SYMPLECTIC MECHANICS AND DIFFEOLOGY

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ref. http://math.huji.ac.il/~piz/documents/ShD-lect-SMD.pdf

In this lecture, we give a short survey on Symplectic Mechanics, the basic foundations, and a way to extend these constructions to Diffeology.

There are a few different approaches to symplectic geometry in mechanics. At the beginning there is three papers from Joseph-Louis Lagrange in 1808, 1809 and 1810 [Lag08, Lag09, Lag10] :

1) *Sur la théorie des variations des éléments des planètes et en particulier des variations des grands axes de leurs orbites* (1808).

2) *Sur la théorie générale de la variation des constantes arbitraires dans tous les problèmes de la mécanique*(1809).

3) *Second mémoire sur la théorie générale de la variation des constantes arbitraires dans tous les problèmes de la mécanique* (1810).

In these papers, Lagrange sets the first elements of what we can call "symplectic calculus". The question was the stability of the great axes of the planets, and Lagrange brought a simplification in the approximation computations of this time, in particular by Laplace and Poisson. I will not detail here the details of his work but I refer (for now) to the paper I wrote, but in French, on the subject [PIZ98] :

   Les Origines du Calcul Symplectique chez Lagrange

That said, I will emphasize — in this overview on symplectic mechanics — the following points relative to the global structure of the spaces of solutions of dynamical (symplectic) systems.

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(1) The presymplectic/symplectic framework.
(2) The symmetries of dynamical systems.
(3) Isolated mechanical/dynamical systems, the Galilean and Poincaré groups.
(4) The moment map associated to a group of symmetries.
(5) The conservation of the moment map (Noether-Souriau theorem).
(6) The Souriau cocycle and the barycentric decomposition.
(7) The elementary particles/systems and the classic spin.
(8) The Geometric Quantization Program, the prequantization.

These few points summarizes, I believe, the main progress on the global structure of dynamical systems made in the 20th century.

Of course, symplectic mechanics does not reduces to these chapters, other constructions like the behavior of hamiltonian vector fields, the geometric optics, reflexion, diffraction, caustics... The structure of the group of symplectomorphisms, etc. All these subjects are a part of modern symplectic mechanics. A whole year of lectures could hardly be enough to cover all the applications of symplectic mechanics.

The short Approach To Symplectic Mechanics

The following presentation of the syplectic structure on the space of solutions of a dynamical (second order differential equation) system is due to Elie Cartan [Car22], followed by some authors, Galissot [Gal52], and especially Jean-Marie Souriau in his book "Structure des Systèmes Dynamique" [Sou70].

<u>1. Construction</u> [To Drive Out the Denominators] Let us recall that a classical dynamical system in Galilean Mechanics, or Newtonian Mechanics, is described by an ordinary second order differential equation. For example, for a material point:

$$m\frac{d^2x}{dt^2} = \mathrm{F}(x, v, t) \quad \text{where} \quad v = \frac{dx}{dt}.$$

The unknown here is the path $[t \mapsto x]$, where t is a real number defined on some interval and $x \in \mathbf{R}^3$.

Then we transform this second order system into a first order differential equation system:

$$m\frac{dv}{dt} = F(x, v, t) \quad \text{and} \quad v = \frac{dx}{dt}.$$

Then we reinterpret the Newton equations by "driving out the denominators":

$$mdv = F(x, v, t)dt \quad \text{and} \quad dx = vdt.$$

We need here to explain this writing. Consider the subset

$$Y \subset \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}$$

where F is defined. We call it the space of initial conditions, or following Souriau: the "evolution space". Let $y = (x, v, t)$ a point in Y, let us denote by dy a tangent vector to Y, at the point y, that is a vector of $\mathbf{R}^7$. We can imagine dy being the shortcut the derivative of a path $s \mapsto y$

$$dy = \frac{dy}{ds} \in T_y Y \quad \text{with} \quad dy = \begin{pmatrix} dx \\ dv \\ dt \end{pmatrix}.$$

Then, the equations of motion writes

$$mdv - F(x, v, t)dt = 0 \quad \text{and} \quad dx - vdt = 0.$$

At this point we use a trick, and considering an other tangent vector

$$\delta y = \begin{pmatrix} \delta x \\ \delta v \\ \delta t \end{pmatrix} \in T_y Y,$$

we define

$$\omega(dy, \delta y) = \langle mdv - Fdt, \delta x - v\delta t \rangle - \langle m\delta v - F\delta t, dx - vdt \rangle.$$

Let us make some comment on the notation: first of all, F denotes at the same time the function F and the value of F at the point $y = (x, v, t)$. Second of all, the 2-form ω should rigorously be indexed by the point y where it is taken, but the vectors dy and δy contains already this information, so it is unnecessary to be redundant.

One can check now that:

(1) ω is a 2-form on Y.

(2) the vector $dy \simeq dy/ds$ satisfies the *equations of motion* if and only if it belongs to the kernel of ω.

The kernel of ω is defined by

$$dy \in \ker(\omega) \quad \Leftrightarrow \quad \omega(dy, \delta y) = 0 \quad \forall \delta y.$$

Therefore, the *integral curves* of the kernel distribution

$$y \mapsto \ker(\omega)$$

are the solutions of the Newton equations.

Proposition 1. *The space $\mathcal{M}$ of integral curves of the kernel distribution is a manifold, that can be, for some kind of force F, non Hausdorff. It is called the* space of motions.

I would like to emphasize the fact the a solution of the differential equation, in this approach, is an integral curve of the distribution $y \mapsto \ker(\omega)$, that is, a subset of Y, the graph of some curve $t \mapsto (x, v, t)$. So the space of motions is indeed a set of subspaces of Y, and this set of subspaces is equipped with a structure of manifold.

And beware to not confuse the motion with the trajectory of the motion. The hourly law of the trajectory is a part of the motion. For example, a circular motion has a circle for trajectory but its motion is a line, precisely an helix.

Proposition 2. *The restrictions $(x, v) \mapsto \mathrm{class}_t(x, v)$, for all $t \in \mathbf{R}$, made a canonical atlas of the manifold $\mathcal{M}$.*

Note that these canonical charts class_t are called *Darboux charts* because the symplectic form takes the canonical expression,

$$\omega_t = m \sum_{i=1}^{3} dx_i \wedge dv_i.$$

Proposition 3. *If the force F is the gradient of some* potential *φ, actually $F = -\mathrm{grad}(\varphi)$, then the 2-form ω is closed*

$$d\omega = 0,$$

and descends on the quotient space $\mathcal{M}$ into a symplectic 2-form.

In that case, the 2-form ω is the exterior derivative of the so-called *Cartan form*

$$\lambda(\delta y) = m\langle v, \delta x \rangle - h\, \delta t \quad \text{with} \quad h = \frac{1}{2}m\|v\|^2 + \varphi.$$

Definition. *We recall that a symplectic form on a manifold is a <u>non degenerate closed</u> 2-form.*

Presymplectic and Symplectic Manifolds

2. Definition [Presymplectic Form] Let M be a manifold, a closed 2-form ω on M is said to be *presymplectic* if its kernel has a constant dimension on M [Sou70]. The vector distribution

$$y \mapsto \ker(\omega)$$

is called the *characteristic distribution*. Thanks to the Frobenious theorem that states that for every differential form ω, the characteristic distribution

$$y \mapsto \ker(d\omega) \cap \ker(\omega)$$

is integrable, since $\ker(d\omega)_y = T_y M$, the characteristic distribution of a closed form $y \mapsto \ker(\omega)$ is integrable.

The leaves of the integral submanifolds of the characteristic distribution are called *characteristics* of the distribution, and the resulting foliation is called the *characteristic foliation*.

The characteristics of the presymplectic form ω are the connected submanifolds $F \subset M$ such that at each point $y \in F$,

$$T_y F = \ker(\omega).$$

3. Definition [Symplectic Dynamical Systems] In *symplectic mechanics*, a dynamical system is defined as a presymplectic manifold (M, ω). By analogy with the case of a particle, we call *motions* of the system the characteristics of the presymplectic form. The *space of motions*, which plays an important rôle in mechanics, is then the set of all characteristics.

Note. There is no reason that, in general, the space of motions inherits a manifold structure by quotient, and it does not. That can be an obstacle in ordinary differential geometry, but not from the general diffeology point of view, where any case can be dealt with. The space of characteristic of a presymplectic manifold can always be equipped with the quotient diffeology, for which the

usual diffeological tools continue to work. We denote this space by

$$\mathcal{M} = M/\ker(\omega).$$

4. Definition [Symplectic Form] A closed 2-form ω on a manifold M is said to be symplectic if it is *non degenerate*, that is, if its kernel is reduced to $\{0\}$

$$\omega \text{ symplectic} \quad \Leftrightarrow \quad d\omega = 0 \text{ and } \ker(\omega) = 0.$$

A symplectic manifold is then a presymplectic manifold with a trivial characteristic foliation:

$$F_y = \{y\}.$$

Proposition. Consider a presymplectic manifold (M, ω), if the quotient space $\mathcal{M} = M/\ker(\omega)$ is a manifold, it inherits a symplectic form for which ω is the pullback.

That is why symplectic geometry is important in physics:

(1) Conservative dynamical systems are identified as presymplectic dynamical systems, for which the characteristic leaves are the solutions, or motions, of the system.
(2) The space of solutions of a conservative dynamical system, if it is a manifold itself, inhertis a symplectic structure.

The word "conservative" will be explained later.

5. Example [The geodesics trajectories on the Sphere] Consider the sphere S^2. Let US^2 be the unitary tangent bundle:

$$US^2 = \{(x, u) \in \mathbf{R}^3 \times \mathbf{R}^3 \mid \|x\| = \|u\| = 1 \text{ and } \langle x, u \rangle = 0\}.$$

Define on US^2 the 1-form

$$\lambda_y(\delta y) = \langle u, \delta x \rangle, \quad \text{with} \quad \begin{cases} y & = & (x, u) \in US^2 \\ \delta y & = & (\delta x, \delta u) \in T_y US^2. \end{cases}$$

let $\omega = d\lambda$, precisely

$$\omega_y(\delta y, \delta' y) = \langle \delta u, \delta' x \rangle - \langle \delta' u, \delta x \rangle.$$

One can check that ω is presymplectic and its charactetistic distribution is defined by

$$\frac{dy}{ds} \in \ker \omega \quad \Leftrightarrow \quad \frac{dx}{ds} = \alpha u \quad \text{and} \quad \frac{du}{ds} = -\alpha x,$$

for all $\alpha \in \mathbf{R}$. The solutions of this sytem are great circles described by the *hourly law*

$$x(s) = e^{sj(\ell)}x_0 \quad \text{and} \quad u(s) = e^{sj(\ell)}u_0 \quad \text{with} \quad \ell = x_0 \wedge u_0$$

Note that $\ell = x \wedge u$ is constant on the solutions, it is called the *kinetic momentum*. It is a particular case of the general moment map theory we talk later in the following.

Thus, the map

$$y = (x, u) \mapsto \ell = x \wedge u$$

from Y to S^2 realizes the quotient space

$$Y/\ker(\omega) \simeq S^2.$$

And the symplectic structure of the space of motions is equal, up to some constant, to the standard *area form*

$$\mathrm{Surf}_\ell(\delta\ell, \delta'\ell) = \langle \ell, \delta\ell \wedge \delta'\ell \rangle.$$

This example is a special case of the general construction of the set of *geodesic trajectories*, which are equipped with a symplectic structure as soon as it is a manifold.

6. **Example** [The geodesics trajectories on the torus] The geodesic trajectories of the 2-torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ are the projections of the affine lines in $\mathbf{R}^2$. Consider the projection which associates with each geodesic trajectory Δ, its direction $u \in S^1$.[1] This is a surjection. We can write it

$$\pi\colon \mathfrak{G}_{\mathrm{traj}} \to S^1 \quad \text{with} \quad \pi(\Delta) = u.$$

Now, the fiber over the direction $u \in S^1$ are all the projections on the 2-torus T^2 of the affine lines in $\mathbf{R}^2$, parallel to the unique line of direction u, and passing through the origin. Depending on the orientation, these lines cut the axis oy or the axis ox according the action of $\mathbf{Z} \oplus \mathbf{Z}$

$$(n, m)\colon y \mapsto y + n + \tau m,$$

with

$$u = (\alpha, \beta), \text{ with } \alpha^2 + \beta^2 = 1, \text{ and } \tau = \frac{\beta}{\alpha}.$$

Thus,

$$\pi^{-1}(u) \simeq T_\tau,$$

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[1]We consider first the oriented geodesic trajectories.

where T_τ is the torus of slope τ. If τ is rational then T_τ is diffeomorphic to the torus S^1, otherwise it is not a manifold but a diffeological space called the *irrational torus* (of slope τ) [PDPI83].

So, in the case of the torus the space of geodesic trajectories is not a manifold but at least a diffeological space. We proved, somewhere else, that the canonical symplectic form on the space of geodesic curves (which is always a manifold) descends, in some sense, to the quotient $\mathfrak{G}_{\text{traj}}$, as a differential closed 2-form, according to the definition in diffeology.

That example shows in particular the necessity to enlarge the category of manifolds if we want to describe a bigger class of dynamical systems than usually.

