimensional manifold Σ with a non-empty boundary, triangulated by U we first label the edges of ∂U by some arbitrary labeling β. Define:

Definition 3.7. $Z_\beta(\Sigma, U) = \langle e_\alpha \mid \alpha_{\Delta=e} : Edg(U) \to G \text{ and } \alpha |_{\partial U} = \beta \rangle$

Notice that the above vector space is spanned by all legal labelings which restrict to β on the boundary. **It is crucial to understand that for a manifold Σ with boundary, $Z(\Sigma, U)$ is meaningless. Only once a boundary labeling is chosen can we apply the theory to the manifold.** Different boundary labelings may very well result in different spaces. There may for example be boundary labelings which are inherently illegal, meaning that they can not be extended at all to a legal labeling, such boundary labelings will result in an empty vector space.

[2]The general proof takes other boundary components into account and will not necessarily result with a closed manifold. It is more technical and would gain the reader nearly no insight into the underlying idea.

4. Constructing the inverse limit theory Z'

4.1. **Z does not depend on inner triangulations.** In this section our aim is to derive from the theory Z that we defined above, a smaller theory, Z', which is independent of triangulations. To do this we use the inverse limit construction described in section 2.2. However, as noted before, the inverse limit construction will only work if the theory we begin with, Z, is independent of inner triangulations.[3] In other words it first must be shown that if a 3-manifold, M, is triangulated by two different triangulations T_1, T_2 such that $\partial T_1 = \partial T_2$, then $Z(M, T_1) = Z(M, T_2)$.

Two local triangulation moves allow us to get from any one triangulation of a 3-manifold to another (see [3]). These moves are illustrated in figure 4.1.

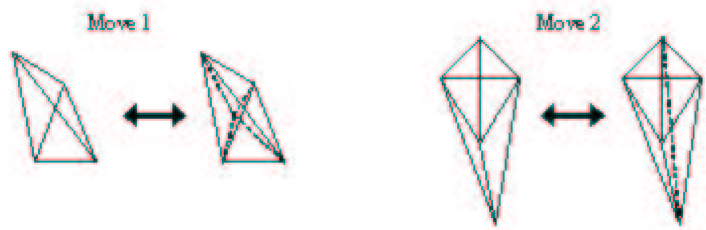

FIGURE 4.1. Local triangulation moves. **Left:** Splitting one tetrahedron into 4 by adding a central vertex, or vice versa. **Right:** Two tetrahedra glued along their base change into 3 tetrahedra with sides glued along the dashed line, or vice versa.

Firstly note that if the triangulations T_1, T_2 are such that $\partial T_1 = \partial T_2$ then by definition $Z(M, T_1)$, $Z(M, T_2)$ live in the same vector space. Secondly, since any triangulation can be changed into any other using the local triangulation moves it will suffice to show that these moves conserve definition 3.4.

Move 2 does not add or remove any vertices, and does not affect the freedom of the labeling (an inner edge is added, but its label is completely determined by existing edges), so it must conserve definition 3.4.

Move 1 does in fact add (remove) a vertex from the triangulation, but as it does so it increases (decreases) the freedom on the labeling by factor $|G|$ (four inner edges are added, one of them can be labeled with complete freedom, but now the 3 remaining ones are uniquely determined). The following equations prove that these two changes cancel out (T denotes the original triangulation, and T' is the triangulation after one inner vertex has been added).

Proof. $Z(M, T')_\alpha$

$$= \sum_{\substack{\sigma : Edg(T') \to G \\ \sigma |_{\partial T'} = \alpha}} \frac{1}{|G|^{|T^0|+1-1/2|\partial T^0|}} \prod_{\Delta \in T'} \delta_{\Delta = e}$$

[3]We actually need something a little weaker, namely that cylinders are only dependent on the triangulation of their boundaries.

$$= \sum_{\substack{\sigma : Edg(T) \to G \\ \sigma|_{\partial T} = \alpha}} \frac{|G|}{|G|^{|T^0|+1-1/2|\partial T^0|}} \prod_{\Delta \in T} \delta_{\Delta = e}$$

$$= \sum_{\substack{\sigma : Edg(T) \to G \\ \sigma|_{\partial T} = \alpha}} \frac{1}{|G|^{|T^0|-1/2|\partial T^0|}} \prod_{\Delta \in T} \delta_{\Delta = e}$$

$$= Z(M,T)_\alpha \qquad \square$$

We have just shown that the two moves can be freely employed without affecting the result, and therefore Z **is dependent on triangulations only up to the triangulation induced on the boundary**. This can be viewed as a generalization of the fact that Z is completely independent of triangulations on closed manifolds.

4.2. **Defining Z'.** We now use the inverse limit construction of section 2.2 to define a new theory Z'. We make the transition from the general inverse limit construction by replacing the general additional data, with triangulations.

Recall that the inverse limit construction gave us $Z'(\Sigma) = \ker(F_{U_0,U_0} - Id)$ where F_{U_0,U_0} denotes the value Z assigns the cylinder $\Sigma \times I$ which has both boundary components triangulated by U_0. Since we have already shown that Z is independent of inner triangulations we are free to choose a standard triangulation [see section 2.3] for the above cylinder. The advantages of using a standard triangulation will become clear over the next few sections. (Note that unless a different triangulation is explicitly given, we will always assume that a cylinder $\Sigma \times I$, where Σ^*, Σ are triangulated in the same way, is triangulated in a standard way).

Definition 2.7 gives us the result of Z for a 3-manifold, but since F_{U_0,U_0} can be thought of as a linear transformation $Z(\Sigma, U_0) \to Z(\Sigma, U_0)$, we can reformulate the result using a matrix.

Let Σ be a 2-manifold triangulated by U_0 , and let S be the number of legal edge labelings on U_0. Define the matrix $A \in M_{S\times S}(\mathbb{Q})$, columns and rows indexed by all legal edge labelings on U_0 as follows:

Definition 4.1. $A^{\alpha}_{\alpha'} = \frac{1}{|G|^{|U_0^0|}} \cdot \# \left(\begin{array}{l} legal\ labelings\ \sigma : Edg(T) \to G\ s.t. \\ \sigma|_{U_0^*} = \alpha \quad and \quad \sigma|_{U_0} = \alpha' \end{array} \right)$ where T is a standard triangulation of $\Sigma \times I$ which restricts to U_0 on Σ, Σ^*. Then $Z'(\Sigma) = \ker(A - Id)$

Let M be a 3-manifold, with boundary Σ, choose a triangulation T of M. From definition 2.7 we have:

Definition 4.2. $Z'(M) = F_{\partial T, U_0}(Z(M,T)) \in Z'(\Sigma)$

Note that although the definitions above make use of specific triangulations, we have, from section 2.2, that they are completely independent of these triangulations in the following sense: Changing the fixed triangulations would result in isomorphic vector spaces for 2-manifolds and vectors which transform to each other for 3-manifolds.

Also, notice that the factor $\frac{1}{|G|^{|U_0^0|}}$ in definition 4.1 originated from $\frac{1}{|G|^{2|U_0^0|-|U_0^0|}}$ corresponding to the factor $\frac{1}{|G|^{|T^0|-1/2|\partial T^0|}}$ in definition 3.4. (This is the first time that using a standard triangulation has helped us, since using a standard triangulation we may assume that there are no vertices that do not lie on the boundary).

