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Dear Serge,

Excuse me for the time I took to reply to your request. I have had a lot of stuff to fix before leaving for the US. And I'm leaving next week.

So let's come back to the question of geodesics in a pseudo riemannian manifold. My motivation was to understand how to glue together all the geodesics in only one structure or description. Because, for example, if we take the space $\mathbb{R}^4$ with the pseudo metric $dx^2 + dy^2 + dz^2 - dt^2$ (the Minkowsky space), we know that the geodesics are just lines and the spaces of lines of $\mathbb{R}^4$ is a manifold diffeomorphic to TS^3. And this manifold is splitted in three parts: the time-like geodesics, the space-like geodesics which are 2 open domains of TS^3 and the light geodesics which is a codimension 1 submanifold. We know that on the time/space geodesics we can put a symplectic structure, but what about the light geodesics? And if there exist a structure on the light geodesics how does it glue with the symplectic structures?

This was my first question.

Now I'll try to describe my answer :

1) first of all I had to understand the difference between "parametrized geodesics" and "unparametrized geodesics":

* "parametrized geodesics" are curves in M (M is the pseudo-riemannian manifold), that is maps $\gamma : t \mapsto x$ such that

$$\frac{\hat{d} v}{dt} = 0 \quad \text{with} \quad v = \frac{dx}{dt}$$ and $\hat{d}$ is the covariant derivative with respect to the Levi-Civita connexion.

So parametrized geodesics live in the space of functions from (a priori) $\mathbb{R}$ to M. They are solutions of a 2nd order ordinary differential equation, then the set of parametrized geodesics is a 2n-dimensional space. Let us assume that the metric is complete and then this space, I denote by $\mathcal{G}_{par}$ is diffeomorphic to TM.

In the example of $M = \mathbb{R}^4$ $(+++-)$:

- $\mathcal{G}_{par} = \{ \gamma_{(x,v)} = [t \mapsto x + tv] \mid x \in \mathbb{R}^4, v \in \mathbb{R}^4 - \{0\} \}$

* "Unparametrized geodesic".

Let me open a parenthesis now: Of course in a pseudo-riemannian manifold we do not want to consider parametrized geodesics since the parametrization by the time is included in the image of the curve $\text{im}(\gamma_{(x,v)})$. It is why we have to clean up the parametrization. So we define a new set:

$$\begin{cases} \mathcal{G}_{unpar} = \mathcal{G}_{par} / \sim \\ \text{with:} \\ \gamma \sim \gamma' \text{ iff } \quad \text{im}(\gamma) = \text{im}(\gamma') \end{cases}$$

If we are lucky the space $\mathcal{G}_{unpar}$ of unparametrized geodesics is a nice manifold. This is the case for $(\mathbb{R}^4, +++-)$ for example. And in this example $\mathcal{G}_{unpar}$ is just the space of lines as we told before.

So for $(\mathbb{R}^4, +++-)$: $\begin{cases} \mathcal{G}_{par} \simeq \mathbb{R}^4 \times \mathbb{R}^4 - \{0\} & (\dim 8) \\ \mathcal{G}_{unpar} \simeq TS^3 & (\dim 6) \end{cases}$

But we can get the quotient $\mathcal{G}_{unpar}$ by the quotient of the action of the affine group of $\mathbb{R}$:

$$\text{Aff}^+(\mathbb{R}) = \{ (a, b) \in]0, \infty[\times \mathbb{R} \}$$

[Note: I consider oriented geodesics, parametrized are of course oriented by the "speed".]

Now, two geodesics γ and γ' have the same "trajectory" iff:

$$\gamma'(t) = \gamma(at + b)$$

So I have an action of $\mathrm{Aff}^+(\mathbb{R})$ on $\mathcal{G}_{par}$:

$$(a,b)(\gamma) = [t \mapsto \gamma(at+b)]$$

And

$$\mathcal{G}^+_{unpar} = \mathcal{G}_{par} / \mathrm{Aff}(\mathbb{R})$$

this is completely true if the geodesic flow is complete, it's more complicate if it is not complete.

So now, I make a difference between the space of "parametrized geodesics" and "geodesic trajectories" moreover oriented. I denote also $\mathcal{G}^+_{traj} = \mathcal{G}^+_{unpar}$.