7. Theorem [Darboux Theorem] An important theorem due to the mathematician Jean Gaston Darboux makes explicit the local nature of presymplectic and symplectic manifolds. Let (M, ω) be a presymplectic manifold of rank $2n$; the rank is defined by

$$\text{rank}(\omega) = \dim(M) - \dim(\ker(\omega)).$$

The rank of a 2-form is always even, let $\dim(M) = 2n + k$. Then: There exists an atlas $\mathcal{A}$ of charts such that, in any chart $F \in \mathcal{A}$, the form ω can be identified with the matrix, if $k > 0$:

$$\begin{pmatrix} 0_n & -\mathbf{1}_n & 0 \\ \mathbf{1}_n & 0_n & 0 \\ 0 & 0 & 0_k \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0_n & -\mathbf{1}_n \\ \mathbf{1}_n & 0_n \end{pmatrix}$$

in the symplectic case $k = 0$.

This is an important, even crucial, theorem with an enormous set of applications. In a few words: a symplectic structure is *flat*, always.

> There is no local invariants.
> All symplectic invariants are globals.

Symmetries And Moment Map

The symmetries of a dynamical system, and its consequences, are certainly at the heart of symplectic mechanics.

8. Definition [Symmetries of a System] Let (M, ω) be a dynamicalsystem, that is, a presymplectic manifold. We call a *symmetry* of the system any diffeomorphism $f \in \text{Diff}(M)$ that preserves the

presymplectic form ω. So, we define the largest group of symmetries of the system

$$\mathrm{Diff}(X, \omega) = \{f \in \mathrm{Diff}(X) \mid f^*(\omega) = \omega\}$$

When the form is symplectic, such a diffeomorphism is called a *symplectomorphism*.

<u>Proposition</u> *Except for trivial cases, the group* $\mathrm{Diff}(X, \omega)$ *is infinite dimensional.*

<u>Theorem</u> *On symplectic manifolds,* $\mathrm{Diff}(X, \omega)$ *is transitive on* M. *Actually* $\mathrm{Diff}(X, \omega)$ *is in that case n-transitive* [Boo69].

9. Definition [Galilean Mechanics] The history has produced three kind of Mechanics, each of them characterized by a group of transformations [PIZ18]. The *Aristotelean group*, the *Galilean group* and the *Poincaré group*. The Aristotelean mechanics has not been much developped.[2] The Galilean group is made of matrices

$$m = \begin{pmatrix} A & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad A \in \mathrm{SO}(3); b, c \in \mathbf{R}^3 \text{ and } e \in \mathbf{R}.$$

ws This group is associated with Galilean/Newtonian mechanics. It acts on $\mathbf{R}^3 \times \mathbf{R}$ by

$$\begin{pmatrix} A & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} Ar + bt + c \\ t + e \\ 1 \end{pmatrix}$$

Now we can anwser the Question:

What is a (Galilean) isolated system?

<u>Principle</u> *An isolated Galilean system is a presymplectic manifold* (M, ω) *with a symmetric action of the Galilean group.*

Later we will see that one adds the condition for this action to be Hamiltonian.

10. Definition [Relativity and the Poincaré Group] The group of Einstein Relativity is the Poincaré Group, that is, the group of affine transformations that preserve the quadratic form

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2,$$

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[2]I have some project on that question.

with $(x, y, z, t) \in \mathbf{R}^3 \times \mathbf{R}$.

$$g : X \mapsto LX + C,$$

where $X \in \mathbf{R}^3 \times \mathbf{R}$, $C \in \mathbf{R}^4$ and L is a linear automorphism of ds^2, a *Lorentz transformation*. Let $X = (r, t) \in \mathbf{R}^3 \times \mathbf{R}$, Every element of the Lorentz group decompose uniquely in the product of 2 matrices

$$L = \begin{pmatrix} \mathbf{1}_3 & \boldsymbol{\beta} \\ \bar{\boldsymbol{\beta}} & 1 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & a \end{pmatrix}, \quad \text{with } \bar{B}B + \boldsymbol{\beta}\bar{\boldsymbol{\beta}} = \mathbf{1}_3 \text{ and } a = \pm\frac{1}{\sqrt{1-\beta^2}}.$$

B is a 3×3 matrix, $\boldsymbol{\beta}$ is a vector in $\mathbf{R}^3$, the bar over B ou $\boldsymbol{\beta}$ denotes the transposition operator, and $\beta = \|\boldsymbol{\beta}\|$.

The Poincaré group is a 6-dimension Lie group with 4 connected-components. It is customary to reduce the Poincaré group to its identity component. Again,

Principle *An isolated Einstein raltivistis system is a presymplectic manifold* (M, ω) *with a symmetric action of the Poincaré group.*

Coadjoint Orbits

11. **Definition** [Coadjoint Action and Orbits] Given a Lie group G, there is a universal model of symplectic manifold. Consider a left invariant 1-form α and let $\mathcal{G}^*$ be the space of left-invariant 1-forms.

$$\mathcal{G}^* = \{\alpha \in \Omega^1(G) \mid L(g)^*(\alpha) = \alpha\},$$

where $L(g)\colon g' \mapsto gg'$ is the left-multiplication. By invariance every element of $\mathcal{G}^*$ is uniquely determined by its value at the origin, that makes $\mathcal{G}^*$ a vector space with dimension $\dim(G)$. This space is the space of momenta of the group G, it is also interpreted as the dual of the Lie algebra.

Now, the group G acts by conjugation on itself and by coadjoint action on $\mathcal{G}^*$:

$$\mathrm{Ad}(g)\colon g' \mapsto gg'g^{-1}$$

and for all $g \in G$:

$$\mathrm{Ad}_*(g)\colon \mathcal{G}^* \to \mathcal{G}^* \quad \text{with} \quad \mathrm{Ad}(g)_*(\alpha) = \mathrm{Ad}(g^{-1})^*(\alpha).$$

We define the *coadjoint orbit* of $\alpha \in \mathcal{G}^*$ by

$$\mathcal{O}_\alpha = \{\mathrm{Ad}_*(g)(\alpha) \mid g \in G\},$$

and we equip $\mathcal{O}_\alpha$ with the quotient diffeology

$$\mathcal{O}_\alpha \simeq G/\mathrm{St}(\alpha),$$

where $\mathrm{St}(\alpha)$ is the stabilizer of α

$$\mathrm{St}(\alpha) = \{g \in G \mid \mathrm{Ad}_*(g)(\alpha) = \alpha\}.$$

<u>Theorem</u> *The exterior derivative $d\alpha$ is a presymplect form on G that have the orbits of $\mathrm{St}(\alpha)$ as characteristics. It follows that $\mathcal{O}_\alpha$ has a natural structure of symplectic manifold. That is, there exists a symplectic form ω on $\mathcal{O}_\alpha$ such that*

$$\mathrm{class}^*(\omega) = d\alpha, \quad \textit{with} \quad \mathrm{class}\colon G \mapsto G/\mathrm{St}(\alpha).$$

<u>12. Definition</u> [Elementary Systems or Particles] The Kirillov-Kostant-Souriau theorem on classification of transitive symplectic manifolds leads to interpret coadjoint orbits of the groups of symmetries of the mechanics as *elementary particles*. Thus, in Galilean Mechanics, elementary particles will be coadjoint orbits of the Galilean group. In Einstein Relativity they will be coadjoint orbits of the Poincaré group.

Note that in GalileanMechanics, a large class of orbits have the type

$$\mathcal{O} = \mathbf{R}^6 \times S^2,$$

with the symplectic form

$$\omega_{m,s} = m\,\mathrm{Can} \oplus s\,\mathrm{Surf},$$

where Can is the canonical symplectic form on $\mathbf{R}^6$ and Surf the surface element on S^2. The S^2 part represents the *classical spin* component of the particle, s is the spin and m the mass of the elementary particle.

The Classic Moment Map

The impact of symmetries in symplectic mechanics are subsumed in a special map called the *Moment Map* and introduced by Souriau in [Sou70].

13. Proposition [The Classic Moment Map] Let (M, ω) be a presymplectic manifold. Let G be a Lie group with an action by symmetries on M, that is, a smooth morphism

$$G \ni g \mapsto g_M \in \text{Diff}(X, \omega).$$

We say that G is a group of symmetries. Let $\mathcal{G}$ be the Lie algebra, that is the space of smooth homomorphisms

$$\mathcal{G} = \text{Hom}^\infty(\mathbf{R}, G).$$

It is identified with the space of invariant vector fields

$$Z_G(g) = \frac{d}{dt}h(t) \cdot g\bigg|_{t=0}, \quad \text{with} \quad Z = Z_G(\mathbf{1}_G) = \frac{dh(t)}{dt}\bigg|_{t=0}$$

Every element of the Lie algebra Z defines a vector field Z_M on M, called the *infinitesimal action* of G on M

$$Z_M(x) = \frac{d}{dt}h(t)_M(x)\bigg|_{t=0},$$

Applying the Cartan formula

$$\mathcal{L}_\xi(\varepsilon) = [d\varepsilon](\xi) + d[\varepsilon(\xi)],$$

where $\mathcal{L}_\xi$ denotes the Lie derivative by ξ, for any differential form ε and any vector field ξ, to ω and Z_M, we get

$$d[\omega(Z_M)] = 0,$$

where $\omega(Z_M)$ is the contraction of ω by the vector field Z_M.

<u>Theorem-Definition</u> *We say that the action of G on M is* Hamiltonian *if $\omega(Z_M)$, which is closed, is exact. Then, there exists a map*

$$\mu\colon M \to \mathcal{G}^* \quad \text{such that} \quad \omega(Z_M) = d[x \mapsto mu(x) \cdot Z].$$

This map is called the moment map, *it is defined up to a constant, the manifold M is assumed to be connected.*

14. Proposition [The Noether-Souriau Theorem] Let (M, ω) be a presymplectic manifold. Let G a Lie group equipped with an Hamiltonian action on M. Then <u>the moment map $\mu\colon M \to \mathcal{G}^*$ is constant on the characteristics.</u> If the quotient $\mathcal{M} = M/\ker(\omega)$ is a manifold, then the moment map descends on $\mathcal{M}$. <u>The moment map μ represents the invariants associated with the symmetries represented by G.</u>

15. Proposition [The Souriau Cocycle] For a presymplectic manifold (M, ω) with a Hamiltonian action of G, with moment μ, we can check the variance of μ according to the action of G on M and the coadjoint action of G on $\mathcal{G}^*$. That gives the following theorem:

Theorem. [Souriau] *The lack of equivariance of the moment map* μ *is a cocycle of the group* G *with values in* $\mathcal{G}^*$, *twisted by the coadjoint action:*

$$\theta(g) = \mu(g_M(x)) - Ad^*(g)(\mu(x)).$$

The choice of another moment map change θ *by a coboundary. Thus, the class is well defined and depends only on the form* ω *and the action of* G.

$$class(\theta) \in H^1(G, \mathcal{G}^*).$$

I call this cocycle the *Souriau cocycle*.

We recall that these cocycles are defined as map $\theta \colon G \to \mathcal{G}^*$ such that

$$\theta(gg') = Ad_*(g)(\theta(g')) + \theta(g).$$

A coboundary is a map

$$\Delta(\varepsilon)(g) = Ad_*(g)(\varepsilon) - c,$$

for all $\varepsilon \in \mathcal{G}^*$.

One of the main consequence of the moment map and its lack of equivariance is the general theorem of barycentric decomposition.

16. Theorem [The Barycentric Decomposition Theorem] [Souriau] Let (M, ω) be an isolated dynamical system, that is, a symplectic manifold with an Hamiltonian action of the Galilean group. First of all let us recall that, considering its cohomology:

Claim *The cohomology of the Galilean group is* 1-*dimensional (Bargmann theorem). Therefore the Souria cocycle* θ *is equivalent to* $m\theta_0$, *where* θ_0 *is a chosen unit.*

The number m is interpreted as the total mass of the system in the unit θ_0.

Theorem *For an isolated dynamical system* (M, ω), *if the total mass of the system is not zero then the manifold* M *is a product*

$$M = \mathbf{R}^6 \times M_0,$$

where $\mathbf{R}^6$ represents the motions of the center of gravity and M_0 the motions around the center of gravity. The group $SO(3) \times \mathbf{R}$ continues to act on M_0.

17. Theorem [The Kostant-Kirillov-Souriau Theorem] The following theorem is due under different formulations to Kostant, Kirillov and Souriau. I give here the Souriau's formulation.

<u>Theorem</u> *Let* (M, ω) *be a symplectic manifold, transitive under a Hamiltonian action of a Lie group* G. *Then, the moment map* $\mu\colon M \to \mathcal{G}^*$ *is a covering onto a coadjoint orbit, maybe affine.*

Let θ be the Souriau cocycle of the system, it modifies the coadjoint action by adjunction to the standard linear coadjoint ation

$$\mathrm{Ad}^{\theta}_*(g)\colon \varepsilon \mapsto \mathrm{Ad}_*(g)(\varepsilon) + \theta(g).$$

This action is called an *affine coadjoint action*.

18. Exemple [The Cylinder and $SL(2, \mathbf{R})$] The group $SL(2, \mathbf{R})$ acts transitively on the cylinder $\mathbf{R}^2 - \{0\}$, preserving the symplectic form $\mathrm{Surf} = dx \wedge dy$. And the moment map is given by

$$\mu(z)(F_\sigma) = \frac{1}{2}\, \mathrm{Surf}(z, \sigma z) \times dt,$$

where $z = (x, y) \in \mathbf{R}^2 - \{0\}$, and

$$F_\sigma = \left[s \mapsto e^{s\sigma} \right]$$

is the one-parameter group defined by

$$\sigma \in \mathfrak{sl}(2, \mathbf{R}),$$

the Lie algebra of $SL(2, \mathbf{R})$, vector space of real 2×2 traceless matrices:

$$\mathfrak{sl}(2, \mathbf{R}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbf{R} \right\}$$

We have clearly $\mu(z) = \mu(-z)$.

19. Definition [Dynamic Variable and Poisson Bracket] Let (M, ω) a symplectic manifold. Every compactly supported real function $[x \mapsto u]$ defines a 1-parameter group of symplectomorphisms. The *symplectic gradient* is uniquely defined by the equation:

$$\omega\big(\mathrm{grad}_\omega(u), \delta x \big) = -\delta u \quad \text{with} \quad \delta u = \frac{\partial u}{\partial x}(\delta x).$$

Since u is compactly supported, $\mathrm{grad}_\omega(u)$ is compactly supported too, and then integrable. The 1-parameter group generated by $\mathrm{grad}_\omega(u)$ is denoted by:

$$s \mapsto e^{s\,\mathrm{grad}_\omega(u)}.$$

We have then,

$$\mathcal{L}_\xi(\omega) = 0, \quad \text{with} \quad \xi = \mathrm{grad}_\omega(u);$$

and

$$\left(e^{s\,\mathrm{grad}_\omega(u)}\right)^*(\omega) = \omega.$$

<u>Note 1.</u> The function $[x \mapsto u]$ is the moment map associated with the symmetry $\left(e^{s\,\mathrm{grad}_\omega(u)}\right)_{s \in \mathbf{R}}$.

<u>Note 2.</u> The *Poisson bracket* of two dynamical variables $[x \mapsto u]$ and $[x \mapsto v]$ is defined and denoted by:

$$\{u, v\} = \omega\big(\mathrm{grad}_\omega(u), \mathrm{grad}_\omega(v)\big).$$

It satisfies the identity

$$\mathrm{grad}_\omega\left(\{u, v\}\right) = \big[\mathrm{grad}_\omega(u), \mathrm{grad}_\omega(v)\big],$$

where the right is the bracket of vector fields. The Poisson bracket is a morphism of algebras.

Geometric Quantization

The program of geometric quantization try to answer the Dirac program of quantization. It consist, for any symplectic manifold (M, ω), to find a Hilbert space $\mathcal{H}$ and a morphism from the algebra of the real functions, for the Poisson bracket, the the albegra of unitary operators:

$$u \mapsto \hat{u} \quad \text{such that} \quad \widehat{\{u, v\}} = [\hat{u}, \hat{v}],$$

and

$$\hat{1} = \mathbf{1}_{\mathcal{H}},$$

where here 1 denotes the constant function $[x \mapsto 1]$.

The expected solution would have been

$$\mathcal{H} = \mathrm{L}^2(M) \quad \text{and} \quad \hat{u}(\varphi) = \mathrm{L}_\xi(\varphi)$$

with

$$\xi = \mathrm{grad}_\omega(u), \quad \text{and for all} \quad \varphi \in \mathcal{H}.$$

Unfortunately, in that case

$$\hat{1} = 0.$$

The first step in the direction of a solution to this problem is given by the *prequentization construction*.

20. Definition [Prequantization] Consider a symplectic manifold (M, ω). Let P_ω be its *group of periods*:

$$P_\omega = \left\{ \int_\sigma \omega \mid \sigma \in H_1(M, \mathbf{Z}) \right\}$$

Then,

<u>Theorem</u> [PIZ95] *There exists always a principal fiber bundle* Y *over* M, *with group* $T_\omega = \mathbf{R}/P_\omega$, *and equipped with a connexion form* λ *of curvature* ω. *That is,*

$$d\lambda = \pi^*(\omega),$$

where $\pi \colon Y \to M$.

Look in [PIZ13, 8.37] for the defintion of a connexion form on a principal bundle with group a diffeological torus $\mathbf{R}/P$, where P is a strict subgroup. Note that there maybe more than 1 such *integration bundles*, the classification is given in the paper cited above.

<u>Definition</u> (Sou70) *A symplectic manifold* (M, ω) *is* quantizable *if its group of periods is* $P_\omega = \hbar\mathbf{Z}$.

In this case the integration bundle Y is a manifold, a S^1-principal bundle. There exist a unique *fundamental vector field* τ on Y such that:

$$\lambda(\tau) = 1 \quad \text{and} \quad d\lambda(\tau) = 0.$$

Consider a dynamical variable $x \mapsto u$, lift it by π, $[y \mapsto u]$. Then, define the quantized lifting $\widehat{\mathrm{grad}}_\omega(u)$ of $\mathrm{grad}_\omega(u)$ on Y by

$$\widehat{\mathrm{grad}}_\omega(u) = u \times \tau + \eta_u,$$

where η_u is defined by

$$d\lambda(\eta_u) = 0 \quad \text{and} \quad \pi_*(\eta_u) = \mathrm{grad}_\omega(u).$$

Now, let $\mathcal{H}$ be the set of L^2 complex valued function on Y satisfying the equivariant condition

$$\varphi(z \cdot y) = z\varphi(y),$$

where $z \cdot y$ is the action of $z \in U(1)$ on $y \in Y$. Let $y \mapsto u$ the pullback on Y of $x \mapsto u$. Define

$$\hat{u}(\varphi) = \frac{\partial \varphi}{\partial y} \left(\widehat{\mathrm{grad}_\omega(u)} \right).$$

We check esealy now that

$$\hat{1} = 1 \times \tau + 0 \Rightarrow \hat{1}(\varphi) = \frac{\partial \varphi}{\partial y} \left(\tau(y) \right) = \varphi.$$

The other part of the condition $\widehat{\{u, v\}} = [\hat{u}, \hat{v}]$ is still satisfied.

That construction is called the *prequantization*.

<u>21. Definition</u> [The Dirac Program, Almost] With the prequantization we have a good quandiate which would be perfect if it satisfied the Dirac conditions. Indeed a wawe function φ in prequatization is, up to a phase, a function on the symplectic manifold, that is, $2n$ variables. There are n variables too much. In the simplest case the symplectic manifold is $\mathbf{R}^n \times \mathbf{R}^n$, space of pairs (q, p), position q and momentum p. The wave function is a function only on the poisition or momentum, or a miw of both, but no more than $n$variables.

So, to resolve that problem, geometric quantization had proposed to use a *polarization*, that is a projection

$$\pi \colon X \to Q$$

which is a *priori* a fibration, where the leaves are Lagrangian subspaces, that is n-dimensional subspaces where ω vanishes.

$$\omega \restriction \pi^{-1}(q) = 0 \quad \text{for all } q \in Q.$$

The wave function would be, more or less, a function on Q. But to represent the 1-parameter group associated with a dynamical variable $x \mapsto u$ the polarization needs to be invariant by that group. Unfortuantely, that almost never happens.

For example in the simplest example of the 2-dimensional *harmonic oscillator*, where the space is just $\mathbf{R} \times \mathbf{R}$ and the group willing to be represented $SO(2)$. The ordinary polarization $(q, p) \mapsto q$, or $(q, p) \mapsto p$, is not invariant by the matrices

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} q \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta)q \\ \sin(\theta)q \end{pmatrix}.$$

The method of geometric quantization is front of a deep problem until now without clear solution.

The case of the harmonic oscillator has been solved by the *pairing method*, which is too long to explain in that kind of short survey notes, but has been satisfactory solved.

In conclusion, the problem has been posed by Dirac, and been solved for some part by the geometric quantization program, but remains largely open until today.

Symplectic Diffeology

In the last decades many examples of wannabe symplectic spaces in infinite dimension came from physics. Also physicists were more and more interested in symplectic constructions involving singularities. These two directions are not covered by traditional symplectic geometry and need a new framework. What was usually done is that to each new example or new situation one creates a specific framework specially adapted to that situation. We call that the *heuristic approach*.

On the other hand, diffeology has all the qualities necessary to fill this program to extend symplectic geometry into the direction of infinite dimensional spaces and singular situations. This is what we shall discuss here. But we need to think and redefine some fundamental definitions to frame correctly the field of symplectic diffeology. Especially, what does mean to be symplectic, for a beginning.

22. Definition [The Darboux Condition in Diffeology] One most striking property symplectic manifolds share is the Darboux theorem. That is, every symplectic manifold (M, ω) is locally equivalent to $\mathbf{R}^{2n}$ equipped with the standard symplectic structure. That can be rephrased as follow:

> The pseudogroup of local automorphisms $\mathrm{Diff}_{\mathrm{loc}}(M, \omega)$
> is transitive.

Of course, that does not say nothing about the local structure itself. We shall come back on that quesiton, and the does not implies that the form ω is non degenerate. It just implies that ω is présymplectic.

Now, since we deal with various properties of closed 2-forms we introduce the geneal definition:

Definition 1. *We call* parasymplectic form *on a diffeological space X, any closed 2-form ω on X.*

Now we introduce:

Definition 2. *We call* presymplectic form *on a diffeological space X, any parasymplectic form ω on X such that the pseudogroup of automorphisms* $\mathrm{Diff}_{\mathrm{loc}}(X, \omega)$ *is transitive.*

If that condition is sufficient to determine the germ of ω at each point as being the germ of $(\mathbf{R}^{2n}\mathbf{R}^k, \mathrm{Can})$, it is not necessarilly the case in diffeology. That leads to a specific invariant in symplectic diffeology

Definition 3. *The* type *of a presymplectic structure in diffeology will be defined as some representant of equivalent* $\mathrm{germ}_x(\omega)$, *where* (X, ω) *run over presymplectic spaces and* $x \in X$.

For example, $(\mathbf{R}^{2n}\mathbf{R}^k, \mathrm{Can})$ is the type of all finite dimension presymplectic manifolds.

The question now is to understand how can we characterize the symplectic forms from the presymplectic ones? For that we need the help of the moment map in diffeology.

23. Construction [The Moment Map in Diffeology]

Next, we consider the group of symmetries (or automorphisms) of ω, denoted by $\mathrm{Diff}(X, \omega)$. Then, to introduce the moment map for any group of symmetries G, we need to clarify some vocabulary and notations:

Definition 1. *A diffeological group is a group that is a diffeological space such that the multiplication and the inversion are smooth.*

Definition 2.. *A* momentum *(Plural* momenta*) of a diffeological group G is any left-invariant 1-form on G. We denote by* $\mathcal{G}^*$ *the space of* momenta *:*

$$\mathcal{G}^* = \{\varepsilon \in \Omega^1(G) \mid \mathrm{L}(g)^*(\varepsilon) = \varepsilon, \ \text{for all } g \in G\}.$$

The set $\mathcal{G}^*$ *is a real vector space. It is also a diffeological vector space for the functional diffeology, but we shall not discuss that point here.*

Next, let (X, ω) be a parasymplectic space and G be a diffeological group.

Definition 3. *A symmetric action of* G *on* (X, ω) *is a smooth morphism*

$$g \mapsto g_X \text{ from } G \text{ to } \mathrm{Diff}(X, \omega),$$

where $\mathrm{Diff}(X, \omega)$ *is equipped with the functional diffeology. That is,*

$$\text{for all } g \in G, \quad g_X^*(\omega) = \omega.$$

Now, to grab the essential nature of the moment map, which is a map from X to $\mathcal{G}^*$, we need to understand it in the simplest possible case. That is, when:

(1) ω is exact, $\omega = d\alpha$,
(2) and when α is also invariant by G, $g_X^*(\alpha) = \alpha$.

In these conditions, the moment map is given by

$$\mu \colon X \to \mathcal{G}^* \quad \text{with} \quad \mu(x) = \hat{x}^*(\alpha),$$

where

$$\hat{x} \colon G \to X$$

is the *orbit map*

$$\hat{x}(g) = g_X(x).$$

We check immediately that,

Proposition 1. *Since* α *is invariant by* G, $\hat{x}^*(\alpha)$ *is left invariant by* G, *and therefore*

$$\mu(x) \in \mathcal{G}^*.$$

 Actually, as we know that:

Not all closed 2-forms are exact, and even if they are exact, they do not necessarily have an invariant primitive.

We shall see now, how we can generally come to a situation, so close to the simple case above, that, modulo some minor subtleties, we can build a good moment map in all cases.

Let us consider now the general case, with X connected. Let $\mathcal{K}$ be the chain-homotopy operator, defined in [PIZ13, 6.83] :

$$\mathcal{K} \colon \Omega^k(X) \to \Omega^{k-1}(X) \quad \text{with} \quad \mathcal{K} \circ d + d \circ \mathcal{K} = \hat{1}^* - \hat{0}^*.$$

Then, the differential 1-form $K\omega$, defined on $\mathrm{Paths}(X)$, satisfies

$$d[\mathcal{K}\omega] = (\hat{1}^* - \hat{0}^*)(\omega),$$

and $\mathcal{K}\omega$ is invariant by G [PIZ13, 6.84]. Considering

$$\bar{\omega} = (\hat{1}^* - \hat{0}^*)(\omega)$$

and

$$\bar{\alpha} = \mathcal{K}\omega,$$

we are in the simple case:

$$\bar{\omega} = d\bar{\alpha}$$

and $\bar{\alpha}$ invariant by G. We can apply the construction above and then:

Definition 4. *We define the paths moment map by*

$$\Psi\colon \mathrm{Paths}(X) \to \mathcal{G}^* \quad with \quad \Psi(\gamma) = \hat{\gamma}^*(\mathcal{K}\omega),$$

where $\hat{\gamma}\colon G \to \mathrm{Paths}(X)$ is the orbit map $\hat{\gamma}(g) = g_X \circ \gamma$ of the path γ.

The paths moment map is additive with respect to the concatenation,

$$\Psi(\gamma \vee \gamma') = \Psi(\gamma) + \Psi(\gamma'),$$

and it is equivariant by G, which acts by composition on $\mathrm{Paths}(X)$, and by coadjoint action on $\mathcal{G}^*$. That is, for all $g, k \in G$ and $\varepsilon \in \mathcal{G}^*$,

$$\mathrm{Ad}(g)\colon k \mapsto gkg^{-1}$$

and

$$\mathrm{Ad}_*(g)\colon \varepsilon \mapsto \mathrm{Ad}(g)_*(\varepsilon) = \mathrm{Ad}(g^{-1})^*(\varepsilon).$$

Then,

Definition 5. *We define the holonomy of the action of G on X as the subgroup*

$$\Gamma = \{\Psi(\ell) \mid \ell \in \mathrm{Loops}(X)\} \subset \mathcal{G}^*.$$

Proposition 2. *The group Γ is made of (closed) Ad_*-invariant momenta. But $\Psi(\ell)$ depends only on the homotopy class of ℓ, so then Γ is a homomorphic image of $\pi_1(X)$, more precisely, its abelianized.*

Definition 6. *If $\Gamma = \{0\}$, the action of G on (X, ω) is said to be Hamiltonian. The holonomy Γ is the obstruction for the action of the group G to be Hamiltonian.*

Now, we can push forward the paths moment map on $\mathcal{G}^*/\Gamma$, as suggested by the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Paths(X)} & \xrightarrow{\;\;\Psi\;\;} & \mathcal{G}^* \\
\big\downarrow{\scriptstyle\mathrm{ends}} & & \big\downarrow{\scriptstyle\mathrm{class}} \\
\mathrm{X}\times\mathrm{X} & \xrightarrow[\;\;\psi\;\;]{} & \mathcal{G}^*\!/\Gamma
\end{array}
$$

and we get then:

Defintion 7. *The two-points moment map is defined by:*

$$
\psi(x, x') = \mathrm{class}(\Psi(\gamma)) \in \mathcal{G}^*/\Gamma,
$$

for any path γ such that $\mathrm{ends}(\gamma) = (x, x')$.

Proposition 3. *The additivity of Ψ becomes the Chasles's cocycle condition on ψ:*

$$
\psi(x, x') + \psi(x', x'') = \psi(x, x'').
$$

 Since the group Γ is invariant by the coadjoint action, the coadjoint action passes to the quotient group $\mathcal{G}^*\!/\Gamma$, and ψ is a natural *group-valued moment map*, equivariant for this quotient coadjoint action.

Definition 8. *Because X is connected, there exists always a map*

$$
\mu: \mathrm{X} \to \mathcal{G}^*\!/\Gamma \quad \text{such that} \quad \psi(x, x') = \mu(x') - \mu(x).
$$

The solutions of this equation are given by

$$
\mu(x) = \psi(x_0, x) + c,
$$

where x_0 is a chosen point in X and c is a constant. These are the one-point moment maps.

But these moment maps μ are a priori no longer equivariant. Their variance introduces a 1-cocycle θ of G with values in $\mathcal{G}^*\!/\Gamma$.

Definition 9. *Let $x_0 \in X$ and $c \in \mathcal{G}^*/\Gamma$. We define:*

$$
\theta(g) = \psi(x_0, g(x_0)) + \Delta c(g),
$$

for all $g \in$ G, with

$$
\Delta c(g) = \mathrm{Ad}_*(g)(c) - c
$$

Then, $\theta(g)$ is a 1-cocycle of G with values in $\mathcal{G}^*/\Gamma$, twisted by Ad_*, and Δc is a coboundary. Moreover,

$$\mu(g(x)) = \mathrm{Ad}_*(g)(\mu(x)) + \theta(g),$$

Changing the base point x_0 and the constant c in μ changes the cocycle θ into a equivalent cocycle.

The cocycle θ capture the lack of invariance of the moment μ, we called it the Souriau's cocycle since it is a generalization of the manifold case. The cohomology class

$$\sigma = \mathrm{class}(\theta) \in H^1(G, \mathcal{G}^*/\Gamma)$$

is uniquely defined by the action of G on X. We say that the action of G on (X, ω) is exact when $\sigma = 0$, that is, when the cocycle θ is a coboundary.

Next, defining

$$\mathrm{Ad}_*^\theta(g) \colon \nu \mapsto \mathrm{Ad}_*(g)(\nu) + \theta(g),$$

then

$$\mathrm{Ad}_*^\theta(gg') = \mathrm{Ad}_*^\theta(g) \circ \mathrm{Ad}_*^\theta(g').$$

The cocycle property of θ, that is,

$$\theta(gg') = \mathrm{Ad}_*(g)(\theta(g')) + \theta(g),$$

makes Ad_*^θ an action of G on the group $\mathcal{G}^*/\Gamma$. This action is called the affine action.

Proposition 4. For the affine action, the moment map μ is equivariant:

$$\mu(g(x)) = \mathrm{Ad}_*^\theta(g)(\mu(x)).$$

This construction extends to the category {Diffeology}, the moment map for manifolds introduced by Souriau in [Sou70]. When X is a manifold and the action of G is Hamiltonian, they are the standard moment maps he defined there.

The remarkable point is that none of the constructions brought up above involves differential equationss, and there is no need for considering a possible Lie algebra either. That is a very important point.

The momenta appear as invariant 1-forms on the group, naturally, without intermediaries, and the moment map as a map in the space of momenta.

Note that the group of automorphisms $G_\omega = \mathrm{Diff}(X, \omega)$ is a legitimate diffeological group. The above constructions apply and give rise to universal objects:

- *universal momenta* $\mathcal{G}_\omega^*$,
- *universal path moment map* Ψ_ω,
- *universal holonomy* Γ_ω,
- *universal two-points moment map* ψ_ω,
- *universal moment maps* μ_ω,
- *universal Souriau's cocycles* θ_ω, and *cohomology class* σ_ω.

Note The universal cohomology class σ_ω is a *parasymplectif invariant* depeding only on (M, ω).

A *parasymplectic action* of a diffeological group G is a smooth morphism $h \colon G \to G_\omega$, and the objects, associated with G, introduced by the above moment maps constructions, are naturally subordinate to their universal counterparts.

Many examples can be found in [PIZ13, Sections $9.27 - 9.34$]

24. Example [The moment of imprimitivity] Consider the cotangent space T^*M of a manifold M, equipped with the standard symplectic form

$$\omega = d\lambda,$$

where λ is the Liouville form:

$$\lambda_{(x,a)} \left(\frac{d(x, a)}{ds} \right) = a \left(\frac{dx}{ds} \right).$$

Let G be the Abelian group

$$G = \mathcal{C}^\infty(M, \mathbf{R}).$$

Consider the action of G on T^*M defined by

$$f \colon (x, a) \mapsto (x, a - df_x),$$

where $x \in M$, $a \in T_x^*M$, and df_x is the differential of f at the point x. Then, the moment map is given by

$$\mu \colon (x, a) \mapsto d[f \mapsto f(x)] = d[\delta_x],$$

where δ denotes here the Dirac distribution

$$\delta_x(f) = f(x).$$

Since δ_x is a smooth function on $\mathcal{C}^\infty(M, \mathbf{R})$, its differential is a 1-form. Let us check that this 1-form is invariant:

Let $h \in \mathcal{C}^\infty(M, \mathbf{R})$,

$$
\begin{aligned}
L(h)^*(\mu(x)) &= L(h)^*(d[\delta_x]) \\
&= d[L(h)^*(\delta_x)] \\
&= d[\delta_x \circ L(h)],
\end{aligned}
$$

but

$$
\delta_x \circ L(h) \colon f \mapsto \delta_x(f + h) = f(x) + h(x).
$$

Then,

$$
\begin{aligned}
d[\delta_x \circ L(h)] &= d[f \mapsto f(x) + h(x)] \\
&= d[f \mapsto f(x)].
\end{aligned}
$$

Therefore:

$$
L(h)^*(\mu(x)) = \mu(x).
$$

We see that in this case, the moment map identifies with a function with values distributions but still has the definite formal statute of a map into the space of momenta of the group of symmetries.

Moreover, this action is Hamiltonian and exact. This example, generalized to diffeological space, is developed in [PIZ13, Exercise 147].

25. Theorem [Symplectic manifolds are coadjoint orbits] Because symplectic forms of manifolds have no local invariants, as we know thanks to Darboux's theorem, they have a huge group of automorphisms. This group is big enough to be transitive [Boo69], so that we will be able to identify the the symplectic manifold with its image by the universal moment map. Then, by equivariance, it will give a coadjoint orbit (affine or not) of its group of symmetries. In other words, coadjoint orbits are the universal models of symplectic manifolds.

Precisely, let M be a connected Hausdorff manifold, and let ω be a closed 2-form on M. Let $G_\omega = \mathrm{Diff}(M, \omega)$ be its group of symmetries and $\mathcal{G}_\omega^*$ its space of momenta. Let Γ_ω be the holonomy, and μ_ω be a universal moment map with values in $\mathcal{G}_\omega^*/\Gamma_\omega$. We have, then, the following:

Theorem 1. [PIZ] *The form ω is symplectic, that is non-degenerate, if and only if:*

1. *the group G_ω is transitive on M;*
2. *the universal moment map $\mu_\omega \colon M \to \mathcal{G}_\omega^*/\Gamma_\omega$ is injective.*

This theorem is proved in [PIZ13, 9.23], but let us make some comments on the keys elements.

Remark Consider the closed 2-form $\omega = (x^2 + y^2)\, dx \wedge dy$; one can show that it has an injective universal moment map μ_ω. But its group G_ω is not transitive, since ω is degenerate in $(0, 0)$, and only at that point. Thus, the transitivity of G_ω is necessary.

The case homogeneous presymplectix is interresting for what it suggest:

Theorem 2. [PIZ] *Let* (M, ω) *be a presymplectic manifold, homogeneous under its group of automorphisms. Then, the characteristcis of* ω *are the preimages of the universal moment map* μ_ω.

◄*Proof.* Let us give some hint about the sequel of the proof. Assume ω is symplectic. Let $m_0, m_1 \in M$ and p be a path connecting these points. For all $f \in \mathcal{C}^\infty(M, \mathbf{R})$ with compact support, let

$$F \colon t \mapsto e^{t \operatorname{grad}_\omega(f)}$$

be the exponential of the symplectic gradient of the f. Then, F is a 1-parameter group of automorphisms, and its value on $\Psi_\omega(p)$ is:

$$\Psi_\omega(p)(F) = [f(m_1) - f(m_0)] \times dt.$$

Now, if $\mu_\omega(m_0) = \mu_\omega(m_1)$, then there exists a loop ℓ in M such that $\Psi_\omega(p) = \Psi_\omega(\ell)$. Applied to the 1-plot F, we deduce $f(m_1) = f(m_0)$ for all f. Therefore $m_0 = m_1$, and μ_ω is injective.

Conversely, let us assume that G_ω is transitive, and μ_ω is injective. By transitivity, the rank of ω is constant. Now, let us assume that ω is degenerate, that is, $\dim(\ker(\omega)) > 0$. Since the distribution $\ker(\omega)$ is integrable, given two different points m_0 and m_1 in a characteristic, there exists a path p connecting these two points and drawn entirely in the characteristic, that is, such that $dp(t)/dt \in \ker(\omega)$ for all t. But that implies $\Psi_\omega(p) = 0$ [PIZ13, Section 9.20]. Hence, $\mu_\omega(m_0) = \mu_\omega(m_1)$. But we assumed μ_ω is injective. Thus, ω is nondegenerate, that is, symplectic. ▶

26. Definition [Symplectic Diffeological Spaces] Thanks to the two previous theorems I proposed a definition of symplectic diffeological space:

Definition *A parasymplectic form* ω *on a diffeological space* X *will be sait symplectic if:*

(1) *its pseudogroup of local automorphisms* $\mathrm{Diff}_{\mathrm{loc}}(X, \omega)$ *is transitive.*
(2) *The universal moment map* μ_ω *is a covering onto its image.*

The first condition means that ω is presymplectic, the <u>Darboux condition</u>. The second condition mimic the situation of manifolds, but we can hardly ask the universal moment map to be injective, we do not know enough. The weaker condition of being a covering would be probably sufficient to insure the symplectic nature od the presymplectic form.

It is possible that we can weaken the second condition by considering an equivalent of the universal moment map for the pseudogroup of automorphisms. But that remain to be investigated.

<u>Note.</u> There is a conflict betweenmy definition above and what is usually regarded as "symplectic orbifold". For example, the symplectic form $\omega = dx \wedge dy$ descends on the quotient space $\mathcal{Q}_m = \mathbf{C}/\mathcal{U}_m$, where $\mathcal{U}_m$ is the group of mth-roots of unity. And this space is regarded as "symplectic" by mathematicians. However, it does not fit the definition above because the pseudogroup of automorphisms fixes the origin $0 \in \mathcal{Q}_m$, but the universal moment is injective.

<u>Example</u> The infinite projective space:

$$\mathrm{CP}^\infty = \mathrm{S}^\infty/\mathrm{S}^1,$$

where $\mathrm{S}^\infty \subset \ell^2$ is the set of infinite complex sequences of norm 1, is a symplectic diffeological space. It is actually a coadjoint orbit of $\mathrm{U}(\mathcal{H})$.

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