This completes the construction of the inverse limit theory Z', a smaller theory, completely independent of triangulations. Actually, we can say much more about Z', but we first give two examples in hope that the reader gains some familiarity with the new theory.

4.3. **A couple of examples.** Let Σ be a closed 2-manifold. What can we say about $\dim(Z'(\Sigma))$? Well, since the matrix A of definition 4.1 corresponds to the result of Z for a cylinder $\Sigma \times I$ it must be an idempotent, and therefore we have that: $\dim(Z'(\Sigma)) = \dim(\ker(A - Id)) = \dim(Im(A))$.

We now calculate the dimensions of $Z'(S^2)$ and of $Z'(\mathrm{T}^2)$, when $G = S_3$ in hope of gaining some intuition.

Example 4.3. $Z'(S^2)$

Figure 4.2 presents a standard triangulation of $S^2 \times I$ (omitting all inner diagonal edges).

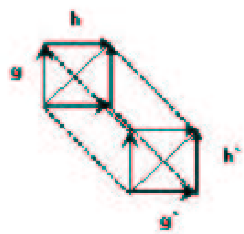

FIGURE 4.2. A standard triangulation of $S^2 \times I$ (omitting all inner diagonal edges). Edges ending with the same arrow shape on each boundary are glued.

Since the constraints on the boundary are given by a single triangle, any choice of g, h uniquely determines a legal labeling of S^2, furthermore since all closed loops on S^2 are represented by triangles, we have complete freedom in choosing a single interboundary edge label, which in turn determines all other labels. So we have that the matrix $A \in M_{36\times 36}(\mathbb{Q})$ is a constant $\frac{|G|}{|G|^3} = |G|^{-2} = \frac{1}{36}$, which of course has dimension one. Notice that A is an idempotent. Keep in mind that $\pi_1(S^2) = \{e\}$.

Example 4.4. $Z'(\mathrm{T}^2)$

The torus is much trickier. Figure 4.3 presents a standard triangulation of $\mathrm{T}^2 \times I$ (omitting all inner diagonal edges).

As we calculated in example 3.6, there are $|G|\ k$ legal labelings on each boundary (where k denotes the number of conjugacy classes), so in our case there are 18 such labelings. Each such labeling corresponds to a pair $(g, h) \in G$ such that g commutes

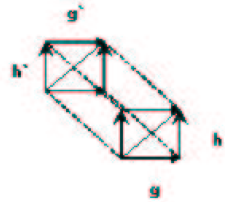

FIGURE 4.3. A standard triangulation of $T^2 \times I$ (omitting all inner diagonal edges). Edges ending with the same arrow shape on each boundary are glued.

with h (we write $[g,h]$). Notice that all vertices on the boundary are actually the same vertex, so all inter-boundary edges are also the same. This means that there is a legal labeling on $T^2 \times I$ only if there exists an element $f \in G$ such that $fgf^{-1} = g'$ and $fhf^{-1} = h'$. The matrix A, is presented here[4]:

$$A = \frac{1}{6}\begin{pmatrix}
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3
\end{pmatrix}$$

The factor $\frac{1}{6}$ is given by $|G|^{|T^0|-1/2|\partial T^0|} = 6^{2-1}$. Notice that A is an idempotent and its image has dimension 8.

Exercise. At this point the reader may find it insightful to try the following simple exercise:

---

[4]The columns and rows of A are indexed by the pairs *(g,h)* in the following order:
Id,Id | Id, (12) | Id, (13) | Id, (23) | Id, (123) | Id, (132) | (12), Id | (13), Id | (23), Id | (123), Id | (132), Id | (12), (12) | (13), (13) | (23), (23) | (123), (123) | (123), (132) | (132), (123) | (132), (132)

Keeping in mind that $\pi_1(\mathrm{T}^2) = \mathbb{Z} \times \mathbb{Z}$, how many homomorphisms $\rho : \mathbb{Z} \times \mathbb{Z} \to S_3$ are there? Could some homomorphisms be identified so that the result matches the dimension of $Im(A)$? The answer is given in the next sections.

4.4. **Extending Z' to 2-manifolds with boundary.** Up until now we followed the safe road of the inverse limit construction. We now attempt to define Z' for 2-manifolds with boundary. This definition is not provided to us by the inverse limit construction since it was not rigorously defined for the original theory. We will, however, attempt to give a definition which stems from the definition given in section 3.5. Since it is not derived directly from the inverse limit construction, we will need to prove that this definition, or rather special cases of this definition, indeed comply with all the rules a TFT should observe, whenever we make use of them.

4.4.1. *Defining Z' for 2-manifolds with boundary.* Let $L_1, \dots, L_n$ be the one dimensional boundary components of a 2-manifold Σ. Note that since we assume Σ to be compact, the boundary components are simply closed loops (copies of S^1). We first triangulate Σ by some triangulation U, and label the boundary edges with elements of G arbitrarily. We denote this boundary labeling β. Let $g_1, \dots, g_n \in G$ be the products of the edge labels given by β around the boundary components $L_1, \dots, L_n$ respectively. Note that in order for $g_1, \dots, g_n$ to be well defined one must choose a point from which to begin the calculation of the product. Temporarily, $g_1, \dots, g_n$ should be thought of as given by some fixed arbitrary choice of starting points. Our aim is to define $Z'_\beta(\Sigma)$, the result of Z' for a manifold Σ which has its boundary components labeled by β. Later we will see that $Z'_\beta(\Sigma)$ depends only on $[g_1], \dots, [g_n]$.

Let Σ be a 2-manifold with boundary triangulated by U, let β be an arbitrary labeling of ∂U, and let S be the number of legal edge labelings on U which restrict to β on ∂U . Define the matrix $A \in M_{S\times S}(\mathbb{Q})$, columns and rows indexed by all legal edge labelings on U, **which restrict to β on ∂U** as follows:

Definition 4.5. $A^{\alpha}_{\alpha'} = \frac{|[g_1]|\cdot\ldots\cdot|[g_n]|}{|G|^{|U^0|}} \cdot \# \begin{pmatrix} \textit{legal labelings}\, \sigma : Edg(T) \to G \; s.t. \\ \sigma\,|_{U^*} = \alpha \quad \textit{and} \quad \sigma\,|_{U} = \alpha' \end{pmatrix}$
where T is a standard triangulation of $\Sigma \times I$ which restricts to U on Σ, Σ^*. Then $Z'_\beta(\Sigma) = \ker(A - Id)$

A few clarifications are in place here:

a. It would seem that the definition given above is dependent on the triangulation U. If this definition is to be included in our new theory it must be shown that it, or at least the dimension of the image of A, actually does not depend on the triangulation. This will be shown shortly.

b. Even once a triangulation is given, $Z'(\Sigma)$ is meaningless. It is not until the boundary labeling β is given that we can apply our theory and get $Z'_\beta(\Sigma)$.

c. This definition is a natural way of extending the inverse limit construction to definition 3.7 except for the extra factor of $|[g_1]|\cdot\ldots\cdot|[g_n]|$, which will be explained shortly.

d. This definition restricts to the original inverse limit construction definition, when $\partial\Sigma = \emptyset$.

Let Σ be a 2-manifold with boundary components $L_1, \ldots, L_n$, and let U be a triangulation of Σ. Let β be a labeling of the boundary components such that it can be extended to some legal labeling α for all the edges of U, and for which the products around L_i (starting at some arbitrary point) are g_i. Recall that each legal labeling induces an homomorphism $\rho : \pi_1(\Sigma, *) \to G$ [see section 3.2]. Consider the loop ζ_i which emerges from $*$, loops around a boundary component L_i and immediately returns to $*$. Note that the homomorphism that α induces will assign to ζ_i, an element in $[g_i]$. We say that a homomorphism which assigns to all loops ζ_i an element in $[g_i]$ is *compatible* with the boundary labels.

Definition 4.6. $HOM_g = \{\rho \in HOM(\pi_1(\Sigma) \to G)\, s.t.\, \rho(\zeta_i) \in [g_i]\}$
or in other words HOM_g is the set of all homomorphisms compatible with the boundary labels g_i given by β.

The idea is illustrated in figure 4.4.

FIGURE 4.4. $x^{-1} g_i x = \rho(\zeta_i)$

Lemma 4.7. *The elements $A^{\alpha}_{\alpha'}$ of the matrix A from definition 4.5 are dependent on legal edge labelings α, α^*, only up to the homomorphisms $\rho, \rho' : \pi_1(\Sigma, *) \to G$ that α, α' induce.*

Proof. Remember that we are using a standard triangulation for the 3-manfiold. We claim that when we triangulate a manifold with a standard triangulation, once legal labelings on the 2-manifolds are placed, a single label, *f*, of an edge going between the 2-manifolds determines all other labels uniquely. To see this, look at the triangulation of the prism given in figure 2.2 and the added tetrahedron in figure 2.3. **Since there are no internal vertices**, and the boundary labels are already fixed, once one edge between the 2-manifolds is labeled, the triangle condition uniquely determines the labels on all other edges between the 2-manfiolds.

Let Σ, Σ^* be 2-manifolds with boundary triangulated by $U = U^*$. Let β be an arbitrary labeling of $\partial U, \partial U^*$ and let α, α' be legal labelings of U, U^*, respectively, which restrict to β on $\partial U, \partial U^*$. Any pair of vertices, (v, v^*), on Σ, Σ^* respectively, which represent the same points $\tilde{v}, \tilde{v}^*$ on the manifolds, will have the edge between them labeled by the same element in G (since they all are the exact same path on the 3-manifold). Fix a point $*$ on Σ and Σ^*, and label the edge between them *f*. By the above paragraph f together with α, α^* uniquely determine the labels of all other edges. The only constraints that still need to be checked are those corresponding to closed loops on the manifolds.

These remaining constraints will be of the form $f\rho(\sigma) = \rho'(\sigma)f$ where σ is a loop on Σ, Σ^*. Figure 4.5 demonstrates the reason why all constraints have the above form. (The loop σ is marked by thick lines, all inter-boundary edges are dashed, v, v^* denote vertices corresponding to the same point on the boundary).

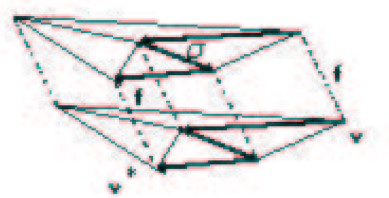

FIGURE 4.5. Follow σ along inner rectangles to check that $f\rho(\sigma) = \rho'(\sigma)f$ must hold

As long as the induced homomorphisms remain the same, the number of labels f satisfying the constraint $f\rho(\sigma)f^{-1} = \rho'(\sigma)$ is unchanged. Thus the lemma is proved. □

We now want to map homomorphisms in HOM_g to legal edge labelings which restrict to β. This can be done in a way very similar to the way it was done in section 3.2, with one exception. Given a homomorphism $\rho \in HOM_g$, consider a loop ζ_i, around the boundary component L_i as described above. On the one hand the element that the homomorphism assigns it, $\rho(\zeta_i)$, is known (and must be an element in the conjugacy class of g_i); on the other hand the label on the boundary is already fixed. This means that we no longer have a freedom of degree $|G|$ for labeling a path from * to a vertex on the boundary. Rather, it must hold that the value, x, that we assign this path satisfies the following equation: $x^{-1}gx = \rho(\zeta_i)$ [see figure 4.4]. This means that we have exactly $\frac{|G|}{|[g_i]|}$ and not $|G|$ choices for this vertex. Note that edges of the boundary are already labeld by β, and therefore once a label x is chosen for a path from * to one vertex on L_i, labels for all paths from * to other vertices on L_i are induced.

Let Σ be a 2-manifold with n boundary components triangulated by T. All in all the freedom we have in constructing a legal edge labeling is then[5]:

$$|G|^{|T^0|-1-n} \cdot \prod_{i=1}^{n} \frac{|G|}{|[g_i]|}$$

Since any legal labeling that restricts to β on the boundary can be constructed in this manner (using the homomorphism it induces), we have just proved that:

Theorem 4.8. $\frac{\#(\alpha_{\Delta=e}:Edges(T)\to G \ s.t. \ \alpha|_{\partial T}=\beta)}{|G|^{|T^0|-1}} = \frac{|Hom:\pi_1(\Sigma)\to G|}{\prod_{i=1}^{n}\frac{|G|}{|[g_i]|}}$

[5]Here we were assuming that the vertex, *, is not on any boundary component. This assumption does not compromise the generality of the proof, since changing the base point would result with an isomorphic fundamental group, and the existence of an internal point can always be guaranteed by applying the second local triangulation move [see figure 4.6 and the discussion that follows].

The following lemma is an immediate result of the way in which we mapped homomorphisms to legal labelings.

Lemma 4.9. *$Z'_\beta(\Sigma)$ depends on the boundary labeling β only up to $[g_1], \dots, [g_n]$, the conjugacy classes of the products around the boundary components.*

Proof. We have already shown that the matrix A of definition 4.5 is dependent on legal edge labelings only up to the induced homomorphisms. Thus it will suffice to show that given two boundary labelings $\beta, \tilde{\beta}$ such that $\beta(L_i) = g_i \in [\tilde{g_i}] = [\tilde{\beta}(L_i)]$, $1 \leq i \leq n$, and a boundary labeling α which restricts to β on the boundary, one can construct a legal labeling $\tilde{\alpha}$ which restricts to $\tilde{\beta}$ on the boundary, such that α and $\tilde{\alpha}$ induce the same homomorphism. Let ρ be the homomorphism that α induces. We construct a legal labeling $\tilde{\alpha}$ as follows:

For each inner vertex, v, we assign the path from * to v the same element that α assigned it. For vertices on the boundary components we label the corresponding paths with an element $\tilde{x_i}$ such that $\tilde{x_i}^{-1}\tilde{g_i}\tilde{x} = x_i^{-1} g_i x_i = \rho(\zeta_i)$. This can always be done since $g_i \in [\tilde{g_i}]$. Obviously α and $\tilde{\alpha}$ induce the same homomorphism. □

Although we have shown that the matrix A of definition 4.5 is dependent on legal edge labelings only up to the induced homomorphisms, we have not yet shown that the definition is independent of triangulations. We prove this in two steps.

Lemma 4.10. *$Z'_\beta(\Sigma)$ is independent of inner triangulations*

Proof. To prove the above we use local triangulation moves to shift from one triangulation to another, as we did in section 4.1, only here we are dealing with 2-manifolds. Two simple moves allow shifting from any 2-dimensional triangulation to another. These are shown in figure 4.6.

FIGURE 4.6. Local triangulation moves (2-dimensional)

Since all finite vector space of dimension d over the complex field are isomorphic, all we need to show is that the dimension of $\ker(A - Id)$ does not change when a local move is applied.

The first move does not add any vertices or edges, and since the final labeling must be legal, the only edge that changes is uniquely determined, and there is a 1 to 1 correspondence between the labelings. The resulting matrices are of the same size and represent exactly the same transformation in isomorphic spaces.

The second move adds (removes) one vertex and one degree of freedom for legal labelings (any one of the three added edges uniquely determines the other two). This means that for any labeling α considered before adding the new vertex there will be $|G|$ corresponding labelings to consider after the vertex is added. We denote such a new labeling by (α, g) where g is the element chosen to label a fixed new edge. Recall that we are using a standard triangulation for the cylinder. Once legal labelings are placed on Σ, Σ^*, setting the label of a single edge between the 2-manifolds, immediately determines the labels of all other inner edges uniquely. The above gives us that the $|G|$ new labelings (α, g) will have the same matrix values as the original labeling α had, except that since an extra vertex was added, their values will be divided by $|G|$. We therefore have that the following map induces an isomorphism between the two spaces $\ker(A - Id)$ of the corresponding matrices: $e_\alpha \to \sum_{x \in G} e_{(\alpha,x)}$. We have thus shown that $Z'_\beta(\Sigma)$ is not dependent on the inner triangulation of Σ. □

Lemma 4.11. *$Z'_\beta(\Sigma)$ is independent of boundary triangulations (as long as the products of labels around the boundary components remain in the same conjugacy classes)*

Proof. Since the boundary of a 2-manifold is of dimension 1, changing the boundary triangulation of Σ, means that the **number of edges** used to triangulate the boundary changes. Also we must change the labeling β accordingly. Figure 4.7 presents the simple trick we use to accommodate for this change, keeping in mind that we also must adhere to the restrictions placed by the labeling β, we simply glue on cylinders with matching labels.

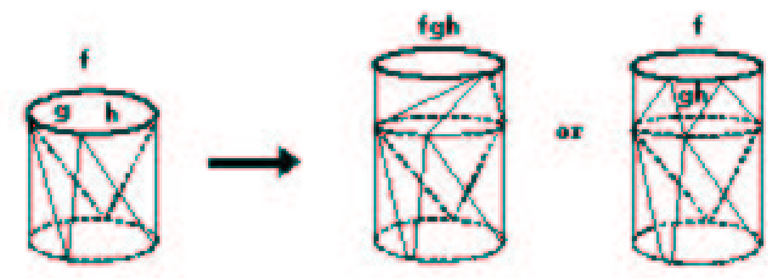

FIGURE 4.7. changing the boundary triangulation of a 2-manifold by gluing on cylinders

We have already shown that changing the inner triangulation does not affect the result, and that the result depends on β only up to the conjugacy classes of the products around the boundary components, which do not change by gluing on the cylinders with matching labels. Thus we are free to change the boundary triangulation as long as the boundary labeling remains in the same conjugacy class, and $Z'_\beta(\Sigma)$ remains unchanged. □

Using the two previous lemmas we have that $Z'_\beta(\Sigma)$ **is in fact independent of any specific triangulation**, and is thus compatible with the rest of Z'.

Notice that using the gluing method presented in figure 4.7, we may assume that each boundary component is triangulated by a single edge. In fact from now on whenever we use a triangulation of a 2-manifold with boundary we will be implicitly assuming that its boundary is labeled by a single edge.

4.4.2. *The dimension of* $Z'_\beta(\Sigma)$. Since in definition 4.5 the matrix A, takes the place of F_{U_0,U_0} from the inverse limit construction [see section 2.2], it ought to be an idempotent, if it is to be part of our theory. This we set out to prove.

Let Σ be a 2-manifold with boundary triangulated by U, let β be an arbitrary labeling of ∂U, and let $S = |HOM_g|$ be the number of homomorphisms compatible with the boundary labels g_i given by β. Define the matrix $A \in M_{S\times S}(\mathbb{Q})$, columns and rows indexed by all homomorphisms in HOM_g as follows:

Definition 4.12. $\hat{A}^{\rho}_{\rho'} = \frac{1}{|G|} \cdot \#\big(\ f \in G \ \ s.t.\ f\rho(\sigma) = \rho'(\sigma)f\ ,\ \forall \sigma \in \pi_1(\Sigma)\big)$

where T is a standard triangulation of $\Sigma \times I$.

Since the matrix A of definition 4.5 was shown to be dependent on edge labelings α, α', only up to the induced homomorphisms $\rho, \rho' : \pi_1(\Sigma) \to G$ in HOM_g, and each such homomorphism has by theorem 4.8 exactly $\frac{|G|^{|T^0|-1}}{\prod_{i=1}^{n}|[g_i]|}$ corresponding edge labelings, we have that $Image(\hat{A}) \cong Image(A)$.

$\hat{A}$ represents an action of G by conjugacy on homomorphisms ρ: $\hat{A}(e_\rho) = |G|^{-1}\sum_{f\in G} e_{f\rho f^{-1}}$. Furthermore, $(\hat{A})$ must be an idempotent. It follows, by the way that A is related to $(\hat{A})$, that A must also be an idempotent, which is what we set out to prove.

Moreover since $\hat{A}$ represents a group action its image has dimension equal to the number of orbits. Here the orbits are simply conjugacy classes of homomorphisms in HOM_g. Two homomorphisms $\rho, \rho' : \pi_1(\Sigma) \to G$ are considered *conjugate* if $\rho(\sigma) = g\rho'(\sigma)g^{-1}$ for some fixed element $g \in G$, and for all closed loops $\sigma \in \pi_1(\Sigma)$. We denote this conjugacy relation by $\sim$. Also since A is an idempotent we have that $\dim(\ker(A - Id)) = \dim(Im(A))$, and thus have

Theorem 4.13. $\dim\Big(Z'_\beta(\Sigma)\Big) = \big|\big(HOM_g(\pi_1(\Sigma) \to G)\big)/\sim\big|$

Recall that the definition for $Z'_\beta(\Sigma)$ restricts to our original inverse limit construction definition for $Z'(\Sigma)$ when $\partial\Sigma = \emptyset$ (in which case we have $HOM_g = HOM : \pi_1(\Sigma) \to G$).

As a corollary for a 2-manifold Σ **without** boundary,

$$\dim\left(Z'(\Sigma)\right) = |HOM(\pi_1(\Sigma) \to G)/\sim|$$

This is of course in accordance with the two examples of section 4.3.

4.5. **Disconnected manifolds.** Up until now, as mentioned in the beginning of this paper, we were always implicitly assuming that the manifolds were connected, recognizing that all definitions could be extended to disconnected manifolds using Axiom 3 given in section 2.1. We now present an explicit formula for disconnected manifolds.

Let Σ be a 2-manifold with connected components Σ_c, $1 \le c \le n$. Using edge labelings, the definition given for connected manifolds holds also for disconnected ones. It should be noted, however, that a legal labeling on any connected component is entirely independent of a legal labeling on a different connected component, so n

inter-boundary edge labels have to be chosen in order for the rest to be uniquely determined. This implies that when we move to homomorphisms, we will deal with $HOM_g\left(\prod_{c=1}^{n}\pi_1(\Sigma_c)\to G\right)$ and therefore must choose a base point $*_i$ for each connected component. Using the same notation as in definition 4.12 the resulting matrix is:

Definition 4.14. $\hat{A}^{\rho}_{\rho'} = |G|^{-n}\cdot\#\{(f_1,\ldots,f_n)\in G^n \text{ s.t. } f_c\rho(\sigma)=\rho'(\sigma)f_c,\ \forall c,\sigma\in \pi_1(\Sigma_c)\}$

This matrix represents the conjugacy group action of G^n on HOM_g. The dimension of the image of this matrix is simply

$$\left|HOM_g\left(\prod_{c=1}^{n}\pi_1(\Sigma_c)\to G\right)\big/\approx\right|$$

where $\approx$ denotes the conjugacy relation resulting from the action of G^n on HOM_g. Namely $\rho\approx\rho'$ if and only if $\exists\{f_c\in G\}_c$ s.t. $\rho\,|_{\Sigma_c}= f_c\rho'\,|_{\Sigma_c} f_c^{-1}$.

We can rephrase the dimension using the action of G as

$$dimZ'(\Sigma)=\prod_{c=1}^{n}\left|\left(HOM_g(\pi_1(\Sigma_c)\to G)\big/\sim\right|\right.$$

where $\sim$ is simply the conjugacy relation of homomorphisms.

5. Various formulations

We have successfully defined a (2+1)-d TFT Z', and given an explicit formula for the dimension of the vector spaces it assigns 2-manifolds. Many TFTs (such as Chern–Simons–Witten theory, see [4]) possess what is called a *Verlinde algebra*, and one can derive a (1+1)-d TFT from them in which the invariant associated to a 2-manifold is the dimension of the vector space associated in the (2+1)-d TFT (so called spaces of "conformal blacks" in CSW theory). If we try to do something analogous here, we would want to derive from Z' a (1+1)-d TFT Z_0 which would be of the following form:

- $Z_0(S^1)$ is a vector space indexed by conjugacy classes of G;
- for a closed 2-manifold Σ : $Z_0(\Sigma)=\dim(Z'(\Sigma))$;
- for a 2-manifold with boundary: $Z_0(\Sigma)_g=\dim(Z'_g(\Sigma))$, where on the left side of the equation the subscript g denotes the basis element as indexed by conjugacy classes of G, and on the right it denotes the boundary labeling in terms of conjugacy classes. So $Z_0(\Sigma)\in Z_0(S^1)^{\otimes|\partial\Sigma|}$

Unfortunately the above formulation does not give a (1+1)-d TFT, since it does not obey the gluing property. Several attempts to normalize the above formulation have failed. Nevertheless much data has been accumulated regarding the behavior of Z' when applied to the trinion, which will constitute a basic component of Z_0. We present the results in hope that they will assist in future attempts to discern the underlying algebra.

5.1. An explicit Formulation of $Z'_{k,l,m}(trinion)$ as an orbit problem.

Let $T_{k,l,m}$ be a trinion whose boundary components are labeled by *k,l,m* as in figure 5.1.

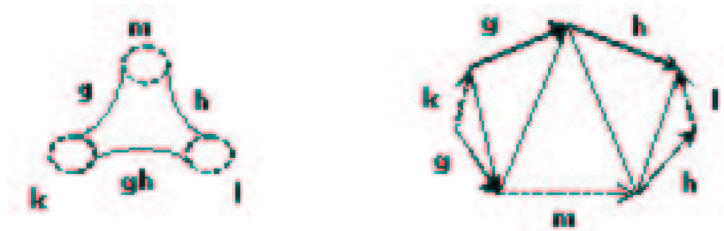

FIGURE 5.1. **Left:** The trinion glued according to the triangulation on the right. **Right:** A partial labeling of the triangulated trinion which uniquely determines all unlabeled edges.

Once the boundary labels *k,l,m* are fixed all legal labelings of $T_{k,l,m}$ are given by the following set.

Definition 5.1. $S_{k,l,m} = \{(g,h) \mid g,h \in G \; and \; h^{-1}l^{-1}hgkg^{-1} = m\} \subseteq G \times G$

Fix *k,l,m* so that $S_{k,l,m} \neq \emptyset$. Let $S = |S_{k,l,m}|$. Definition 4.5 gives us that $Z'_{k,l,m}(T) = \ker(A - Id)$ where $A \in M_{S\times S}(\mathbb{Q})$ is a matrix whose columns and rows are indexed by elements of $S_{k,l,m}$ and whose matrix elements are defined as follows.

Definition 5.2. $A^{g,h}_{g',h'} = \frac{|[k]||[l]||[m]|}{|G|^3} \cdot \left|\{\; f \in G \mid [f,k] \quad and \quad [l, hgfg'^{-1}h'^{-1}]\}\right|$

where $[x,y]$ denotes that x commutes with y, and all constraints on legal labelings of $T \times I$ were translated into the constraints given on *f*.

Now define two new sets.

Definition 5.3. $\tilde{S}_{k,l,m} = \{hg \mid (g,h) \in S_{k,l,m}\}$

Definition 5.4. $U_{k,l,m} = \{\alpha \in G \mid l^{-1}\alpha k \alpha^{-1} \in [m]\}$

Obviously $\tilde{S}_{k,l,m} \subseteq U_{k,l,m}$, we will now show that these two sets are actually equal.

Proof. Let $\alpha \in U_{k,l,m}$ such that $l^{-1}\alpha k\alpha^{-1} = t^{-1}mt$ for some $t \in G$. Denote $g = t\alpha$, $h = t^{-1}$, then $l^{-1}hgkg^{-1}h^{-1} = t^{-1}mt = hmh^{-1}$ so $hg = \alpha \in \tilde{S}_{k,l,m}$, and the sets are therefore equal. □

For a given $\alpha \in U_{k,l,m}$, how many pairs $(g,h) \in S_{k,l,m}$ are there such that $hg = \alpha \in \tilde{S}_{k,l,m} = U_{k,l,m}$? Once h is determined, g is determined as well (since $hg = \alpha$), so the question is reduced to how many elements h, are there such that $h^{-1}(l^{-1}\alpha k\alpha^{-1})h = m$? The answer to this is simply $\frac{|G|}{|[m]|}$. The matrix A depends on the pairs (g,h) only up to hg, therefore the following matrix $\hat{A}$, columns and rows indexed by elements of $U_{k,l,m}$, has a range with the same dimension[6] as the matrix A:

Definition 5.5. $\hat{A}^{\alpha}_{\alpha'} = \frac{|[k]||[l]||[m]||G|}{|G|^3|[m]|} \cdot \left|\{\; f \in G \mid [f,k] \quad and \quad [l, \alpha f \alpha'^{-1}]\}\right|$
$= \frac{|[k]||[l]|}{|G|^2} \cdot \left|\{\; f \in G \mid [f,k] \quad and \quad [l, \alpha f \alpha'^{-1}]\}\right|$

Next we reformulate the definition of $U_{k,l,m}$ by conjugating by α, to get $U_{k,l,m} = \{\alpha \in G \mid \alpha^{-1}l^{-1}\alpha k \in [m]\}$. From this formulation it is obvious that $U_{k,l,m}$ depends on α only up to $\alpha^{-1}l^{-1}\alpha$, which encourages us to define a new set.

Definition 5.6. $\tilde{U}_{k,l,m} = \{\beta \in [l] \mid \beta^{-1}k \in [m]\}$

[6] We have already shown that in the general case A is an idempotent and therefore have that $\dim(\ker(A - Id)) = \dim(Image(A))$

For each $\beta \in \tilde{U}_{k,l,m}$ there are exactly $\frac{|G|}{|[l]|}$ elements $\alpha \in U_{k,l,m}$ such that $\alpha^{-1}l\alpha = \beta$. Now $[l, \alpha f \alpha'^{-1}] \Leftrightarrow \alpha^{-1}l\alpha f = f\alpha'^{-1}l\alpha' \Leftrightarrow \beta f = f\beta'$, where $\beta = \alpha^{-1}l\alpha$ and $\beta' = \alpha'^{-1}l\alpha'$.

We can therefore replace the matrix $\hat{A}$ without changing the dimension of the image, by the matrix $\bar{A}$, columns and rows indexed by elements of $\tilde{U}_{k,l,m}$:

Definition 5.7. $\bar{A}_{\beta'}^{\beta} = \frac{|[k]||[l]||G|}{|G|^2|[l]|} \cdot \left|\{\ f \in G \mid [f,k] \quad and \quad \beta f = f\beta'\}\right|$
$= \frac{|[k]|}{|G|} \cdot \left|\{\ f \in G \mid [f,k] \quad and \quad \beta = f\beta' f^{-1}\}\right|$

$\bar{A}$ represents a group action, namely the group $G_k = \{g \in G \mid [g,k]\}$ acts on the set $\tilde{U}_{k,l,m}$ by conjugation. This action is well-defined, since if $\beta \in \tilde{U}_{k,l,m}$, $f \in G_k$ we have

$$(f\beta f^{-1})^{-1}k = f\beta^{-1}f^{-1}k \underset{[f,k]}{=} f\beta^{-1}kf^{-1} \underset{\beta \in \tilde{U}_{k,l,m}}{\in} [m]$$

so that $f\beta f^{-1} \in \tilde{U}_{k,l,m}$.

We have thus proved the following.

Theorem 5.8. $Z'_{k,l,m}(T) = Image(A) = \left\langle |G_k|^{-1} \sum_{f \,\in\, G_k} e_{f^{-1}\beta f} \right\rangle$ *and*
$\dim(Z'_{k,l,m}(T)) = \#\big($*orbits of conjugacy action of* G_k *on* $\tilde{U}_{k,l,m}\big)$

A more symmetric formulation of the above action can be given as an action of $G_k \times G_l \times G_m$ on the set $S_{k,l,m}$.

(1) $g^{-1}h^{-1}lhg = \beta \in \tilde{U}_{k,l,m}$ can be conjugated by G_k using the action $f : (g,h) \longmapsto (gf^{-1},h)$, where $f \in G_k$
(2) Changing $hg = \alpha \in U_{k,l,m}$ while $\beta \in \tilde{U}_{k,l,m}$ remains fixed, is done using the action $q : (g,h) \longmapsto (g, q^{-1}h)$, where $q \in G_l$.
(3) Finally, changing $(g,h) \in S_{k,l,m}$ while $\alpha \in U_{k,l,m}$ remains fixed, is done using the action $r : (g,h) \longmapsto (r^{-1}g, rh)$, where $r \in G_m$

We can thus formulate the original action as the action of $G_k \times G_l \times G_m$ on the set $S_{k,l,m}$ where $(f,q,r).(g,h) = (r^{-1}gf^{-1}, q^{-1}hr)$. That this action is well-defined follows from

$$m = (h^{-1}l^{-1}h)(gkg^{-1}) \longmapsto (r^{-1}h^{-1}ql^{-1}q^{-1}hr)(r^{-1}gf^{-1}kfg^{-1}r) = m$$

since $q \in G_l$ and $f \in G_k$ and $r \in G_m$. Again $\dim Z'_{k,l,m}(T)$ will be the number of orbits of this action of $G_k \times G_l \times G_m$ on $S_{k,l,m}$.

Definition 5.9. $M_{k,l,m} = \left\{(\tilde{k},\tilde{l}) \in [k] \times [l] \;\middle|\; \tilde{l}^{-1}\tilde{k} = m\right\} \big/ \sim$ where $\sim$ denotes the diagonal conjugation of $\tilde{k}, \tilde{l}$ by elements of G_m.

Theorem 5.10. $\dim Z'_{k,l,m}(T) = |M_{k,l,m}|$

Proof. Define $\psi : S_{k,l,m} \to M_{k,l,m}$ by $\psi(g,h) = [(gkg^{-1}, h^{-1}lh)]$ where [.] denotes an equivalence class in $[k] \times [l]$ given by diagonal conjugation by elements of G_m. That this map is surjective follows immediately from the definitions of $S_{k,l,m}$ and $M_{k,l,m}$.

Now suppose $(g_1, h_1), (g_{2,}, h_2) \in S_{k,l,m}$ lie in the same orbit under the action of $G_k \times G_l \times G_m$ described above. Then $g_2 = r^{-1} g_1 f^{-1}$, $h_2 = q^{-1} h_1 r$ for some $q \in G_l$, $f \in G_k$ and $r \in G_m$. Thus $g_2 k g_2^{-1} = r^{-1} g_1 f^{-1} k f g_1 r = r^{-1} g_1 k g_1 r$ and $h_2^{-1} l h_2 = r h_1^{-1} q l q^{-1} h_1 r = r h_1^{-1} l h_1 r$ and we have $\psi(g_{1,}, h_1) = \psi(g_{2,}, h_2)$. Hence ψ induces a well-defined map $S_{k,l,m}/ \sim \longrightarrow M_{k,l,m}$.

Conversely we show that if $(g_{1,}, h_1) \not\sim (g_{2,}, h_2)$ in $S_{k,l,m}$ then $\psi(g_{1,}, h_1) \neq \psi(g_{2,}, h_2)$; or equivalently that if $\psi(g_{1,}, h_1) = \psi(g_{2,}, h_2)$ then $(g_1, h_1) \sim (g_2, h_2)$.

We have that $r^{-1} h_1^{-1} l h_1 r = h_2^{-1} l h_2$ and $r^{-1} g_1 k g_1^{-1} r = g_2 k g_2^{-1}$. Denote $q = h_1 r h_2^{-1}$ and $f = g_2^{-1} r^{-1} g_1$. Then $lq = ql$ and $kf = fk$, while $g_2 = r^{-1} g_1 f^{-1}$, $h_2 = q^{-1} h_1 r$. Thus $(f, q, r).(g_1, h_1) = (g_2, h_2)$ as required. □

This result should be compared with the general formula from the previous section for dim $Z'_g(\Sigma)$. In the case of the trinion, $\pi_1(T) = F_2$, the free group on two generators.

5.2. **Formulating Z' for two trinions glued as an orbit problem.** We now turn to the problem of two trinions glued together along one boundary component. Figure 5.2 illustrates the resulting manifold along with a partially labeled triangulation of it (which nevertheless determines all unlabeled edges).

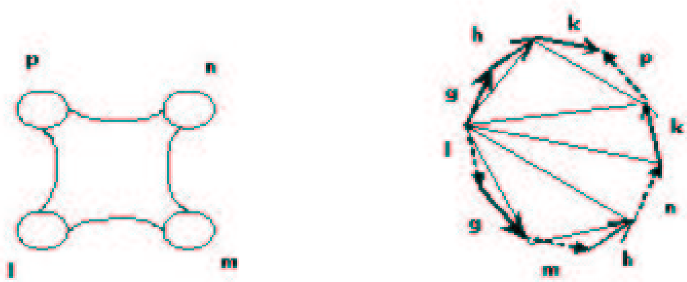

FIGURE 5.2. **Left:** Two trinions glued along one boundary component. **Right:** A triangulation of the two trinions partially labeled.

The resulting manifold is homeomorphic to S^2 with 4 spheres removed, hence we denote it S^2_{-4}. Let $S^2_{-4_{l,m,n,p}}$ be a manifold homeomorphic to S^2 with 4 discs removed, whose boundaries are labeled by $l, m, n, p \in G$ as in figure 5.2.

All legal labelings on $S^2_{-4_{l,m,n,p}}$ are given by the following set:

Definition 5.11. $S_{l,m,n,p} = \{(g,h,k) \in G^3 \mid khgl^{-1}g^{-1}m^{-1}h^{-1}n^{-1}k^{-1} = p\}$.

Throughout the following formulation fix $l, m, n, p \in G$ so that $S_{l,m,n,p} \neq \emptyset$ (or rather there exists at least one legal labeling on the manifold). Let $S = |S_{l,m,n,p}|$.

Definition 4.5 gives us that $Z'_{l,m,n,p}(S^2_{-4}) = \ker(A - Id)$ where $A \in M_{S\times S}(\mathbb{Q})$ is a matrix, columns and rows given by elements of $S_{l,m,n,p}$ and:

Definition 5.12. $A^{g,h,k}_{g',h',k'} = \frac{|[l]||[m]||[n]||[p]|}{|G|^4} \cdot \left| \left\{ \begin{array}{l} f \in G_l \ s.t. \\ {[m, gfg'^{-1}] \wedge [n, hgfg'^{-1}h'^{-1}]} \end{array} \right\} \right|$

Note that the constraints on the 2-manifolds are given by the set $S_{l,m,n,p}$, whereas all inter-boundary constraints are formulated in terms of a single label f (connecting the 1-dimensional segments labeled by l).

Recall that $Z'_{l,m,n,p}(S^2_{-4})$ is dependent on l, m, n, p only up to $[l], [m], [n], [p]$. Also each triple $(g, h, k) \in S_{l,m,n,p}$ depends on k only by the conjugation. Define the following set:

Definition 5.13. $T_{l,m,n,p} = \{(g, h) \mid g, h \in G,\ hgl^{-1}g^{-1}m^{-1}h^{-1}n^{-1} \in [p]\}$

Each pair $(g, h) \in T_{l,m,n,p}$ is given by exactly $\frac{|G|}{|[p]|}$ triples $(g, h, k) \in S_{l,m,n,p}$. We can therefore reformulate the matrix A as the matrix $\hat{A}$, which has an image with the same dimension, rows and columns of which are indexed by pairs in $(g, h) \in T_{l,m,n,p}$ and:

Definition 5.14. $\hat{A}^{g,h}_{g',h'} = \frac{|[l]||[m]||[n]|}{|G|^3} \cdot \left| \left\{ \begin{array}{l} f \in G_l \ s.t. \\ [m, gfg'^{-1}] \wedge [n, hgfg'^{-1}h'^{-1}] \end{array} \right\} \right|$

Note that

- $[m, gfg'^{-1}] \Leftrightarrow f^{-1}(g^{-1}mg)f = (g'^{-1}mg')$
- $[n, hgfg'^{-1}h'^{-1}] \Leftrightarrow f^{-1}(g^{-1}h^{-1}nhg)f = (g'^{-1}h'^{-1}nh'g')$
- $hgl^{-1}g^{-1}m^{-1}h^{-1}n^{-1} \in [p] \Leftrightarrow l^{-1}g^{-1}m^{-1}h^{-1}n^{-1}hg \in [p] \Leftrightarrow$ $l^{-1}(g^{-1}mg)^{-1}g^{-1}h^{-1}n^{-1}hg \in [p]$

Let $\alpha = g^{-1}mg$ and $\beta = g^{-1}h^{-1}n^{-1}hg$, it is evident from the relations above that $T_{l,m,n,p}$ is dependent on (g, h) only up to (α, β).

Definition 5.15. $U_{l,m,n,p} = \{(\alpha, \beta) \mid \alpha \in [m],\ \beta \in [n],\ l^{-1}\alpha^{-1}\beta^{-1} \in [p]\}$

Each pair $(\alpha, \beta) \in U_{l,m,n,p}$ is given by exactly $\frac{|G|}{|[m]|} \cdot \frac{|G|}{|[n]|}$ pairs $(g, h) \in T_{l,m,n,p}$. We can therefore reformulate the matrix $\hat{A}$ as the matrix $\bar{A}$, which has an image with the same dimension, rows and columns of which are indexed by pairs in $(\alpha, \beta) \in U_{l,m,n,p}$ and:

Definition 5.16. $\bar{A}^{\alpha,\beta}_{\alpha',\beta'} = \frac{|[l]|}{|G|} \cdot \left| \left\{ \begin{array}{l} f \in G \ s.t. \\ [f, l] \wedge f^{-1}\alpha f = \alpha' \wedge f^{-1}\beta f = \beta' \end{array} \right\} \right|$

$\bar{A}$ represents the group action of $G_l = \{g \in G \mid [g, l]\}$ on the set $U_{l,m,n,p}$ by diagonal conjugation, $f(\alpha, \beta) = (f^{-1}\alpha f, f^{-1}\beta f)$. This action is well-defined since if $(\alpha, \beta) \in U_{l,m,n,p}$ then $f\alpha f^{-1} \in [m]$ and $f\beta f^{-1} \in [n]$ while

$$l^{-1}(f\alpha f^{-1})^{-1}(f\beta f^{-1})^{-1} = l^{-1}f\alpha^{-1}\beta^{-1}f^{-1} \underset{[f,l]}{=} fl^{-1}\alpha^{-1}\beta^{-1}f^{-1} \in [p]$$

so that $(f\alpha f^{-1},\ f\beta f^{-1}) \in U_{l,m,n,p}$ as required.

We have thus proved that

Theorem 5.17. *$Z'_{l,m,n,p}(S^2_{-4})) = Image(A) = \left\langle |G_l|^{-1} \sum_{f \in G_l} e_{f^{-1}\alpha f, f^{-1}\beta f} \right\rangle$ and $\dim(Z'_{l,m,n,p}(S^2_{-4})) = \#($ orbits of diagonal conjugacy action of G_l on $U_{l,m,n,p})$.*

A more symmetric formulation of the above action can be given as an action of $G_l \times G_m \times G_n \times G_p$ on the set $S_{l,m,n,p}$.

(1) $(g^{-1}mg,\ g^{-1}h^{-1}nhg)(\alpha, \beta) \in U_{l,m,n,p}$ can be conjugated by G_l using the action $f : (g, h, k) \longrightarrow (gf^{-1}, h, k)$, where $f \in G_l$.

(2) Changing $g, h \in T_{k,l,m}$ while $\alpha, \beta \in U_{l,m,n,p}$ and k remain fixed, is done using the action $(q, r) : (g, h, k) \longrightarrow (q^{-1}g,\ r^{-1}hq,\ kr)$, where $q \in G_m$, $r \in G_n$.

(3) Finally, changing k while $g, h \in T_{k,l,m}$ remain fixed, is done using the action $s : (g, h, k) \longrightarrow (g, h, sk)$, where $s \in G_p$.

We can thus reformulate the original action as the action of $G_l \times G_m \times G_n \times G_p$ on the set $S_{l,m,n,p}$ where $(f, q, r, s).(g, h, k) = (q^{-1}gf^{-1},\ r^{-1}hq,\ skr)$. This action is well-defined since

$$\begin{aligned} p &= (khgl^{-1}g^{-1}m^{-1}h^{-1}n^{-1}k^{-1}) \\ &\longrightarrow (skr)(r^{-1}hq)(q^{-1}gf^{-1})l^{-1}(fg^{-1}q)m^{-1}(q^{-1}h^{-1}r)n^{-1}(r^{-1}k^{-1}s^{-1}) \\ &= s(khgl^{-1}g^{-1}m^{-1}h^{-1}n^{-1}k^{-1})s^{-1} = sps^{-1} = p \text{ by } [s,p]. \end{aligned}$$

Thus $\dim Z'_{k,l,m,p}(S^2_{-4})$ is the number of orbits of this action of $G_l \times G_m \times G_n \times G_p$ on $S_{l,m,n,p}$. As in the previous subsection, this can be reformulated again, to express $\dim Z'_{k,l,m,p}(S^2_{-4})$ as the number of ways of writing p in the form

$$p = l_1^{-1}m_1^{-1}n_1^{-1}$$

where $l_1 \in [l]$, $m_1 \in [m]$ and $n_1 \in [n]$, up to diagonal conjugation by elements commuting with p. Again, this formula can be compared with the general formula for $\dim Z'_g(\Sigma)$, noting that $\pi_1(S^2_{-4}) = F_3$, the free group on three generators.

5.3. **Some calculated data.** Some data regarding the trinion and the sphere with four disks removed was generated by a computer program. The group used was S_3. We adopt the following notation:

$$a = [e],\ b = [(12)],\ c = [(123)].$$

Since results are only dependent on the conjugacy class of the label, we make use of the above notation both for the conjugacy class itself and for an arbitrary representative of the class. The following table gives the results for the trinion. Note that the symmetry of the manifold tells us that the ordering of the boundary labelings is of no significance[7].

Boundary labeling	Dimension of $Z'_{k,l,m}(T)$
c,c,c	1
a,a,a	1
a,c,c	1
a,b,b	1
b,b,c	1
Any other labeling	0

The next table gives the results for the sphere with four discs removed, the group used is still S_3.

[7]In general it does not hold that $\dim(Z'_{k,l,m}(T)) = 1$. For example, the results for S_4 , which we will not present here, show this.

Boundary labeling	Dimension of $Z'_{l,m,n,p}(S^2_{-4})$
a,a,c,c	1
c,c,c,c	3
c,a,c,c	1
a,a,a,a	1
a,a,b,b	1
b,b,a,c	1
b,c,b,c	2
b,b,b,b	5
Any other labeling	0

We can see where the gluing formula for Z_0 fails in this case, by comparing Z_0 evaluated on S^2_{-4} with all boundaries labelled by c, against the result of gluing two trinions [see figure 5.3].

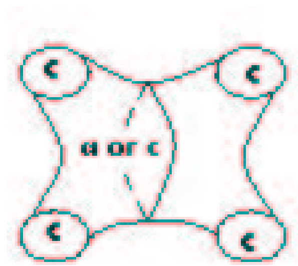

FIGURE 5.3. An example showing the gluing property does not hold.

The label x on the interior glued boundary may be either a or c in order to be compatible with the other lables (both c) on each trinion. Thus one has $Z_0(S^2_{-4})_{c,c,c,c} = 3$ while the result of gluing two trinions would yield

$$Z_0(T)_{c,c,a}Z_0(T)_{a,c,c} + Z_0(T)_{c,c,c}Z_0(T)_{c,c,c} = 1 + 1 = 2.$$

6. Conclusion

We defined a (2+1)-d TFT, Z for triangulated manifolds starting from a finite group G in terms of 'legal labelings' of edges by elements of G. The value Z assigns to closed 3-manifolds M is

$$Z(M) = |HOM(\pi_1(M) \to G)|/|G|$$

We then used the inverse limit construction to produce a new (2+1)-d TFT Z', independent of triangulations. The associated (closed) 3-manifold invariant is the same as for Z. A formula was found for the dimension of the vector space assigned by Z' to a closed 2-manifold, namely

$$\dim(Z'(\Sigma)) = \left|\big(HOM(\pi_1(M) \to G)\big)/\sim\right|$$

where the equivalence $\sim$ is defined by the action of G on homomorphisms by conjugation. An extension of Z' to 2-manifolds with boundary was presented and shown to be consistent with the original definition. In particular, it was seen that the labels required on boundary components are conjugacy classes in G. For a 2-manifold Σ with boundary so labeled by conjugacy classes, g_i on the i-th boundary component, we showed that the dimension of the vector space assigned by Z' is given by

$$\dim(Z'_g(\Sigma)) = \left|\left(HOM_g(\pi_1(M) \to G)\right)/\sim\right|$$

where now HOM_g is the subset of all homomorphisms consisting of those which map the loops defined by the i-th component of $\partial\Sigma$ to an element of the conjugacy class g_i.

We had hoped to use this explicit formula to define a (1+1)-d TFT Z_0 and investigate the corresponding algebra (analogous to the Verlinde algebra of conformal blocks in Chern–Simons–Witten theory) but were unsuccessful since the required gluing formula was not obeyed. However much data was gathered on the way.

In fact it seems that our theory Z' bears some resemblance to Turaev-Viro theory [5] (which is the 'square' of CSW theory in some suitable sense) rather than to CSW theory itself. This explains why a Verlinde-type algebra was not present in our example, and also points to two interesting open questions which we hope to investigate in the future. Firstly, to try to define a 'square-root' of the theory discussed in this paper, which should be more like CSW theory and thus possess a Verlinde-type algebra. Secondly, it would be interesting to identify general categorical conditions on a theory which permit the existence of Verlinde-type algebras.

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