Lyon 13 May 2006

Dear Serge,

I want to detail the last construction of my previous notes:

The following hypothesis are not the most general we can hope but they are convenient. So, let us consider a Lorentzian manifold (M, g) geodesically complet. Let us denote by $\mathcal{G}$ the space of parametrized geodesics, in our case $\mathcal{G} \sim TM$, which is equiped with the inherited symplectic form ω. Let $\mathcal{T}$ the space of geodesical trajectories, as I said previously:

$$\begin{cases} \mathcal{T} = \mathcal{G} / \mathrm{Aff}^{+}(\mathbb{R}) \\ \gamma' \sim \gamma \text{ iff } \gamma'(t) = \gamma(at+b) \quad a > 0 \quad b \in \mathbb{R} \end{cases}$$

Let us consider: $\omega_\gamma : T_\gamma \mathcal{G} \to T^*_\gamma \mathcal{G}$ the symplectic form is viewed as a linear map from the tangent to the cotangent.

Then,

$$\omega^{-1}_\gamma : T^*_\gamma \mathcal{G} \to T_\gamma \mathcal{G}$$

So ω^{-1} is a section of some bundle over $\mathcal{G}$ with fiber $L^*(T^*_\gamma \mathcal{G}; T_\gamma \mathcal{G})$, L^* means non-degenerate. Let us denote $\mathcal{L}^*$ this bundle over $\mathcal{G}$.

Now, let us consider the projectivization of $\mathcal{L}^*$, that is

$$\begin{tikzcd} \mathcal{L}^* \arrow[r] \arrow[dr] & \mathbb{P}\mathcal{L}^* = \mathcal{L}^*/]0,\infty[\arrow[d] \\ & \mathcal{G} \end{tikzcd}$$

the section $\omega^{-1} : \mathcal{G} \to \mathcal{L}^*$ give a section $[\omega^{-1}] : \mathcal{G} \to \mathbb{P}\mathcal{L}^*$ with $\omega^{-1} \sim a\,\omega^{-1}$ $a \in]0,\infty[$

the section $[\omega^{-1}]$ factorizes (since $\omega^{-1} \sim (a,b)^* \omega^{-1}$ $(a,b) \in \mathrm{Aff}^+(\mathbb{R})$)

$$\begin{tikzcd} \mathbb{P}\mathcal{L}^*(\mathcal{G}) \arrow[r] \arrow[d] & \mathbb{P}\mathcal{L}^*(\mathcal{T}) \arrow[d] \\ \mathcal{G} \arrow[u, bend left, "{[\omega^{-1}]}"] \arrow[r, "\pi"'] & \mathcal{T} \arrow[u, bend right, "\beta"'] \end{tikzcd}$$

The section β is what we remember of the symplectic structure ω on $\mathcal{G}$.

Now we can consider β as an operator on the sub spaces of the cotangent space of $\mathcal{T}$:

let $V \subset T^*_\tau \mathcal{T}$ $\qquad \left.\begin{array}{l} \beta_\tau(V) = \pi\left(\omega^{-1}_\gamma(\tilde{V})\right) \\ \text{with } \tilde{V} = \pi^* V,\ \pi(\gamma) = \tau \end{array}\right\}$

this is well defined.

Now we can consider the distribution of subspaces:

$$\tau \longmapsto \beta\left(T^*_\tau \mathcal{T}\right) = F_\tau$$

The fact is that :

a) if τ is not a light geodesic $\beta(T^*_\tau \mathcal{G}) = T_\tau \mathcal{G}$

b) if τ is a light geodesic $\beta(T^*_\tau \mathcal{G}) \subset T_\tau$ light Rays and $= \mathrm{Orth}(T_\gamma \mathrm{Aff}^+(\mathbb{R})(\gamma)) / \mathrm{Aff}^+(\mathbb{R})$ in the contact distribution on the space of light rays of M.

What I want to emphasize now is that physicists do the following :

a) they choose an hypersurface $\Sigma_c = \{(x,v) \in TM \mid v.v = c\}$

b) they make the symplectic reduction $\Sigma_c / \ker(\omega|\Sigma_c)$

c) they get a symplectic manifold of dimension $2n-2$ even for $c = 0$

d) with the above construction the space of light rays has dimension $2n-3$ and is contact.

e) what I guess is that the space of light rays obtained by physicists is the symplectization of the contact manifold

f) the supplementary parameter obtained by physicists is the "color" of the "photon"

That's all for now Serge, All the best Yours Patrick