DIFFEOLOGY, THE AXIOMATIC...

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ref. http://math.huji.ac.il/~piz/documents/ShD-lect-DTA.pdf

In this lecture we will present the short axiomatic that founds diffeology. We will describe the basic set theoretic constructions of the theory and give a few examples.

Diffeologies are defined on arbitrary sets without any preexisting structure, neither topology nor anything else. That is important enough to be underlined and remembered. Diffeology is based on the notion of parametrizations, and will consists in declaring which parameterizations in a set will be regarded as smooth, provided that a small set of axioms is satisfied. Then, the development of diffeology will consist in transferring, through these specific parametrizations, significative properties and constructions *a priori* defined in the category of smooth domains, such as homotopy groups, fiber bundles, differential form. . . for examples.

What is a Diffeology?

The theory of diffeology begins with the idea of parametrization. The first step in this direction was taken by K.T. Chen in his paper on "Iterated Path Integrals" [Che77], but the parametrizations were defined on convex subsets of Euclidean domains. In 1980, in his paper on "Groupes différentiels" [Sou80], J.M. Souriau keeps the same axiomatic but with parametrizations defined on Euclidean domains, open subsets of Euclidean spaces. That is what founds today's diffeology:

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1. Definition [Parametrizations] *We call parameterization in a set* X *any map* $P\colon U \to X$ *where* U *is some Euclidean domain, that is, any open subset of an Euclidean space. If we want to be specific, we say that* P *is an n-parameterization when* U *is an open subset of* $\mathbf{R}^n$*. The set of all parameterizations in* X *is denoted by*

$$\mathrm{Params}(X) = \{P\colon U \to X \mid U \in \mathrm{Domains}(\mathbf{R}^n), n \in \mathbf{R}\}$$

Note that there is no condition of injectivity on P, and as we said, neither any topology precondition on X a priori.

2. Definition [Diffeology] *A diffeology on a set* X *is any subset*

$$\mathcal{D} \subset \mathrm{Params}(X)$$

that satisfies the following axioms:

1. *Covering:* $\mathcal{D}$ *contains the constant parameterizations.*
2. *Locality: Let* $P\colon U \to X$ *be parametrization. If, for all* $r \in U$*, there is an open neighbourhood* V *of* r *such that* $P \restriction V \in \mathcal{D}$*, then* $P \in \mathcal{D}$*.*
3. *Smooth compatibility: For all* $P\colon U \to X$ *in* $\mathcal{D}$*, for all* $F \in \mathcal{C}^\infty(V, U)$*, where* V *is a Euclidean domain,* $P \circ F \in \mathcal{D}$*.*

A space equipped with a diffeology is called a diffeological space. The elements of the diffeology $\mathcal{D}$ *of a diffeological space* X *are called the plots of (or in) the space.*[1]

3. Note Formally, a diffeological space is a pair $(X, \mathcal{D})$ where X is the underlying set and $\mathcal{D}$ the chosen diffeology, but we generally use a single letter to make the reading lighter. For example, we can use the letter $\mathcal{X}$ for the pair $(X, \mathcal{D})$, or anything else suggestive.

4. Example [Smooth Diffeology on $\mathbf{R}^n$] The first and foremost examples of diffeological spaces are the Euclidean domains. The plots of a domain $\mathcal{O}$ are all the smooth parametrizations $F\colon U \to \mathcal{O}$, where U is any other domain, of any dimension. We call this diffeology the *smooth diffeology*, or the *standard diffeology*. It is clear that the three axioms are satisfied. They have been chosen exactly because they are the fundamental properties which we want to replicate on sets, to define a *smooth structure*. Note that this

[1]There is a discussion about diffeology as a sheaf theory in [Igl87, Annex]. But we do not develop this formal point of view in general, because the purpose of diffeology is to minimize the technical tools in favour of a direct, more geometrical, intuition.

is not the only way to imagine smooth structure on sets. We may compare diffeology with other approaches in the future.

5. Definition [Smooth Maps] *After defining the structure, the main ingredient in diffeology id the notion of smooth map. Let* X *and* X' *be two diffeological spaces. A map* $f\colon \mathrm{X} \to \mathrm{X}'$ *is said to be smooth if (and only if) the composite with any plot in* X *is a plot in* X'*, which can be summarize by*

$$f \circ \mathcal{D} \subset \mathcal{D}',$$

where $\mathcal{D}$ *and* $\mathcal{D}'$ *denotes the respective diffeologies. The set of smooth maps from* X *to* X' *is denoted by*[2]

$$\mathcal{C}^\infty(\mathrm{X},\mathrm{X}') = \{f \in \mathrm{Maps}(\mathrm{X},\mathrm{X}') \mid f \circ \mathrm{P} \in \mathcal{D}',\ \forall \mathrm{P} \in \mathcal{D}\}.$$

6. Example [Smooth Parametrizations] The plots of a diffeology are the first examples of smooth maps. Indeed, let $\mathrm{P}\colon \mathrm{U} \to \mathrm{X}$ be a plot of X, let $\mathrm{F}\colon \mathrm{V} \to \mathrm{U}$ be a plot of the smooth diffeology on the domain U, that is $\mathrm{F} \in \mathcal{C}^\infty(\mathrm{V},\mathrm{U})$. Then, the composite $\mathrm{P} \circ \mathrm{F}$ is a plot of X, that is the third axiom of diffeology, the "smooth compatibiity". Therefore,

$$\mathcal{C}^\infty(\mathrm{U},\mathrm{X}) = \{\mathrm{P} \in \mathcal{D} \mid \mathrm{dom}(\mathrm{P}) = \mathrm{U}\}.$$

In particular, for the domains U and V, the notation $\mathcal{C}^\infty(\mathrm{U},\mathrm{V})$ understood in the usual sense and in the diffeological sense coincide. Hence, there is no need to introduce a special notation to denote the plots of a diffeological space.

7. Proposition [Category {Diffeology}] Consider X, X' and X'' be three diffeological spaces, with diffeologies $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$. Let $f \in \mathcal{C}^\infty(\mathrm{X},\mathrm{X}')$ and $f' \in \mathcal{C}^\infty(\mathrm{X}',\mathrm{X}'')$, then $f' \circ f \in \mathcal{C}^\infty(\mathrm{X},\mathrm{X}'')$. Therefore, diffeological spaces together with smooth maps define a category we denote by *{Diffeology}*.

The isomorphisms of this category are called *diffeomorphisms*, they are bijective maps, smooth as well as their inverse. In the case of a diffeomorphism $f \circ \mathcal{D} = \mathcal{D}'$. The set of diffeomorphisms from X to X' is denoted by $\mathrm{Diff}(\mathrm{X},\mathrm{X}')$.

8. Remark {Euclidean Domains} is a full subcategory of {Diffeology}, which is a strict extension of it on sets. We could call

[2] $\mathrm{Maps}(\mathrm{X},\mathrm{X}')$ denotes the set of all maps from X to X'.

such extensions "smooth categories", but that is just to identify the general context of the theory.

9. Definition [Comparing Diffeologies] *Inclusion defines a partial order in diffeology, called fineness. If $\mathcal{D}$ and $\mathcal{D}'$ are two diffeologies on a set* X*, one says that $\mathcal{D}$ is finer than $\mathcal{D}'$ if $\mathcal{D} \subset \mathcal{D}'$. We denote by*

$$\mathcal{D} \preceq \mathcal{D}'.$$

We say also that $\mathcal{D}'$ is coarser than $\mathcal{D}$. Every set has two extreme diffeology:

1. *The discrete diffeology where plots are only local constant parametrizations is the finest.*
2. *The coarse diffeology where plots are all the parametrizations is the coarsest.*

10. Proposition [Infimum and Supremum] *Diffeologies are stable by intersection. Any family $(\mathcal{D}_i)_{i\in\mathcal{I}}$ of diffeologies on a set* X *has an infimum, it is the intersection:*

$$\inf(\mathcal{D}_i)_{i\in\mathcal{I}} = \bigcap_{i\in\mathcal{I}} \mathcal{D}_i.$$

It is the coarsest diffeology contained in every element of the family $(\mathcal{D}_i)_{i\in\mathcal{I}}$.

The family $(\mathcal{D}_i)_{i\in\mathcal{I}}$ has also a supremum, it is the finest diffeology containing every element of the family:

$$\sup(\mathcal{D}_i)_{i\in\mathcal{I}} = \inf\{\mathcal{D}' \in \mathrm{Difflg(X)} \mid \forall i \in \mathcal{I}, \mathcal{D}_i \subset \mathcal{D}'\}$$

where Difflg(X) *denotes the set of all diffeologies on the set* X*.*

This partial order, fineness, makes the set of diffeologies on a set X*, a lattice. That is, a set such that every subset of diffeologies has an infimum and a supremum.*

As usual, if the infimum of a family belongs to the family, then it is called a minimum; and if the supremum belongs to the family, then it is called the maximum.

This property of being a lattice is very useful in defining diffeologies by means of properties. We shall see that most of diffeologies are the finest or coarsest diffeologies such that some property is satisfied. Because mimina and maxima are always distinguished elements in a set when they exist. The following examples will illustrate the point.

