GENERATING FAMILIES, DIMENSION

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ref. http://math.huji.ac.il/~piz/documents/ShD-lect-GFD.pdf

In this lecture we introduce first the notion of *generating family*, and we give an application in the definition of *dimension in diffeology*. Applied to the quotients $\Delta_m = \mathbf{R}^m/\mathrm{O}(\mathbf{R}^m)$, we show that and Δ_m and Δ_n are not diffeomorphic if $n \neq m$, and not diffeomorphic to the half-line $\Delta_\infty = [0, \infty[\subset \mathbf{R}$.

In this lecture we introduce first the notion of *generating family*, and we give an application in the definition of dimension.

The notion of *dimension* in diffeology, that was introduced in [?], is a quick and easy answer to the question: *For two different integers n and m, are the diffeological spaces $\Delta_n = \mathbf{R}^n/\mathrm{O}(n)$ and $\Delta_m = \mathbf{R}^m/\mathrm{O}(m)$ diffeomorphic?* We will show that since $\dim(\Delta_n) = n$ and since the dimension is a diffeological invariant, the answer is *No, they are not*. This method simplifies a partial result, obtained in a more complicated way in [?], stating that Δ_1 and Δ_2 are not diffeomorphic. The half-line $\Delta_\infty = [0, \infty[\subset \mathbf{R}$ is a similar example for which $\dim(\Delta_\infty) = \infty$. Hence, Δ_m is not diffeomorphic to the half-line Δ_∞ for any integer m. Dimension is a simple but powerful diffeological invariant. Thankfully, the diffeological dimension coincides with the usual definition when the diffeology space is an Euclidean domain, and a manifold. That is, when the diffeology is generated by local diffeomorphisms with $\mathbf{R}^n$, for some integer n.

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Generating Famillies

Diffeologies can be built by *generating families*. Any family of parametrizations of a set generates a diffeology. Conversely, any diffeology is generated by some set of parametrizations. This mode of construction of diffeologies is very useful because it can reduce the analysis of the properties of a diffeological space to a subset of its plots, hopefully smaller than the whole diffeology. The definition of generating diffeology leads to the definition of the dimension of a diffeological space, which is a first global invariant of the category {Diffeology}. But this construction also leads to the introduction of important subcategories of diffeological spaces, for example the category of manifolds, or the category of orbifolds and others.

1. Proposition *Let $\mathcal{F}$ be a familly of parametrizations of a set X. There exists a finest diffeology on X containing $\mathcal{F}$, it is called the diffeology generated by $\mathcal{F}$. It is denoted by $\langle\mathcal{F}\rangle$. The family $\mathcal{F}$ is said to be a generating family of the diffeological space $(\mathrm{X}, \langle\mathcal{F}\rangle)$. It is the intersection of the diffeologies containing $\mathcal{F}$*

$$\langle\mathcal{F}\rangle = \bigcap_{\substack{\mathcal{D} \in \mathrm{Diffl g}(\mathrm{X}) \\ \mathcal{F} \subset \mathcal{D}}} \mathcal{D}.$$

Let X be a diffeological space, the set of families generating X will be denoted by $\mathrm{Gen}(\mathrm{X})$.

If the family $\mathcal{F}$ covers X, that is, if for all $x \in \mathrm{X}$ there exists a parametrization $\mathrm{F} \in \mathcal{F}$ such that $x = \mathrm{F}(r)$, with $r \in \mathrm{dom}(\mathrm{F})$, then a plot of $\langle\mathcal{F}\rangle$ is any parametrization $\mathrm{P} \colon \mathrm{U} \to \mathrm{X}$ such that:

(♣) *For all $r \in \mathrm{U}$ there exist an open neighborhood $\mathrm{V} \subset \mathrm{U}$ of r, a parametrization $\mathrm{F} \in \mathcal{F}$, and a smooth parametrization $\mathrm{Q} \colon \mathrm{V} \to \mathrm{dom}(\mathrm{F})$ such that $\mathrm{P} \restriction \mathrm{V} = \mathrm{F} \circ \mathrm{Q}$.*

If the family $\mathcal{F}$ does not cover X, the easiest way is to add the constant parametrizations

$$\hat{x} \colon \mathbf{R}^0 \to \mathrm{X} \quad \text{with} \quad \hat{x}(0) = x,$$

to $\mathcal{F}$. That gives a family $\hat{\mathcal{F}}$ that covers X and

$$\langle\mathcal{F}\rangle = \langle\hat{\mathcal{F}}\rangle.$$

Note 1. *The empty family $\varnothing$ generates the discrete family,*

$$\langle\varnothing\rangle = \mathcal{D}_\circ.$$

Note 2. *The diffeology $\mathcal{D}$ of a diffeological space X belongs to* $\mathrm{Gen}(X)$. *Generating family is a projector:*

$$\langle\mathcal{D}\rangle = \mathcal{D}$$

Note 3. *Generating families is an increasing function of fineness*

$$\mathcal{F} \subset \mathcal{F}' \quad \Rightarrow \quad \langle\mathcal{F}\rangle \subset \langle\mathcal{F}'\rangle.$$

2. Proposition [Pushing Forward Families] *Let X and X' be two sets. Let $\mathcal{F}$ be a family of parametrizations of X, and let $f : X \to X'$ be a map.*

The pushforward $f_*(\mathcal{F})$ of the family $\mathcal{F}$ by f is defined by

$$f_*(\mathcal{F}) = \left\{ f \circ F \mid F \in \mathcal{F} \right\}.$$

Then, the diffeology generated by the pushforward of the family $\mathcal{F}$ by f is the pushforward by f of the diffeology generated by $\mathcal{F}$, that is,

$$\langle f_*(\mathcal{F})\rangle = f_*(\langle\mathcal{F}\rangle).$$

In particular, let X and X' be two diffeological spaces, and let $f : X \to X'$ be a subduction. The pushforward $f_*(\mathcal{F})$ of any generating family $\mathcal{F}$ for X is a generating family for X'.

3. Proposition [Pulling Back Families] *Let X and X' be two sets. Let $\mathcal{F}'$ be a family of parametrizations of X', and let $f : X \to X'$ be any map. Let us define the pullback of the family $\mathcal{F}'$ by f as the family $f^*(\mathcal{F})$ of parametrizations $F : U \to X$ satisfying the following property:*

- (♦) *Either $f \circ F$ is constant or there exist an element $F' : U' \to X'$ of $\mathcal{F}'$, and a smooth parametrization $\varphi : U \to U'$, such that $F' \circ \varphi = f \circ F$.*

Then, the diffeology generated by the pullback $f^*(\mathcal{F}')$ is the pullback by f of the diffeology generated by $\mathcal{F}$, that is,

$$\langle f^*(\mathcal{F}')\rangle = f^*(\langle\mathcal{F}'\rangle).$$

In particular, let X *and* X' *be two diffeological spaces, and let* $f : X \to X'$ *be an induction. The pullback* $f^*(\mathcal{F}')$ *of any generating family* $\mathcal{F}'$ *for* X' *is a generating family for* X*. Unfortunately, compared with the pushforward of a family, pulling back a small generating family may lead to a huge family, almost as big as the diffeology itself. That will be the next example.*

Note. The choice of a generating family is relatively arbitrary. For example, the empty family is equivalent to the family of constant parametrizations. If the family $\mathcal{F}'$ *is empty, its pullback is not empty, but is the set of the parametrizations of* X *with values in the preimages of points* $f^{-1}(x')$*,* $x' \in X'$*. This is not surprising since the pullback of the discrete diffeology is the sum of the preimages of points, equipped with the coarse diffeology.*

4. Example [Generating the Half-Line] *Let the half line* $[0,\infty[\subset \mathbf{R}$ *be equipped with the subset diffeology of* $\mathbf{R}$*. Let* $\mathcal{F}$ *be the generating family of* $\mathbf{R}$ *reduced to the identity,* $\mathcal{F} = \{1_{\mathbf{R}}\}$*. Then the pullback of the generating family* $\mathcal{F}$ *by the inclusion* $j\colon [0,\infty[\to \mathbf{R}$ *is the whole diffeology of* $[0,\infty[$*.*

The pullback $j^*(\{1_{\mathbf{R}}\})$ *is the set of parametrizations* $F\colon U \to [0,\infty[$ *such that* $j\circ F$ *is constant, or there exist an element* $F' \in \{1_{\mathbf{R}}\}$ *and a smooth parametrization* $\varphi : U \to \mathrm{dom}(F')$ *with* $j \circ F = F' \circ \varphi$*. Thus,* $F' = 1_{\mathbf{R}}$*,* $\mathrm{dom}(F') = \mathbf{R}$*, and then* $F = \varphi$*. Therefore,* F *is any smooth parametrization of* $\mathbf{R}$ *with values in* $[0,\infty[$*, and* $j^*(\{1_{\mathbf{R}}\})$ *is the whole diffeology of the half-line.*

Local Diffeology

Be aware that, if the concept of locality does exists in diffeology and can be of some help, the diffeology is never weaker than its associated D-topology.

5. Definition [Local smooth maps and local diffeomorphisms] *Let* X *and* X' *be two sets equipped with the diffeologies* $\mathcal{D}$ *and* $\mathcal{D}'$ *respectively. Let* f *be a map defined on a subset* $A \subset X$ *to* X'*. We denote:*

$$f\colon X \supset A \to X'.$$

We say that f *is a local smooth map if for every* $P \in \mathcal{D}$*,* $f\circ P \in \mathcal{D}'$*. This implies in particular that* $P^{-1}(A)$ *is an Euclidean domain. That is,* A *is open for the D-topology of* X *(see below).*

Proposition. The composite of local smooth maps is a local smooth map.

Definition. The map f is said to be a local diffeomorphism if f is an injective local smooth map as well as its inverse f^{-1}, defined on $f(\mathrm{A}) \subset \mathrm{X}'$.

Note In particular, manifolds are diffeological spaces generated by local diffeomorphisms with $\mathbf{R}^n$, for some integer n.

6. Definition [D-Topology] *Let* X *be a diffeological space, and* $\mathcal{D}$ *be its diffeology. there exists on* X *a finest topology such that the plots* $\mathrm{P} \in \mathcal{D}$ *are continuous. It is called the D-Topology* [?].

A subset $\mathcal{O}$ *of* X *is D-open, that is, open for the D-topology, if and only if* $\mathrm{P}^{-1}(\mathcal{O})$ *is an open subset of* $\mathrm{dom}(\mathrm{P})$.

1. Dimension of a diffeology

In this paragraph, we define the *global dimension* of a diffeology or a diffeological space. This dimension is a diffeological invariant. Further we will see a finer invariant, the dimension map.

7. Definition [Dimension of a parametrization] Let P be a n-parametrization of some set X, $n \in \mathbf{N}$. We say that n is the *dimension* of P, and we denote it by dim(P). That is,

$$\forall \mathrm{P} \in \mathrm{Params}(\mathrm{X}),\ \dim(\mathrm{P}) = n \ \Leftrightarrow\ \mathrm{P} \in \mathrm{Params}_n(\mathrm{X}).$$

8. Definition [Dimension of a Family of Parametrizations] Let X be a set and let $\mathcal{F}$ be any family of parametrizations of X. We define the *dimension* of $\mathcal{F}$ as the supremum of the dimensions of its elements,

$$\dim(\mathcal{F}) = \sup\{\dim(\mathrm{F}) \mid \mathrm{F} \in \mathcal{F}\}.$$

Note that $\dim(\mathcal{F})$ can be infinite if for all $n \in \mathbf{N}$ there exists an element F of $\mathcal{F}$ such that $\dim(\mathrm{F}) = n$. In this case we denote $\dim(\mathcal{F}) = \infty$.

9. Definition [Dimension of a Diffeology] Let $\mathcal{D}$ be a diffeology. The *dimension* of $\mathcal{D}$ is defined as the infimum of the dimensions of its generating families:

$$\dim(\mathcal{D}) = \inf\{\dim(\mathcal{F}) \mid \langle \mathcal{F} \rangle = \mathcal{D}\}.$$

Let X be a diffeological space and $\mathcal{D}$ be its diffeology, we define the dimension of X as the dimension of $\mathcal{D}$:

$$\dim(\mathrm{X}) = \dim(\mathcal{D}) \in \mathbf{N} \cup \{\infty\}.$$

10. Proposition [Dimensions of Euclidean Domains] The diffeological dimension of a Euclidean domain $\mathrm{U} \subset \mathbf{R}^n$, equipped with the standard diffeology, is equal to n:

$$\forall \mathrm{U} \in \mathrm{Domains}(\mathbf{R}^n), \quad \dim(\mathrm{U}) = n.$$

◀*Proof.* Let 1_U be the identity map of U. The singleton $\{1_\mathrm{U}\}$ is a generating family of U, therefore, $\dim(\mathrm{U}) \leq \dim\{1_\mathrm{U}\}$. Since $\dim\{1_\mathrm{U}\} = n$, $\dim(\mathrm{U}) \leq n$. Now let us assume that $\dim(\mathrm{U}) < n$. Then, there exists a generating family $\mathcal{F}$ for U such that $\dim(\mathcal{F}) < n$. Since the identity map 1_U is a plot in U, it lifts locally at every point along some element of $\mathcal{F}$. Thus, for any $r \in \mathrm{U}$ there exist a superset V of r, a parametrization $\mathrm{F} \colon \mathrm{W} \to \mathrm{U}$, element of $\mathcal{F}$ (that is $\mathrm{F} \in \mathcal{C}^\infty(\mathrm{W}, \mathrm{U})$) and a smooth map $\mathrm{Q} \colon \mathrm{V} \to \mathrm{W}$, such that $1_\mathrm{U} \restriction \mathrm{V} = 1_\mathrm{V} = \mathrm{F} \circ \mathrm{Q}$. But $\dim(\mathcal{F}) < n$ implies that $\dim(\mathrm{F}) = \dim(\mathrm{W}) < n$. Now, the rank of the linear tangent map $\mathrm{D}(\mathrm{F} \circ \mathrm{Q})$ is less or equal to $\dim(\mathrm{W}) < n$, but $\mathrm{D}(\mathrm{F} \circ \mathrm{Q}) = \mathrm{D}(1_\mathrm{V}) = 1_{\mathbf{R}^n}$, thus $\mathrm{rank}(\mathrm{D}(\mathrm{F} \circ \mathrm{Q})) = \mathrm{rank}(1_{\mathbf{R}^n}) = n$. Therefore, there is no generating family $\mathcal{F}$ of U with $\dim(\mathcal{F}) < n$, and $\dim(\mathrm{U}) = n$. ▶

11. Proposition [Dimension zero spaces are discrete] A diffeological space has dimension zero if and only if it is discrete.

◀*Proof.* Let X be a set equipped with the discrete diffeology. Any plot $\mathrm{P} \colon \mathrm{U} \to \mathrm{X}$ is locally constant. Then, for any $r \in \mathrm{U}$, P lifts locally along the 0-plot $\mathbf{x} = [0 \mapsto x]$, where $x = \mathrm{P}(r)$. Hence, the 0-plots form a generating family and $\dim(\mathrm{X}) = 0$. Conversely, let X be a diffeological space such that $\dim(\mathrm{X}) = 0$. Then, the 0-plots generate the diffeology of X. But, any plot lifting locally along a 0-plot is locally constant. Therefore, X is discrete. ▶

12. [The dimension is a diffeological invariant] If two diffeological spaces are diffeomorphic, then they have the same dimension.

◀*Proof.* Let X and X′ be two diffeological spaces and let $f \in \mathrm{Diff}(\mathrm{X}, \mathrm{X}')$. Let $\mathcal{F}$ be a generating family of X. The pushforward $\mathcal{F}' = f_*(\mathcal{F})$ made of the plots $f \circ \mathrm{F}$, where $\mathrm{F} \in \mathcal{F}$, is clearly a generating family of X'. Conversely we have f^{-1}. Therefore, $\dim(X) = \dim(X')$. ▶

<u>13. Exemple</u> [Has the set $\{0,1\}$ dimension 1?] Let us consider the set $\{0,1\}$. Let $\pi \colon \mathbf{R} \to \{0,1\}$ be the parametrization defined by:

$$\pi(x) = 0 \text{ if } x \in \mathbf{Q}, \text{ and } \pi(x) = 1 \text{ otherwise.}$$

Let $\{0,1\}_\pi$ be the set $\{0,1\}$ equipped with the diffeology generated by π. Since $\{\pi\}$ is a generating family, the dimension of $\{0,1\}_\pi$ is less than or equal to $1 = \dim(\{\pi\})$. But, since the plot π is not locally constant, by density of the rational (or irrational) numbers in $\mathbf{R}$, the space $\{0,1\}_\pi$ is not discrete. Hence, $\dim\{0,1\}_\pi \neq 0$, and finally $\dim\{0,1\}_\pi = 1$. Thus, a finite diffeological space may have a dimension non zero.

<u>14. Exemple</u> [Dimension of tori] Let $\Gamma \subset \mathbf{R}$ be any strict subgroup of $(\mathbf{R}, +)$ and let T_Γ be the quotient $\mathbf{R}/\Gamma$, whose diffeology is generated by the projection class: $\mathbf{R} \to \mathbf{R}/\Gamma$. Then,

$$\dim(T_\Gamma) = 1.$$

This applies in particular to the circles $\mathbf{R}/a\mathbf{Z}$, with perimeter $a > 0$, or to *irrational tori* when Γ is generated by more than one generators, rationally independent.

◀*Proof.* Since $\mathbf{R}$ is a Euclidean domain, class is a plot of the quotient, and $\mathcal{F} = \{\text{class}\}$ is a generating family of $\mathbf{R}/\Gamma$, and $\dim(\mathcal{F}) = 1$. Thus, as a direct consequence of the definition $\dim(\mathbf{R}/\Gamma) \leq 1$. Now, if $\dim(\mathbf{R}/\Gamma) = 0$, then the diffeology of the quotient is generated by the constant parametrizations. But π is not locally constant, therefore $\dim(\mathbf{R}/\Gamma) = 1$. ▶

2. Dimension Map of a Diffeological Space

Because diffeological spaces are not necessarily homogeneous, the global dimension of a diffeological space is a too rough invariant. It is necessary to refine this definition and to introduce the dimension function of a diffeological space, defined at each of its points.

The dimension function of diffeological spaces is the simplest numeral invariant in diffeology.

<u>15. Definition</u> [Pointed plots and germ of a diffeological space] Let X be a diffeological space, let $x \in X$. Let $P \colon U \to X$ be a plot. We say that P is *pointed at x* if $0 \in U$ and $P(0) = x$. We will agree

that the set of *germs* of the pointed plots of X at x represents the *germ of the diffeology* at this point, and we shall denote it by $\mathcal{D}_x$.

16. Definition [Local generating families] Let X be a diffeological space and let x be a point in X. We shall call *local generating family at x* any family $\mathcal{F}$ of plots of X such that:

(1) Every element P of $\mathcal{F}$ is pointed at x, that is, $0 \in \mathrm{dom}(\mathrm{P})$ and $\mathrm{P}(0) = x$.
(2) For all plots $\mathrm{P}\colon \mathrm{U} \to \mathrm{X}$ pointed at x, there exists a superset V of $0 \in \mathrm{U}$, a parametrization $\mathrm{F}\colon \mathrm{W} \to \mathrm{X}$ belonging to $\mathcal{F}$ and a smooth parametrization $\mathrm{Q}\colon \mathrm{V} \to \mathrm{W}$ pointed at $0 \in \mathrm{W}$, such that $\mathrm{F} \circ \mathrm{Q} = \mathrm{P} \upharpoonright \mathrm{V}$.

We shall say also that $\mathcal{F}$ *generates the germ* $\mathcal{D}_x$ of the diffeology $\mathcal{D}$ of X at the point x. And we denote

$$\mathcal{D}_x = \langle \mathcal{F} \rangle.$$

Note that, for any x in X, the set of local generating families at x is not empty, since it contains the set of all the plots pointed at x, and this set contains the constant parametrizations with value x.

17. Proposition [Union of local generating families] Let X be a diffeological space. Let us choose, for every $x \in \mathrm{X}$, a local generating family $\mathcal{F}_x$ at x. The union $\mathcal{F}$ of all these local generating families,

$$\mathcal{F} = \bigcup_{x \in \mathrm{X}} \mathcal{F}_x,$$

is a generating family of the diffeology of X.

◀*Proof.* Let $\mathrm{P}\colon \mathrm{U} \to \mathrm{X}$ be a plot, let $r \in \mathrm{U}$ and $x = \mathrm{P}(r)$. Let T_r be the translation $\mathrm{T}_r(r') = r' + r$. Let $\mathrm{P}' = \mathrm{P} \circ \mathrm{T}_r$ defined on $\mathrm{U}' = \mathrm{T}_r^{-1}(\mathrm{U})$. Since the translations are smooth, the parametrization P' is a plot of X. Moreover P' is pointed at x, $\mathrm{P}'(0) = \mathrm{P} \circ \mathrm{T}_r(0) = \mathrm{P}(r) = x$. By definition of a local generating family, there exists an element $\mathrm{F}\colon \mathrm{W} \to \mathrm{X}$ of $\mathcal{F}_x$, a superset V' of $0 \in \mathrm{U}'$ and a smooth parametrization $\mathrm{Q}' : \mathrm{V}' \to \mathrm{W}$, pointed at 0, such that $\mathrm{P}' \upharpoonright \mathrm{V}' = \mathrm{F} \circ \mathrm{Q}'$. Thus, $\mathrm{P} \circ \mathrm{T}_r \upharpoonright \mathrm{V}' = \mathrm{F} \circ \mathrm{Q}'$, that is, $\mathrm{P} \upharpoonright \mathrm{V} = \mathrm{F} \circ \mathrm{Q}$, where $\mathrm{V} = \mathrm{T}_r(\mathrm{V}')$ and $\mathrm{Q} = \mathrm{Q}' \circ \mathrm{T}_r^{-1}$. Hence, P lifts locally, at every point of its domain, along an element of $\mathcal{F}$. Therefore, $\mathcal{F}$ is a generating family of the diffeology of X. ▶

18. Definition [The dimension map] Let X be a diffeological space and let x be a point of X. By analogy with the global dimension of X, we define the *dimension of X at the point x* by:

$$\dim_x(X) = \inf\{\dim(\mathcal{F}) \mid \langle\mathcal{F}\rangle = \mathcal{D}_x\}.$$

The map $x \mapsto \dim_x(X)$, with values in $\mathbf{N} \cup \{\infty\}$, will be called the *dimension map* of the space X.

19. Proposition [Global dimension and dimension map] Let X be a diffeological space. The global dimension of X is the supremum of the dimension map of X:

$$\dim(X) = \sup_{x \in X} \left\{\dim_x(X)\right\}.$$

◀*Proof.* Let $\mathcal{D}$ be the diffeology of X. Let us prove first that for every $x \in X$, $\dim_x(X) \leq \dim(X)$, which implies that $\sup_{x\in X} \dim_x(X) \leq \dim(X)$. For that we shall prove that for any $x \in X$ and any generating family $\mathcal{F}$ of $\mathcal{D}$, $\dim_x(X) \leq \dim(\mathcal{F})$. Then, since $\dim(X) = \inf\{\dim(\mathcal{F}) \mid \mathcal{F} \in \mathcal{D} \text{ and } \langle\mathcal{F}\rangle = \mathcal{D}\}$ we shall get, $\dim_x(X) \leq \dim(X)$. Now, let $\mathcal{F}$ be a generating family of $\mathcal{D}$. For any plot $P : U \to X$ pointed at x, let us choose an element F of $\mathcal{F}$ such that: there exists a superset V of $0 \in U$ and a smooth parametrization $Q : V \to \mathrm{def}(F)$, such that $F \circ Q = P \restriction V$. Then, let $r = Q(0)$ and T_r be the translation $T_r(r') = r' + r$. Let $F' = F \circ T_r$, defined on $T_r^{-1}(\mathrm{def}(F))$. Thus, $F'(0) = x$, and F' is a plot of X, pointed at x, such that $\dim(F') = \dim(F)$. Let $Q' = T_r^{-1} \circ Q$, then Q' is smooth and $P \restriction V = F' \circ Q'$. Thus, the set $\mathcal{F}'_x$ of all these plots F' associated with the plots pointed at x is a generating family of $\mathcal{D}_x$, and for each of them $\dim(F') = \dim(F) \leq \dim(\mathcal{F})$. Therefore, $\dim(\mathcal{F}'_x) \leq \dim(\mathcal{F})$. But, $\dim_x(X) \leq \dim(\mathcal{F}'_x)$, hence $\dim_x(X) \leq \dim(\mathcal{F})$. And we conclude that $\dim_x(X) \leq \dim(X)$, for any $x \in X$, and $\sup_{x\in X} \dim_x(X) \leq \dim(X)$.

Now, let us prove that $\dim(X) \leq \sup_{x\in X} \dim_x(X)$. Let us assume that $\sup_{x\in X} \dim_x(X)$ is finite. Otherwise, according to the previous part we have $\sup_{x\in X} \dim_x(X) \leq \dim(X)$, and then $\dim(X)$ is infinite and $\sup_{x\in X} \dim_x(X) = \dim(X)$. Now, for any $x \in X$, $\dim_x(X)$ is finite. And since the sequence of the dimensions of the generating families of $\mathcal{D}_x$ is lower bounded, there exists for any x a generating family $\mathcal{F}_x$ such that $\dim_x(X) = \dim(\mathcal{F}_x)$. For every x in X let us choose one of them. Now, let us define $\mathcal{F}_m$ as the union of all these chosen families. According to a proposition above, $\mathcal{F}_m$ is a generating family of $\mathcal{D}$. Hence, $\dim(X) \leq \dim(\mathcal{F}_m)$. But

$\dim(\mathcal{F}_m) = \sup_{F \in \mathcal{F}_m} \dim(F) = \sup_{x \in X} \sup_{F \in \mathcal{F}_x} \dim(F) = \sup_{x \in X} \dim(\mathcal{F}_x) = \sup_{x \in X} \dim_x(X)$. Therefore, $\dim(X) \leq \sup_{x \in X} \dim_x(X)$. And we can conclude, from the two parts above, that $\dim(X) = \sup_{x \in X} \dim_x(X)$. ▶

20. Proposition [The dimension map is a local invariant] Let X and X′ be two diffeological spaces. If $x \in X$ and $x' \in X'$ are two points related by a local (a fortiori global) diffeomorphism, then $\dim_x(X) = \dim_{x'}(X')$.

21. Proposition [Dimensions of Manifolds] A n-manifold is a diffeological space X generated by local diffeomorphisms with $\mathbf{R}^n$. Each such local diffeomorphism is called a *chart* of X. Note that, each point is locally equivalent to any other point: there exists always a local diffeomorphism from one point to any other. The dimension function is constant, equal to the global dimension. Now, since there is always a chart mapping $0 \in \mathbf{R}^n$ to $x \in X$, $\dim_x(X) = \dim_0(\mathbf{R}^n) = n$. Therefore, $\dim(X) = n$. Which is coherent with the ususal dimension in differential geometry.

22. Proposition [Klein decomposition] Let X be a diffeological space. The local diffeomorphisms of X split the space into classes, according to the relation: $x \sim x'$ if and only if there exists a local diffeomorphism f mapping x to x'. These classes are called the *orbits of the local diffeomorphisms* of X. We call these classes *Klein's orbit*. The dimension map is constant on each Klein's orbit.

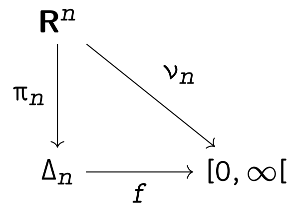
3. Examples of the half-lines

23. Proposition [The half-lines Δ_n] Let $\Delta_n = \mathbf{R}^n/\mathrm{O}(n)$ equipped with the quotient diffeology, $n \in \mathbf{N}$. Then, $\dim_0(\Delta_n) = n$, and $\dim_x(\Delta_n) = 1$ if $x \neq 0$. Thus, $\dim(\Delta_n) = n$ and for $n \neq m$ the half-lines Δ_n and Δ_m are not diffeomorphic.

◀*Proof.* Let $n > 0$, and let us denote by $\mathrm{class}_n : \mathbf{R}^n \to \Delta_n$ the projection from $\mathbf{R}^n$ onto its quotient. There is a natural bijection $f \colon \Delta_n \to [0, \infty[$ such that

$$f \circ \mathrm{class}_n = \nu_n \quad \text{with} \quad \nu_n(x) = \|x\|^2.$$

Now, thanks to the uniqueness of quotients, we use f to identify Δ_n with $[0, \infty[$, equipped with the diffeology $\mathcal{D}_n$ generated by ν_n.



The elements of $\mathcal{D}_n$ consist of the parametrizations which can be lifted locally along ν_n by smooth parametrizations of $\mathbf{R}^n$. Thus, since $\dim(\nu_n) = n$, we get

$$\dim(\Delta_n) \leq n.$$

Let us prove now that $\dim(\Delta_n) \geq n$:

Let us assume that ν_n, which is an element of $\mathcal{D}_n$, can be lifted locally, at the point 0_n, along a plot

$$P \in \mathcal{D}_n \quad \text{with} \quad \dim(P) = p < n.$$

Then, there exists a smooth parametrization

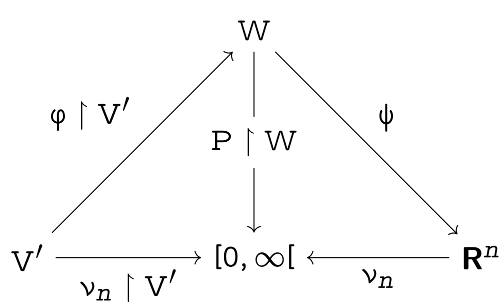
$$\varphi : V \to \mathrm{dom}(P) \quad \text{such that} \quad P \circ \varphi = \nu_n \restriction V.$$

We can assume without loss of generality that

$$0_p \in \mathrm{dom}(P),\ P(0_p) = 0 \quad \text{and} \quad \varphi(0_n) = 0_p.$$

Now, since P is an element of $\mathcal{D}_n$, there exists a smooth parametrization

$$\psi\colon W \to \mathbf{R}^n \quad \text{such that} \quad 0_p \in W \quad \text{and} \quad \nu_n \circ \psi = P \restriction W.$$



Let $V' = \varphi^{-1}(W)$, we get

$$\nu_n \restriction V' = \nu_n \circ F \quad \text{with} \quad F = \psi \circ \varphi \restriction V',$$

and

$$F \in \mathcal{C}^\infty(V', \mathbf{R}^n),\ 0_n \in V' \quad \text{and} \quad F(0_n) = 0_n,$$

that is

$$\|x\|^2 = \|F(x)\|^2.$$

The second derivative of this identity computed at the point 0_n gives

$$1_n = M^t M \quad \text{with} \quad M = D(F)(0_n).$$

But

$$M = AB \quad \text{with} \quad A = D(\psi)(0_p) \quad \text{and} \quad B = D(\varphi)(0_n).$$

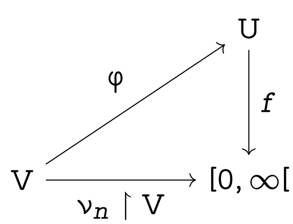
Thus, $1_n = B^t A^t AB$, which is impossible because $\text{rank}(B) \leq p < n$. Therefore, $\dim(\Delta_n) = n$. And, since the dimension is a diffeological invariant, Δ_n is not diffeomorphic to Δ_m for $n \neq m$. ▶

24. Proposition [The half-line Δ_∞] The dimension of a diffeological subspace $A \subset X$ can be less, equal, or even greater than the dimension of X. The following example is an illustration of this phenomenon. Let $\Delta_\infty = [0, \infty[\subset \mathbf{R}$, equipped with the subset diffeology. Then,

$$\dim_0(\Delta_\infty) = \infty \quad \text{and} \quad \dim_x(\Delta_\infty) = 1 \quad \text{if} \quad x \neq 0.$$

Thus, $\dim(\Delta_\infty) = \infty$, and for any integer m, Δ_∞ is not diffeomorphic to Δ_m.

◀*Proof.* Let us assume that $\dim(\Delta_\infty) = N < \infty$. For any integer n, the map $\nu_n : \mathbf{R}^n \to \Delta_\infty$, defined by $\nu_n(x) = \|x\|^2$, belongs to $\mathcal{D}_\infty$, the subset diffeology on $[0, \infty[$. Hence, ν_n lifts locally at the point 0_n along some $P \in \mathcal{D}_\infty$, where $\dim(P) = p \leq N$. Now, let us choose $n > N$. Then, P belongs to some $\mathcal{C}^\infty(U, \mathbf{R})$ with $\text{Values}(P) \subset [0, \infty[$, and there exists a smooth parametrization $\varphi : V \to U$ such that $P \circ \varphi = \nu_n \upharpoonright V$.



We can assume, without loss of generality, that $0_p \in U$, $\varphi(0_n) = 0_p$, and thus: $P(0_p) = 0$. Now, the first derivative of ν_n at a point $x \in V' = \varphi^{-1}(V)$ is given by $x = D(P)(\varphi(x)) \circ D(\varphi)(x)$. But, since P is smooth and positive, and since $P(0) = 0$ we have $D(P)(0_p) = 0$. Hence, the second derivative of ν_n computed at the point 0_n

gives $1_n = M^t HM$, where $M = D(\varphi)(0)$ and $H = D^2(P)(0)$. But since $\mathrm{rank}(M) \leq p \leq N$ and $n > N$, this is impossible. Therefore, $\dim(\Delta_\infty) = \infty$. ►

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