2) What about the structure?

a) We know that there exist a symplectic structure on $\mathcal{G}_{par}$ since $\mathcal{G}_{par}$ is the set of solutions of a variational problem :

$$\delta \int_{t_0}^{t_1} \frac{1}{2} v \cdot v \, dt \quad \text{with } v = \frac{dx}{dt} \text{ and } \cdot \text{ is}$$

the pseudo scalar product.

thus we have the Cartan form

$$\begin{cases} \varpi = v \cdot dx - \frac{1}{2} v \cdot v \, dt & \text{on } TM \times \mathbb{R} = Y \\ y \in Y \quad y = (x, v, t) & \dim Y = 2n+1 \end{cases}$$

and $\mathcal{G}_{par}$ = Characteristics of $d\varpi$

To be complete means just that a solution of the distribution $y \mapsto \ker d\varpi$ projects surjectively on $\mathbb{R}$ by $y \mapsto t$ and then $\mathcal{G}_{par} = Y/\ker d\varpi = TM \times \mathbb{R}/\ker d\varpi$ and then $\mathcal{G}_{par} \simeq TM \times \{0\}$ for example. The symplectic form ω on $\mathcal{G}_{par}$ defined by $\pi^* \omega = d\varpi$ where $\pi : TM \times \mathbb{R} \to \mathcal{G}_{par}$ is the projection. Hence

$$(\mathcal{G}_{par}, \omega) \simeq (TM, d\varpi | TM \times \{0\})$$

b) now, what about the symplectic form and the action of the group $Aff(\mathbb{R})$!

let us look at the $(\mathbb{R}^4, +++-)$ example :

$$\left.\begin{array}{l} \gamma' = [t \mapsto \gamma(at+b)] = [t \mapsto x + (at+b)v] \\ \qquad\qquad\qquad\qquad\qquad \text{with } \gamma(t) = x + tv \end{array}\right\}$$

$\gamma' = [t \mapsto x + bv + t(av)$

Hence

$$(a,b)(\gamma_{(x,v)}) = \gamma_{(x+bv,\, av)}$$

Therefore on TM the action of $Aff(\mathbb{R})$ is described by

$$(a,b)(X,V) = (X + bV, aV)$$

[On a more general manifold $X + bV$ represents the action of the geodesic flow.]

Hence
$$(a,b)^* \varpi_{(X,V)} = a v \cdot (dx + b\, dv)$$
$$= a\, v \cdot dx + \tfrac{1}{2} a b\, d(v \cdot v)$$

$$\Rightarrow \quad \boxed{(a,b)^* \omega = a\, \omega}$$

R 1. The symplectic form is multiplied and not preserved by the group $\mathrm{Aff}^+(\mathbb{R})$.

What about the orbits of $\mathrm{Aff}^+(\mathbb{R})$? Let $\mathcal{O} = \{ (a,b)(X,V) \mid X, V \in TM \}$ an orbit of the group. We get:

$$\omega \big|_{\mathcal{O}} = (v \cdot v) \times \omega_{\mathrm{aff}} \quad \text{with } \omega_{\mathrm{aff}} = da \wedge db$$

R2. The Orbits of $\mathrm{Aff}^+(\mathbb{R})$ are:
- symplectic if $v \cdot v \neq 0$
- isotropic if $v \cdot v = 0$

R3. Note that the action of $\mathrm{Aff}^+(\mathbb{R})$ is always free since $v \neq 0$.

R4. We get a distinguished distribution of $2n-2$ spaces:

$$\gamma \longmapsto \mathrm{Orth}\left(T_\gamma [\mathrm{Aff}^+(\mathbb{R})(\gamma)] \right) \; (= F_\gamma)$$

F_γ is the orthogonal with respect to the symplectic form ω to the orbit of γ under the action of $Aff^+(\mathbb{R})$

we have: a) if $v \cdot v \neq 0$ F_γ is symplectic and

$$T_\gamma \mathcal{G}_{par} = T_\gamma \mathcal{O}_\gamma \oplus F_\gamma$$

$$\mathcal{O}_\gamma = Aff^+(\mathbb{R})(\gamma)$$

b) if $v \cdot v = 0$ F_γ is ω-isotropic and

$$T_\gamma \mathcal{O}_\gamma \subset F_\gamma$$

c) in each case $\dim F_\gamma = 2n-2$

R 5. let $\pi : \mathcal{G}_{par} \to \mathcal{G}_{traj}$

a) if γ is space/time geodesic: $D\pi_\gamma(F_\gamma) = T_{\pi(\gamma)} \mathcal{G}_{traj}$

b) if γ is light: $D\pi_\gamma(F_\gamma) \subsetneq T_{\pi(\gamma)} \mathcal{G}_{traj}$ and

$\dim D\pi_\gamma(F_\gamma) = \dim F_\gamma - 2$ (since the action of $Aff^+(\mathbb{R})$ is free) $= 2n-4$.

c) let $\mathcal{G}_*^{light}$ denote the space of light rays we have:

$$Orth(T_\gamma \mathcal{G}^{light}_{par}) \subset T_\gamma \mathcal{O}_\gamma \quad , \gamma \in \mathcal{G}^{light}_{par}$$

Hence $F_\gamma \subset T_\gamma \mathcal{G}^{light}_{par}$

$$\Rightarrow \underbrace{D\pi_\gamma(F_\gamma)}_{\dim = 2n-4} \subset \underbrace{D\pi_\gamma(T_\gamma \mathcal{G}^{light}_{par})}_{\dim = 2n-3} \quad , \gamma \in \mathcal{G}^{light}_{par}$$

d) the distribution $\tau \mapsto \mathcal{F}_\tau$ $\begin{cases} \mathcal{F}_\tau = D\Pi_\gamma F_\gamma \\ \tau = \Pi_\gamma \end{cases}$

is :

- if τ is space/time : the tangent space
- if τ is light ; a codimension 1 subspace of the tangent space to $\mathcal{G}^{light}_{traj}$ And it is a contact distribution.

R6. On the space/time geodesic subspace the distribution $\gamma \mapsto F_\gamma$ is a connexion distribution with respect to the action of $Aff^+(\mathbb{R})$

R7. In all the case, since the fibration $\mathcal{G}_{par} \to \mathcal{G}_{traj}$ is a principal fibration of group $Aff^+(\mathbb{R})$

and $Aff^+(\mathbb{R})$ is contractible this fibration is trivial, that is $\mathcal{G}_{par} \simeq \mathcal{G}_{traj} \times Aff^+(\mathbb{R})$

3) The Souvenir of the symplectic structure :

Since not everywhere $T_\gamma \mathcal{G}_{par} = T_\gamma O_\gamma \oplus F_\gamma$ we have no chance to find a symplectic structure on $\mathcal{G}_{traj}$ but we can try to remember the most of the symplectic structure on $\mathcal{G}_{par}$?

let consider ω^{-1} the 2 contravariant form defined by

$$\omega^{-1}(\alpha, \beta) = \beta(\omega^{-1}(\alpha)) \quad \text{with} \quad \omega : T_\gamma \mathcal{G}_{par} \to T_\gamma^* \mathcal{G}_{par}$$

$\alpha, \beta \in T_\gamma^* \mathcal{G}_{par}$, $\omega^{-1}(\alpha) \in T_\gamma \mathcal{G}_{par}$

this structure is not invariant by $Aff^+(\mathbb{R})$ but it's conformal class is :

$$[\omega^{-1}] = \text{class with respect to } \Lambda \sim c\Lambda$$

$c \neq 0$, $\Lambda \in T_\gamma^{\wedge 2}(\mathcal{G}_{par})$

Now $[\omega^{-1}]$ can by pushed forward to $\mathcal{G}_{traj}$ in a 2 contravariant class of some tensor, let :

$$\Lambda = \pi_* [\omega^{-1}]$$

Λ is a section of some fiber bundle. The kernel of Λ is well defined :

- on the subspace of time/space geodesics Λ has no kernel
- on the light rays the kernel of Λ is exactly by the contact distribution above.

Note that on the subspace of time/space rays the section Λ can be realized as a cosymplectic class which gives back the symplectic structure.

Last remark: there are a lot of questions still unresolved associated to these construction. From time to time I'am looking at them...