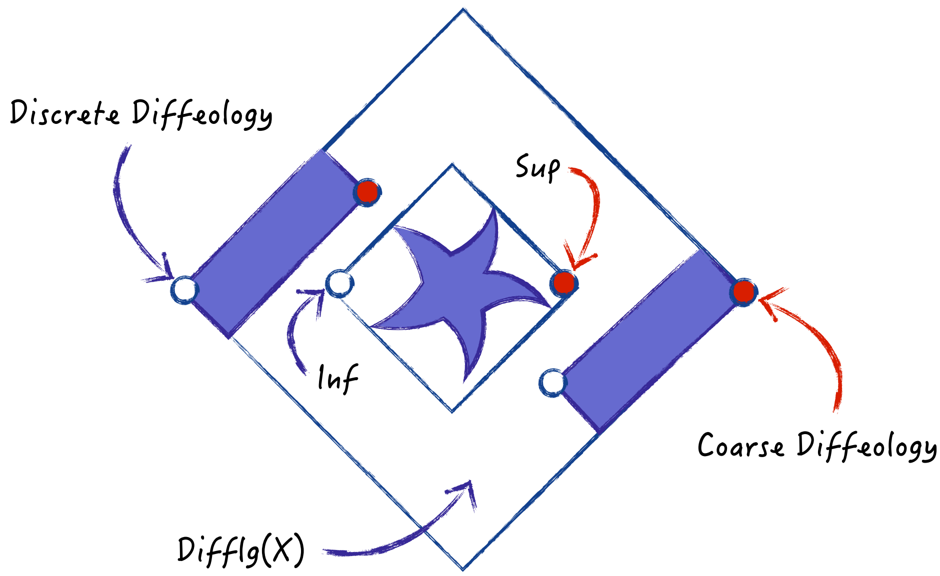

Figure 1. Comparing Diffeologies.

11. Proposition [Pushing Forward Diffeologies] *Let $f : \mathrm{X} \to \mathrm{X}'$ be a map, and let* X *be a diffeological space, with diffeology* $\mathcal{D}$*. Then, there exists a finest diffeology on* X' *such that* f *is smooth. It is called the pushforward of the diffeology of* X*. We denote it by* $f_*(\mathcal{D})$*.*

If f *is surjective, its plots are the parameterizations* P' *in* X' *that can be written* $\mathrm{Sup}_i\, f \circ \mathrm{P}_i$*, where the* P_i *are plots of* X *such that the* $f \circ \mathrm{P}_i$ *are compatible, that is, coincide on the intersection of their domains, and* Sup *denotes the smallest common extension of the family* $\{f \circ \mathrm{P}_i\}_{i \in \mathcal{I}}$*. Formally speaking, a parametrization* $\mathrm{P}' : \mathrm{U} \to \mathrm{X}$ *belongs to* $f_*(\mathcal{D})$*, if (and only if) there exists a family* $(\mathrm{P}_i)_{i \in \mathcal{I}}$ *of plots of* X*,* $\mathrm{P}_i \in \mathcal{D}$*, defined on an open covering* $(\mathrm{U}_i)_{i \in \mathcal{I}}$ *of* U*, such that* $f \circ \mathrm{P}_i = \mathrm{P}' \restriction_{\mathrm{U}_i}$*.*

$$f_*(\mathcal{D}) = \{\mathrm{P}' \in \mathrm{Params}(\mathrm{X}') \mid \exists \mathrm{P}_i \in \mathcal{D}, i \in \mathcal{I}, \mathrm{P}' = \mathrm{Sup}_i(f \circ \mathrm{P}_i)\}.$$

It is equivalent to say that for all $r \in \mathrm{U}$ *there exist an open neighbourhood* V *and a plot* Q *in* X*, such that* $\mathrm{P} \restriction \mathrm{V} = f \circ \mathrm{Q}$*.*

12. Definition [Subductions] *Let* $\pi : \mathrm{X} \to \mathrm{X}'$ *be a map between diffeological spaces. We say that* π *is a subduction if (and only if)*

(1) *The map* π *it is surjective.*

(2) *The pushforward of the diffeology of* X *coincides with the diffeology of* X'.

We can check that the composite of two subductions is again a subduction, that makes the subcategory {Subductions}.

Let's talk about quotients.

13. Definition [What a Quotient is and where does it lives] *There is sometimes an ambiguity about the construction of quotient sets that needs to be adressed once and for all. They are too often identified with some sets of representants in a way that can be regarded as arbitrary. Let us begin with a set* X *and an equivalence relation* $\sim$ *on* X*, that is, a binary relation which is reflexive, symetric and transitive. Let* $x \in \mathrm{X}$*, the equivalence class of* x*, denoted by* $\mathrm{class}(x)$ *is by definition the subset*

$$\mathrm{class}(x) = \{x' \mid x' \sim x\} \subset \mathrm{X}.$$

It lives then in the powerset

$$\mathfrak{P}(\mathrm{X}) = \{\mathrm{A} \mid \mathrm{A} \subset \mathrm{X}\},$$

set of all the subset of X*. The quotient set* Q *of* X *by* $\sim$*, denoted generally by* $\mathrm{X}/\sim$*, can always be regarded as a subset of* $\mathfrak{P}(\mathrm{X})$*:*

$$\mathrm{Q} = \{\mathrm{class}(x) \mid x \in \mathrm{X}\} \subset \mathfrak{P}(\mathrm{X}).$$

The canonical projection class *is then an application*

$$\mathrm{class}\colon \mathrm{X} \to \mathfrak{P}(\mathrm{X}) \quad \textit{with} \quad \mathrm{X}/\sim\, = \mathrm{class}(\mathrm{X}).$$

When it comes to quotient space in these lectures, it is always the way we look at it.

14. Definition [Quotienting Spaces] *Let* X *be a diffeological space and let* $\sim$ *be an equivalence relation on* X*. Let* $\mathrm{Q} = \mathrm{X}/\sim$ *be the quotient set. The pushforward* $\mathrm{class}_*(\mathcal{D})$ *on* Q *of the diffeology of* X*, by the canonical projection, is called the quotient diffeology. Equipped with the quotient diffeology,* Q *is called the quotient space of* X *by* $\sim$.

This is the first important property of the category {Diffeology}, it is closed by quotient, and we shall see not trivially closed.

Note that Considering the quotient $\mathrm{Q} = \mathrm{X}/\sim$ *for what it is, as described above, does not prevent us to identifiy it with some smooth* *representant $\mathcal{Q}$, according to the diagram where* class *and* π *are two subductions and f a bijection, and therefore a diffeomorphism.*

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{class}"'] \arrow[dr, "\pi"] & \\ Q \arrow[rr, "f"'] & & \mathcal{Q} \end{tikzcd}$$

<u>**15. Proposition**</u> [Pulling Back Diffeologies] *Let $f \colon X \to X'$ be a map, and let X' be a diffeological space with diffeology $\mathcal{D}'$. Then, there exists a coarsest diffeology on X such that f is smooth. It is called the pullback of the diffeology of X'. We denote it by $f^*(\mathcal{D}')$. Its plots are the parameterizations P in X such that $f \circ P$ is a plot of X'.*

$$f^*(\mathcal{D}') = \{P \in \mathrm{Params}(X) \mid f \circ P \in \mathcal{D}'\}.$$

<u>**16. Definition**</u> [Induction] *Let X and X' be be two diffeological spaces and $f \colon X \to X'$ be a map. We say that f is an induction if (and only if):*

(1) *The map f is injective.*
(2) *The pullback $f^*(\mathcal{D}')$ of the diffeology of X' coincide with the diffeology $\mathcal{D}$ of X.*

<u>**17. Definition**</u> [Subset Diffeology] *Pulling back diffeologies gives to any subset $A \subset X$, where X is a diffeological space, a subset diffeology $j^*(\mathcal{D})$, where $j \colon A \to X$ is the inclusion and $\mathcal{D}$ is the diffeology of X. A subset equipped with the subset diffeology is called a diffeological subspace. Plots of the subset diffeology are simply plots of X taking their values in A.*

This is a second important property of the category {Diffeology}, <u>it is closed by inclusion</u>.

<u>**18. Remark**</u> [Discrete Subspaces] *We have defined discrete diffeological spaces as diffeological spaces equipped with the discrete diffeology. That is, the diffeology consisting in locally constant parametrization. It happens that subspaces of diffeological spaces inherit the discret diffeology.*

The <u>best example</u> is $\mathbf{Q} \subset \mathbf{R}$ which equipped with the subset diffeology is discrete. That corresponds perfectly to what what we understand to be discrete. But we have to be careful because discrete *in diffeology does not coincide always with discrete in topology, in particular for* $\mathbf{Q}$ *in* $\mathbf{R}$ *that topologists do not consider as discrete, which is a little bit exagerate. But anyway, we better specify in which sense we used the word "discrete" when there is a doubt, to avoid confusion with topologists.*

Proof. Consider a plot $\mathrm{P}\colon \mathrm{U} \to \mathbf{R}$ but with values in $\mathbf{Q}$. