MODELING, ORBIFOLDS AND QUASIFOLDS

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ref. http://math.huji.ac.il/~piz/documents/ShD-lect-MOQ.pdf

In this lecture we build the caregory of orbifolds, and also quasifolds, by modeling locally these spaces according to what they should look like.

Orbifolds have been introduced by Ishiro Satake as *V-Manifolds* in 1956 and 1957 [Sat56] [Sat56]. They have been introduced as smooth structures for describing quotient spaces by a finite group of transformations. We recall Satake construction first and show then how we can rethink these spaces as diffeological spaces.

By the way we sho how diffeology solves a problem unsolved by Satake and successors about what are smooth maps between orbifolds, and buid then the subcategory {Orbifolds} of the catagory {Diffeology}.

Let us mention that the word *orbifold* has been substituted to V-manifold by Thurston in 1978 [Thu78], but it recovers the same original Satake concept without modification.

Orbifolds, the Satake Definition

The elementary brick in Satake construction is the Local Uniformization System. It is a topological construction.

1. Definition [Local Uniformization System] Let M be a Hausdorff space and $\mathrm{U} \subset \mathrm{M}$ an open subset. A *local uniformizing system for* U (l.u.s) is a triple $(\tilde{\mathrm{U}}, \mathrm{G}, \varphi)$, where $\tilde{\mathrm{U}}$ is a connected open subset of $\mathbf{R}^n$ for some n, where G is a finite group of diffeomorphisms of $\tilde{\mathrm{U}}$,[1]

Date: December 4, 2020.

Shantou 2020 Lecture Notes.

[1]in original definition, the fixed point sets have codimension ≥ 2.

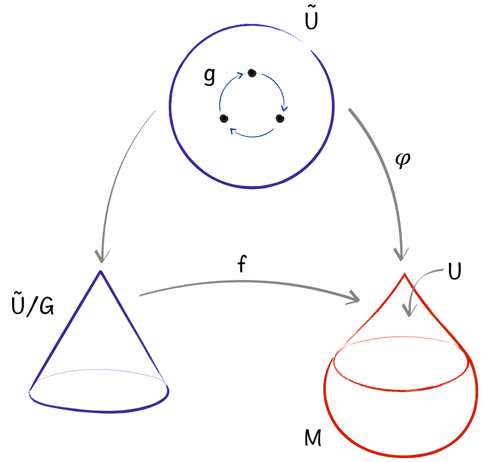

Figure 1. Local Uniformizing System.

and where $\varphi\colon \tilde{U} \to U$ is a map which induces a homeomorphism between $\tilde{U}/G$ and U.

Note. In the figure 1, we denote that homeomorphism from U/G to U by f.

Local uniformizing systems are patched together by *injections*; these can be thought of as the "transition maps". The following definition is taken from [Sat57, p. 466]:

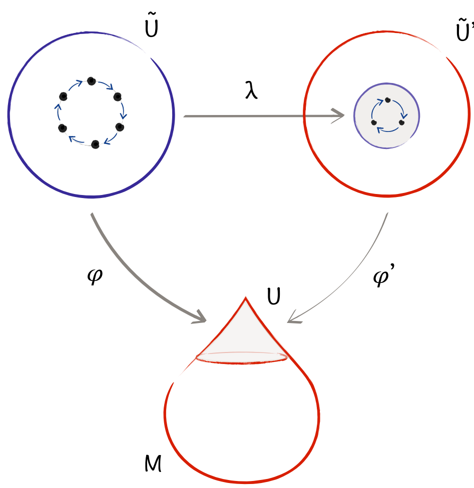

Figure 2. Injection.

2. Definition [Injections] An *injection* from an l.u.s $(\tilde{U}, G, \varphi)$ to an l.u.s $(\tilde{U}', G', \varphi')$ is a diffeomorphism λ from $\tilde{U}$ onto an open subset of $\tilde{U}'$ such that

$$\varphi = \varphi' \circ \lambda.$$

3. Definition [Defining Family] Let M be a Hausdorff space. A *defining family* on M is a family $\mathcal{F}$ of l.u.s for open subsets of M, satisfying conditions below. An open subset $U \subset M$ is said to be $\mathcal{F}$-*uniformized* if there exists an l.u.s. $(\tilde{U}, G, \varphi)$ in $\mathcal{F}$ such that $\varphi(\tilde{U}) = U$.

(1) Every point in M is contained in one $\mathcal{F}$-uniformized open set, at least. If a point p is contained in two $\mathcal{F}$-uniformized open sets U_1 and U_2, then there exists an $\mathcal{F}$-uniformized open set U_3 such that $p \in U_3 \subset U_1 \cap U_2$.
(2) If $(\tilde{U}, G, \varphi)$ and $(\tilde{U}', G', \varphi')$ are l.u.s in $\mathcal{F}$ and $\varphi(\tilde{U}) \subset \varphi'(\tilde{U}')$, then there exists an injection $\lambda \colon \tilde{U} \to \tilde{U}'$.

In other words, if $p \in U \cap U'$ there exist a third l.u.s. $(\tilde{U}'', G'', \varphi'')$, two injections $\lambda \colon \tilde{U}'' \to \tilde{U}$ and $\lambda' \colon \tilde{U}'' \to \tilde{U}'$ such that $\varphi' \circ \lambda' = \varphi'' = \varphi \circ \lambda$, as shows the Figure 3.

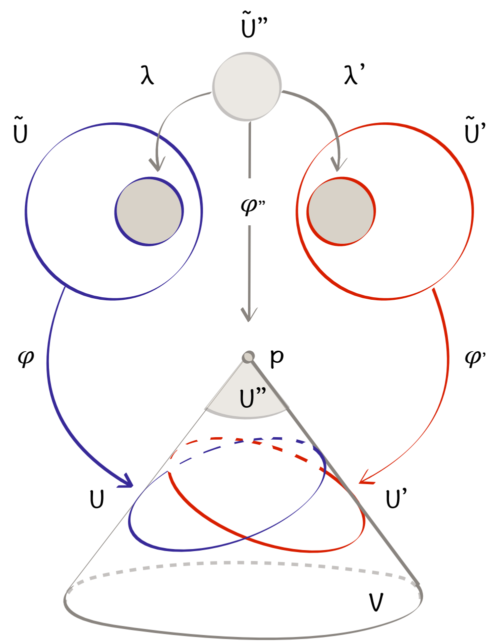

Figure 3. Defining Family.

4. Definition [V-Manifold] The following definition is taken from [Sat57, p. 467, Definition 1].

A V-manifold is a composite concept formed of a Hausdorff topological space M *and a defining family* $\mathcal{F}$.

Two defining families $\mathcal{F}$ and $\mathcal{F}'$ are said to be *directly equivalent* if there exists a third defining family $\mathcal{F}''$ containing both of them. Two defining families are said to be *equivalent* if they are the ends of a chain of directly equivalent defining families. Equivalent families are regarded as defining one and the same V-manifold structure on M.

In [Sat57, p. 467, footnote 1] Satake write:

> *But in the following we consider a V-manifold* M *with a fixed defining family* $\mathcal{F}$. *(i.e. a "coordinate V-manifold"* $(\mathrm{M}, \mathcal{F})$*).*

That is the convention we have made and when we say *V-manifold* it is always a coordinate V-manifold $(\mathrm{M}, \mathcal{F})$ we have in mind.

Orbifolds as Diffeologies

5. Definition [Diffeological Orbifolds] Let X be a diffeological space. We say that X is an *diffeolgical n-orbifold* (or a D-orbifold) if X is everywhere locally diffeomorphic to some $\mathbf{R}^n/\Gamma$, with Γ a finite subgroup of $\mathrm{GL}(n, \mathbf{R})$, possibly different from point to point. The diffeological n-orbifolds are modeled on quotient spaces of type $\mathbf{R}^n/\Gamma$.

More precisely, for every point $x \in \mathrm{X}$, there exist a finite group $\Gamma \subset \mathrm{GL}(n, \mathbf{R})$, a (connected) Γ-invariant Euclidean domain $\tilde{\mathrm{U}} \subset \mathbf{R}^n$ and a local diffeomorphism $\varphi \colon \mathbf{R}^n \supset \tilde{\mathrm{U}} \to \mathrm{X}$.

The situation looks like the previous defintion of l.u.s, as illustrated in Figure 4, except that here the quotient $\tilde{\mathrm{U}}/\Gamma$ is equipped with the quotient diffeology, the map $\mathrm{class} \colon \mathbf{R}^n \to \mathbf{R}^n/\Gamma$ is the canonical subduction and F is a local diffeomorphism.

These local diffeomorphisms are called *charts* of the D-orbifol X. An atlas of X is any covering set $\mathcal{A}$ of charts. Of course there exists a *saturated atlas* made of all charts.

Given an atlas $\mathcal{A}$ of X, that defines a generating family

$$\mathcal{F} = \{\mathrm{F} \circ \mathrm{class} \mid \mathrm{F} \in \mathcal{A}\},$$

where class is relative to the group Γ associated with F. We call $\mathcal{F}$ the *strict generating family* associated with the atlas $\mathcal{A}$.

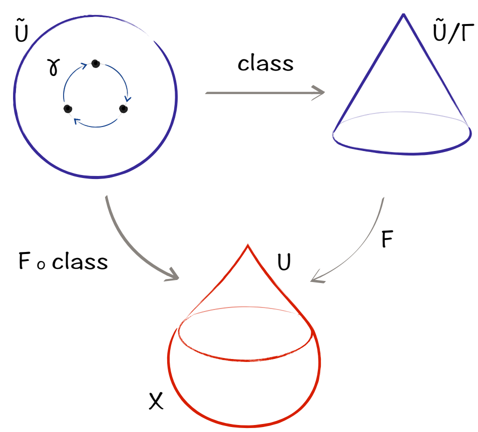

Figure 4. Generating Family for a D-orbifold.

6. Remark [Linear Action or Not?] In the previous definition, it is equivalent to ask Γ to be a finite group of diffeomorphisms or to be linear. Indeed, we define a Γ-invariant Riemmannian metric by $\langle u, v\rangle_\Gamma = \sum_{\gamma\in\Gamma}\langle \gamma u, \gamma v\rangle$, then the *slice theorem* states that this action is equivalent to an orthogonal action.

7. Examples [Some Diffeological Orbifolds] It is important to clarify a similar point we made for the diffeological definition of manifold. Here again an D-orbifold come a priori as a diffeological space, that is, equipped with a diffeology $\mathcal{D}$. The fact that X is an orbifold is a property of the diffeology, not the set itself, as the following example will show.

(1) The first example, the simplest is certainly $\Delta_1 = \mathbf{R}/\{\pm 1\}$, which is equivalent to the half-line $[0,\infty[$ equipped with the pushforward of the standard diffeology of $\mathbf{R}$ by the squae map $x \mapsto x^2$. The D-topology is the subset topology on $[0,\infty[$.

(2) The second classical example is the *cone orbifold* $\mathcal{C}_m = \mathbf{C}/\mathrm{U}_m$, where $\mathrm{U}_m = \{\exp(2i\pi k/m) \mid k = 1\ldots m\}$. The cone orbifold $\mathcal{C}_m$ is equivalent to the set of complex numbers $\mathbf{C}$ equipped with the pushforward of the standard diffeology by the map $z \mapsto z^m$. The D-topology is the standard topology of $\mathbf{C}$.

(3) The third example is more elaborated, the *waterdrop*. This is a diffeology defined on the sphere $\mathrm{S}^2 \subset \mathbf{R}^3$. A plot of the

waterdrop diffeology is a smooth parametrization ζ in S^2 which is identified to $\mathbf{C}\times\mathbf{R}$, with $\mathrm{N}=(0,1)$ the *North Pole*, satisfying:

$$\zeta : \mathrm{U} \to \mathbf{C}\times\mathbf{R} \quad \text{with} \quad \begin{cases} \zeta(r) = \begin{pmatrix} z(r) \\ t(r) \end{pmatrix}, \\ |z(r)|^2 + t(r)^2 = 1. \end{cases}$$

such that, for all $r_0 \in \mathrm{U}$:

• if $\zeta(r_0) \neq \mathrm{N}$, then there exists a small ball $\mathcal{B}$ centered at r_0 such that $\zeta \restriction \mathcal{B}$ is smooth.

• If $\zeta(r_0) = \mathrm{N}$, then there exist a small ball $\mathcal{B}$ centered at r_0 and a smooth parametrization z in $\mathbf{C}$ defined on $\mathcal{B}$ such that, for all $r \in \mathcal{B}$,

$$\zeta(r) = \frac{1}{\sqrt{1+|z(r)|^{2m}}} \begin{pmatrix} z(r)^m \\ 1 \end{pmatrix}.$$

Note that this orbifold diffeology on S^2 is a *subdiffeology* of the standard diffeology of manifold, embedded in $\mathbf{R}^3$. Note also that how it is possible to multiply the number of conical points on the sphere.

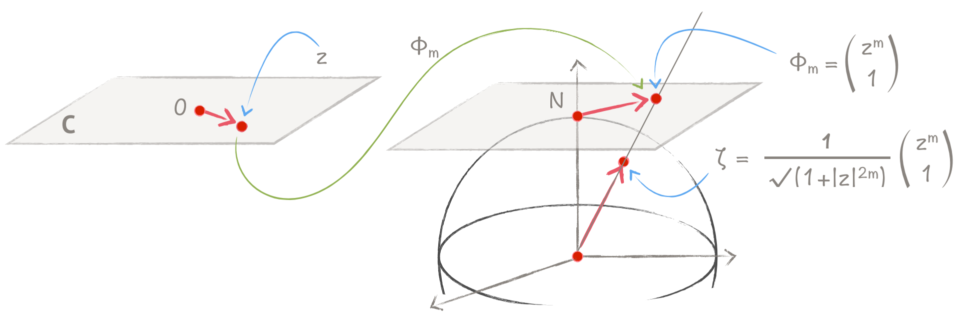

Figure 5. The Waterdrop Orbifold.

8. Example [Smooth Maps Between Orbifolds] We consider the orbifold $\mathcal{C}_m = \mathbf{C}/\mathcal{U}_m$. Let $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } r > 1 \text{ or } r = 0 \\ e^{-1/r}\rho_n(r)(r,0) & \text{if } \frac{1}{n+1} < r \leq \frac{1}{n} \text{ and } n \text{ is even} \\ e^{-1/r}\rho_n(r)(x,y) & \text{if } \frac{1}{n+1} < r \leq \frac{1}{n} \text{ and } n \text{ is odd,} \end{cases}$$

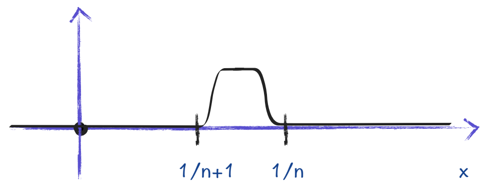

Figure 6. The function ρ_n.

where $r = \sqrt{x^2 + y^2}$ and ρ_n is a function vanishing flatly outside the interval $]1/(n+1), 1/n[$ and not inside, see Figure 6.

Remark now that

$$f(AX) = h_X(A) f(X), \quad \text{with} \quad X \in \mathbf{R}^2 \quad \text{and} \quad A \in SO(2).$$

On the annulus

$$\frac{1}{n+1} < r \leq \frac{1}{n}, \quad \text{with} \quad \begin{cases} h_X(A) = 1_{\mathbf{R}^2} & \text{if } n \text{ is even, and} \\ h_X(A) = A & \text{if } n \text{ is odd.} \end{cases}$$

Now, the function f descend onto a smooth map φ from the cone orbifold $\mathcal{C}_m$ to itself. In particular because the homomorphism h_X flips from the identity to trivial on any successive anulus, φ has no local equivariant smooth lifting.

This is a big difference with diffeomorphisms for which it is proven that the stabilizer of one point is locally mapped equivariantly into the stabilizer of the image by a homomorphism.

$$\begin{tikzcd}
\mathbf{R}^2 \arrow[r, "f"] \arrow[d, "\mathrm{class}_m"'] & \mathbf{R}^2 \arrow[d, "\mathrm{class}_m"] \\
\mathcal{C}_m = \mathbf{R}^2/\mathcal{U}_m \arrow[r, "\varphi"'] & \mathcal{C}_m = \mathbf{R}^2/\mathcal{U}_m
\end{tikzcd}$$

This example is the very illustration of the unsuccessful attempt to define smooth maps between orbifolds as locally equivariant maps, on the level of local symmetry group, and that answer Satake footnote in [Sat57, page 469],

> "*The notion of* C^∞*-map thus defined is inconvenient in the point that a composite of two* C^∞*-maps defined in a different choice of defining families is not always a* C^∞ *map.*"

Embedding of orbifolds into a category such as {Diffeology} could has solved this question. The existence of good smooth maps between orbifolds is crucial for having a covariant satisfactory theory of orbifolds.

Equivalence between V-Manifolds and D-orbifolds

9. Proposition [V-manifolds are D-orbifolds] Let M be a Hausdorff topological space. A defining family $\mathcal{F}$ on M determines a diffeology of orbifold. Namely, the diffeology generated by the parametrizations $\varphi : \tilde{U} \to U$, for all $(\tilde{U}, G, \varphi) \in \mathcal{F}$.

10. Proposition [Equivalence of Defining Families] Let M be a Hausdorff topological space. If two defining families $\mathcal{F}$ and $\mathcal{F}'$ on M generate the same diffeology then they are equivalent. More precisely, the union $\mathcal{F}'' = \mathcal{F} \cup \mathcal{F}'$ is a defining family.

11. Proposition [D-orbifolds are V-manifolds] Conversely, let X be a D-orbifold, then equip X with the D-topology (assumed Hausdorff). Let A be an atlas of X, the strict generating family of the atlas $\mathcal{A}$ is a defining family in the sense of Satake, and equip X with a structure of V-manifold for its D-topology.

Two different atlases $\mathcal{A}$ and $\mathcal{A}'$ give equivalent Satake defining families, essentially because they are sub-atlases of the maximal atlas.

12. Proposition [Equivalence of definitions] The two constructions above are, up to an equivelence, inverse one from each other.

Note. Actually, Satake defined only what ca we call *reflexion free V-manifolds*. But there was no technical obstacles to extend the definition to any V-manifold.

The proofs of these late propositions are consequence of the precise description of what we can call now the *internal structure* of a D-orbifold.

Internal Structure of a D-orbifold

13. Definition [The Groupoid $\mathbf{G}$ of Germs of Local Diffeomorphisms] Let X be any diffeological space. We define the groupoid

$\mathbf{G}$ of *germs of local diffeomorphisms* of X as follow:

$$\begin{cases} \mathrm{Obj}(\mathbf{G}) & = & \mathrm{X}, \\ \mathrm{Mor}(\mathbf{G}) & = & \{\,\mathrm{germ}(\varphi)_x \mid \varphi \in \mathrm{Diff_{loc}}(\mathrm{X}) \text{ and } x \in \mathrm{dom}(\varphi)\}. \end{cases}$$

The *source* maps, the *target* maps and the composition of germs of local diffeomorphisms are defined as follows:

$$\begin{cases} \mathrm{src}(\mathrm{germ}(\varphi)_x) = x, \quad \mathrm{trg}(\mathrm{germ}(\varphi)_x) = \varphi(x). \\ \mathrm{germ}(\varphi)_x \cdot \mathrm{germ}(\varphi')_{x'} = \mathrm{germ}(\varphi' \circ \varphi)_x, \text{ with } x' = \varphi(x). \end{cases}$$

The pseudo group of local diffeomorphisms of X is equipped with the functional diffeology of pseudo group, that is, the diffeology of local smoth map for the pairs (f, f^{-1}), where $f \in \mathrm{Diff_{loc}}(\mathrm{X})$. Let then define the germ map by:

$$\begin{cases} \mathfrak{G} = \{(\varphi, x) \mid \varphi \in \mathrm{Diff_{loc}}(\mathrm{X}) \text{ and } x \in \mathrm{dom}(\varphi)\}. \\ \mathrm{germ} : (\varphi, x) \mapsto \mathrm{germ}(\varphi)_x. \end{cases}$$

We equip $\mathrm{Mor}(\mathbf{G})$ with the *pushforward* of the diffeology of $\mathfrak{G}$ by the map germ. That makes $\mathbf{G}$ a diffeological groupoid. That means essentially that:

(a) the multiplication and the inversion are smooth maps

(b) and the inclusion $\mathrm{Obj}(\mathbf{G}) \hookrightarrow \mathrm{Mor}(\mathbf{G})$ by the identities is an induction.

<u>14. Definition</u> [The Structure Groupoid of an Orbifold] Let X be an n-orbifold, $\mathcal{A}$ be an atlas, $\mathcal{F}$ be the strict generating family over $\mathcal{A}$, $\mathcal{N}$ be the nebula and ev be the evaluation map, that is:

$$\mathcal{N} = \coprod_{\mathrm{F} \in \mathcal{F}} \mathrm{dom}(\mathrm{F}) \quad \text{and} \quad \mathrm{ev} : \mathcal{N} \to \mathrm{X} \quad \text{with} \quad \mathrm{ev}(\mathrm{F}, r) = \mathrm{F}(r).$$

We call *Structure Groupoid* $\mathbf{G}$ of the orbifold X the subgroupoid of the groupoid of germs of local diffeomorphisms of $\mathcal{N}$ that descend on the identity of X along ev. That is,

$$\mathrm{Mor}_{\mathbf{G}}((\mathrm{F}, r), (\mathrm{F}', r')) = \left\{ \mathrm{germ}(\varphi)_r \,\middle|\, \begin{array}{l} \varphi \in \mathrm{Diff_{loc}}(\mathbf{R}^n), r' = \varphi(r) \\ \mathrm{F}' \circ \varphi = \mathrm{F} \restriction \mathrm{dom}(\varphi) \end{array} \right\}$$

<u>Note</u>. In order to show the dependency of the structure groupoid with respect to the atlas $\mathcal{A}$ we need the two following lemma.

<u>15. Proposition</u> [Lifting the identity] Let $\mathcal{Q} = \mathbf{R}^n/\Gamma$. Consider a local smooth map F from $\mathbf{R}^n$ to itself, such that $\mathrm{class} \circ \mathrm{F} = \mathrm{class}$. In other words, F is a local lifting of the identity on $\mathcal{Q}$. Then,

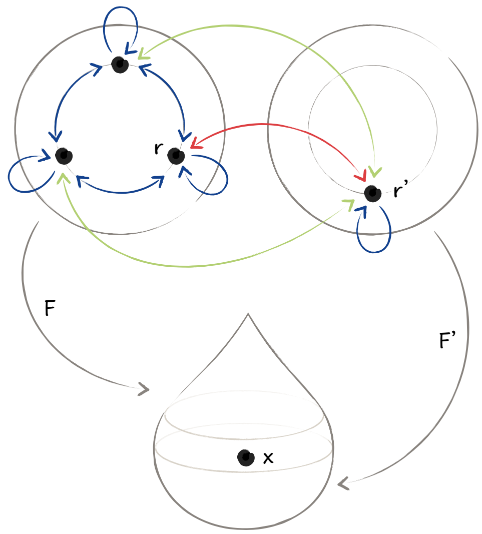

Figure 7. The Groupoid of the Waterdrop.

$$\begin{tikzcd} \mathbf{R}^n \supset \tilde{U} \arrow[rr, "F"] \arrow[dr, "\text{class}"'] & & \mathbf{R}^n \arrow[dl, "\text{class}"] \\ & \mathcal{Q} & \end{tikzcd}$$

Theorem. *F is locally equal to some group action* $F(r) =_{\rm loc} \gamma \cdot r$, *where* $\gamma \in \Gamma$.

◀*Proof.* Let us assume first that F is defined on an open ball $\mathcal{B}$. Then, for all r in the ball, there exists a $\gamma \in \Gamma$ such that $F(r) = \gamma \cdot r$. Next, for every $\gamma \in \Gamma$, let

$$F_\gamma : \mathcal{B} \to \mathbf{R}^n \times \mathbf{R}^n \quad \text{with} \quad F_\gamma(r) = (F(r), \gamma \cdot r).$$

Let $\Delta \subset \mathbf{R}^n \times \mathbf{R}^n$ be the diagonal and let us consider

$$\Delta_\gamma = F_\gamma^{-1}(\Delta) = \{r \in \mathcal{B} \mid F(r) = \gamma \cdot r\}.$$

Lemma 1. *There exist at least one* $\gamma \in \Gamma$ *such that the interior* $\mathring{\Delta}_\gamma$ *is non-empty.*

◀ Indeed, since F_γ is smooth (thus continuous), the preimage Δ_γ by F_γ of the diagonal is closed in $\mathcal{B}$. However, the union of all the preimages $F_\gamma^{-1}(\Delta)$ — when γ runs over Γ — is the ball $\mathcal{B}$. Then, $\mathcal{B}$ is a finite union of closed subsets. According to Baire's theorem, there is at least one γ such that the interior $\mathring{\Delta}_\gamma$ is not empty. ▶

Lemma 2. *The union* $\mathring{\Delta}_\Gamma = \cup_{\gamma\in\Gamma}\mathring{\Delta}_\gamma$ *is an open dense subset of* $\mathcal{B}$.

◀ Indeed, let $\mathcal{B}' \subset \mathcal{B}$ be an open ball. Let us denote with a prime the sets defined above but for $\mathcal{B}'$. Then, $\Delta'_\gamma = (F_\gamma \restriction \mathcal{B}')^{-1}(\Delta) = \Delta_\gamma \cap \mathcal{B}'$, and then $\mathring{\Delta}'_\gamma = \mathring{\Delta}_\gamma \cap \mathcal{B}'$. Thus, $\mathcal{B}' \cap \mathring{\Delta}_\Gamma = \mathcal{B}' \cap (\cup_{\gamma\in\Gamma}\mathring{\Delta}_\gamma) = \cup_{\gamma\in\Gamma}\mathring{\Delta}'_\gamma$, which is not empty for the same reason that $\cup_{\gamma\in\Gamma}\mathring{\Delta}_\gamma$ is not empty. Therefore, $\mathring{\Delta}_\Gamma$ is dense. ▶

In conclusion: the tangent linear map $D(F)\colon \mathcal{B} \to GL(n, \mathbf{R})$ is smooth, then continuous, thus $D(F)^{-1}(\Gamma)$ is closed. But, $\mathring{\Delta}_\Gamma$, which is an open dense subset of $\mathcal{B}$, is contained in $D(F)^{-1}(\Gamma)$. Hence, $\mathcal{B}$ is contained in $D(F)^{-1}(\Gamma)$ (its own closure) which is contained in $\mathcal{B}$. Thus, $D(F)^{-1}(\Gamma) = \mathcal{B}$. Then, since $\mathcal{B}$ is connected, $D(F)(\mathcal{B}) \subset \Gamma$ is connected. But $\Gamma \subset GL(n, \mathbf{R})$ is discrete, then $D(F)(\mathcal{B}) = \{\gamma\}$, for some $\gamma \in \Gamma$. ▶

16. Proposition [Lifting Local Diffeomorphisms] Let $\mathcal{Q} = \mathbf{R}^n/\Gamma$ and $\mathcal{Q}' = \mathbf{R}^{n'}/\Gamma'$, Then,

Theorem. *Every local smooth lifting* $\tilde{f}$ *of any local diffeomorphism* f*, from* $\mathcal{Q}$ *to* $\mathcal{Q}'$*, is necessarily a local diffeomorphism, from* $\mathbf{R}^n$ *to* $\mathbf{R}^{n'}$*. In particular* $n = n'$*. Moreover, let*

$$x \in \mathrm{dom}(f), \ \ x' = f(x)$$

and $r, r' \in \mathbf{R}^n$ *such that*

$$\mathrm{class}(r) = x \quad \textit{and} \quad \mathrm{class}(r') = x'.$$

Then, the local lifting $\tilde{f}$ *can be chosen such that*

$$\tilde{f}(r) = r'.$$

◀*Proof.* Let the local diffeomorphism f be defined on U with values in U'. By definition of local diffeomorphism, they are both open for the D-topology. Then $\tilde{U} = \mathrm{class}^{-1}(U)$ is open in $\mathbf{R}^n$. Since the composite $f \circ \mathrm{class}\colon \tilde{U} \to U'$ is a plot in $\mathcal{Q}'$, for all $r \in \tilde{U}$ there exists a smooth local lifting $\tilde{f}\colon \tilde{V} \to \mathbf{R}^{n'}$, defined on an open neighborhood of r, such that $\mathrm{class}' \circ \tilde{f} = f \circ \mathrm{class} \restriction \tilde{V}$.

$$\begin{array}{ccc} \mathbf{R}^n \supset \tilde{U} \supset \tilde{V} & \xrightarrow{\ \tilde{f}\ } & \mathbf{R}^{n'} \\ \Big\downarrow{\scriptstyle \mathrm{class}} & & \Big\downarrow{\scriptstyle \mathrm{class}'} \\ \mathcal{Q} \supset U & \xrightarrow[\ f\ ]{} & \mathcal{Q}' \end{array}$$

Let $x = \mathrm{class}(r)$, $x' = f(x)$, $r' = \tilde f(r)$, and then $x' = \mathrm{class}'(r')$. Next, let $\tilde{\mathrm{U}}' = \mathrm{class}'^{-1}(\mathrm{U}')$. Since the composite $f^{-1}\circ\mathrm{class}'$ is a plot in $\mathcal{Q}$, there exists a smooth lifting $\hat f\colon \tilde{\mathrm{V}}' \to \mathbf{R}^n$, defined on an open neighborhood of r', such that $\mathrm{class}\circ\hat f = f^{-1}\circ\mathrm{class}'\restriction\tilde{\mathrm{V}}'$. Let $r'' = \hat f(r')$, which can be different from r.

$$\begin{tikzcd}
\mathbf{R}^n \arrow[d, "\mathrm{class}"'] & \tilde{\mathrm{V}}' \subset \tilde{\mathrm{U}}' \subset \mathbf{R}^{n'} \arrow[l, "\hat f"'] \arrow[d, "\mathrm{class}'"] \\
\mathcal{Q} & \mathrm{U}' \subset \mathcal{Q}' \arrow[l, "f^{-1}"]
\end{tikzcd}$$

Now, we consider the composite $\hat f\circ\tilde f\colon \tilde{\mathrm{W}} \to \mathbf{R}^n$, where $\tilde{\mathrm{W}} = \tilde f^{-1}(\tilde{\mathrm{V}}')$ is a non-empty open subset of $\mathbf{R}^n$ since it contains r. Moreover, $\hat f\circ\tilde f(r) = r''$. It also satisfies $\mathrm{class}\circ(\hat f\circ\tilde f) = \mathrm{class}$. Indeed, $\mathrm{class}\circ(\hat f\circ\tilde f) = (\mathrm{class}\circ\hat f)\circ\tilde f = (f^{-1}\circ\mathrm{class}')\circ\tilde f = f^{-1}\circ(\mathrm{class}'\circ\tilde f) = f^{-1}\circ(f\circ\mathrm{class}) = (f^{-1}\circ f)\circ\mathrm{class} = \mathrm{class}$. Thus, thanks to §15, there exists, locally, $\gamma\in\Gamma$ such that $\hat f\circ\tilde f = \gamma\restriction\tilde{\mathrm{W}}$. By the way, $r'' = (\hat f\circ\tilde f)(r) = \gamma\cdot r$. Let $\bar f = \gamma^{-1}\circ\hat f$, then: $\mathrm{class}\circ\bar f = \mathrm{class}\circ\gamma^{-1}\circ\hat f = \mathrm{class}\circ\hat f = f^{-1}\circ\mathrm{class}'$, and $\bar f$ is still a local lifting of f^{-1}. Thus $\bar f\circ\tilde f = 1_{\tilde{\mathrm{W}}}$, that is, $\bar f = \tilde f^{-1}\restriction\tilde{\mathrm{W}}$. We conclude that, around r, $\tilde f$ is a local diffeomorphism. Now, if we consider any another point r''' over x', there exists γ' such that $\gamma'\cdot r' = r'''$; changing $\tilde f$ to $\gamma'\circ\tilde f$ and $\bar f$ to $\bar f\circ\gamma'^{-1}$, we get $\tilde f(r) = r'''$, and $\tilde f$ and $\bar f$ still remain inverse of each other. Thus, for any $r\in\mathbf{R}^n$ over x and any $r'\in\mathbf{R}^n$ over $x' = f(x)$, we can locally lift f to a local diffeomorphism $\tilde f$ such that $\tilde f(r) = r'$. ▶

17. Definition [Equivalence Between Categories, Groupoids] Let A and C be two categories. Let us recall that, according to [SML78, Chap. 4 § 4 Thm. 1], a *functor*

$$\mathrm{S}\colon \mathrm{A}\to\mathrm{C}$$

is an *equivalence of categories* if and only if,

(1) S is full and faithful,
(2) each object c in C is isomorphic to S(a) for some object a in A.

If A and C are groupoids, the last condition means that,

(2') for each object c of C, there exist an object a of A and an arrow from S(a) to c.

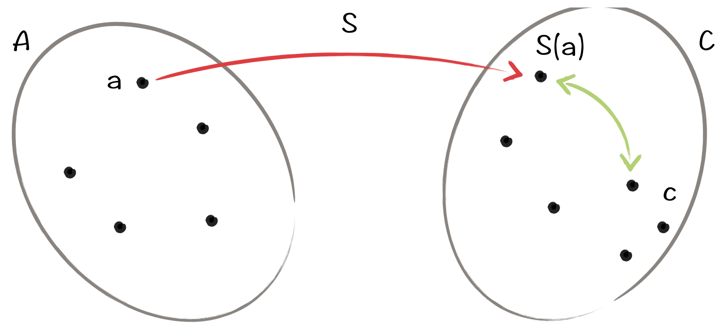

Figure 8. Equivalence of Categories.

In other words: let the *transitivity-components* of a groupoid be the maximal full subgroupoids such that each object is connected to any other object by an arrow. The functor S is an equivalence of groupoids if

(1) it is full and faithful,
(2) it descends surjectively on the set of transitivity-components.

$$\begin{tikzcd}
\mathrm{A} \arrow[r, "\mathrm{S}"] \arrow[d, "\mathrm{comp}"'] & \mathrm{C} \arrow[d, "\mathrm{comp}"] \\
\mathrm{Comp}(\mathrm{A}) \arrow[r, "\mathrm{S}_*"'] & \mathrm{Comp}(\mathrm{C})
\end{tikzcd}$$

18. Proposition [Equivalence of Structure-Groupoids] Consider an n-orbifold X. Let $\mathcal{A}$ be an atlas, let $\mathcal{F}$ be the associated strict generating family, let $\mathcal{N}$ be the nebula of $\mathcal{F}$ and let $\mathbf{G}$ the associated structure groupoid.

Proposition. *The fibers of the subduction* $\mathrm{ev}\colon \mathrm{Obj}(\mathbf{G}) \to \mathrm{X}$ *are exactly the transitivity-components of* $\mathbf{G}$. *In other words, the space of transitivity components of the groupoid* $\mathbf{G}$ *associated with any atlas of the orbifold* X, *equipped with the quotient diffeology, is the orbifold itself.*

Theorem. *Different atlases of* X *give equivalent structure groupoids. The structure groupoids associated with diffeomorphic orbifold are equivalent.*

In other words, the equivalence class of the structure groupoids of a orbifold is a diffeological invariant.

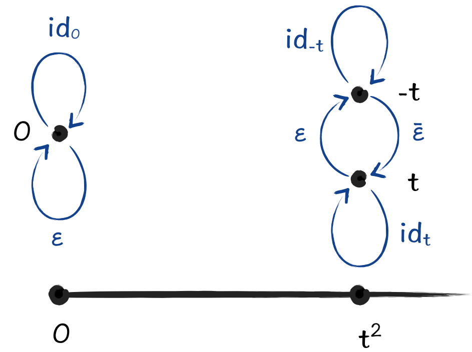

Figure 9. The Groupoid of Δ_1.

The proof of this theorem is based on the propositions §15 and §16.

◀*Proof.* Let us start by proving the proposition. Let $F: U \to X$ and $F': U' \to X'$ be two generating plots from the strict family $\mathcal{F}$, and $r \in U \subset \mathbf{R}$ and $r' \in U' \subset \mathbf{R}'$. Assume that

$$\mathrm{ev}(F, r) = \mathrm{ev}(F', r') = x,$$

that is,

$$x = F(r) = F'(r').$$

Note that

$$F = f \circ \mathrm{class} \restriction U \quad \text{and} \quad F' = f' \circ \mathrm{class}' \restriction U',$$

where $f, f' \in \mathcal{A}$. Then, let

$$\psi = f'^{-1} \circ f \quad \text{with} \quad \psi: f^{-1}(f'(U')) \to U',$$

is a local diffeomorphism that maps

$$\xi = f(\mathrm{class}(r)) \quad \text{to} \quad \xi' = f'(\mathrm{class}'(r')).$$

Then, according to §16:

(1) $n = n'$,
(2) there exists a local diffeomorphism φ of $\mathbf{R}^n$, lifting locally ψ and mapping r to r'. That is

$$\mathrm{class}' \circ \varphi =_{\mathrm{loc}} \psi \circ \mathrm{class} \quad \text{and} \quad \psi(r) = r'.$$

Its germ realizes a morphism (an arrow) of the groupoid $\mathbf{G}$ connecting (F, r) to (F', r'), which are then on the same transitivity component:

$$\mathrm{F}(r) = \mathrm{F}'(r') \Rightarrow \mathrm{comp}(\mathrm{F}, r) = \mathrm{comp}(\mathrm{F}', r').$$

Of course, when $\mathrm{F}(r) \neq \mathrm{F}'(r')$ there cannot be an arrow, by definition. Therefore, the fibers of the evaluation map are the transitive components of the structure groupoid $\mathbf{G}$ of the orbifold.

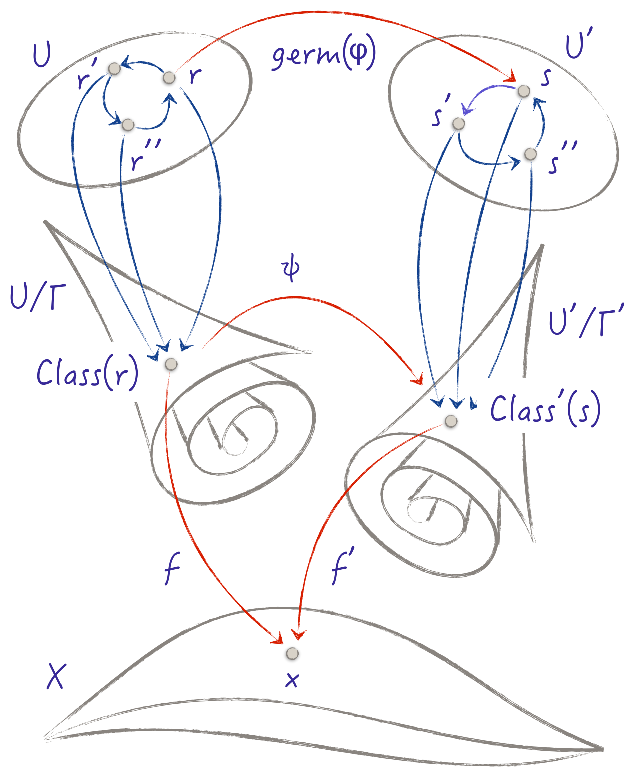

Figure 10. Lifting the identity.

Now, for the theorem: let $\mathcal{A}$ and $\mathcal{A}'$ be two atlases of X and consider

$$\mathcal{A}'' = \mathcal{A} \coprod \mathcal{A}'.$$

With an obvious choice of notation:

$$\mathrm{Obj}(\mathbf{G}'') = \mathrm{Obj}(\mathbf{G}) \coprod \mathrm{Obj}(\mathbf{G}'),$$

and $\mathbf{G}''$ contains naturally $\mathbf{G}$ and $\mathbf{G}'$ as full subgroupoids. The question then is: how does the adjunction of the crossed arrows between $\mathbf{G}$ and $\mathbf{G}'$ change the distribution of transitivity-components?

According to the previous proposition, it changes nothing since, for $\mathbf{G}$, $\mathbf{G}'$ or $\mathbf{G}''$, the set of transitivity-components are always exactly the fibers of the respective subductions ev. In other words, the set of groupoid components is always X, for any atlas of X. Thus $\mathbf{G}$ and $\mathbf{G}'$ are equivalent to $\mathbf{G}''$, therefore $\mathbf{G}$ and $\mathbf{G}'$ are equivalent. ▶

Quasifolds as Diffeologies

19. Definition [Quasifolds] The notion de *quasifold* has been proposed by Elisa Prato in [Pra01], it extend the notion of orbifold. The original definition, which has been modified once or twice, has been revisited by diffeology as follow:

Definition *A diffeological space* X *is said to be a n-quasifold if it is locally diffeomorphic everywhere to a quotient* $\mathbf{R}^n/\Gamma$, *where* Γ *is a countable subgroup of* $\mathrm{Aff}(\mathbf{R}^n)$.

The main difference is that the group Γ can be infinite, but countable.

The first example would certainly be the irrational torus $T_\alpha = \mathbf{R}/(\mathbf{Z}+\alpha\mathbf{Z})$ with $\alpha \in \mathbf{R}-\mathbf{Q}$.

20. Proposition [Quasifolds versus Orbifolds] The construction of the structure groupoid on orbifolds and the lemma attached is similar in the case of quasifolds. We get the same lemmas and results. For the details see [IZP20].

The map f, defined in §8, descends also on the quotient

$$\mathcal{C}_\alpha = \mathbf{C}/\{e^{2i\pi k\alpha}\}_{k\in\mathbf{Z}},$$

where $\alpha \in \mathbf{R}-\mathbf{Q}$, into a smooth map φ. The same phenomena that occurs for the orbifolds there occurs also for the quasifolds.

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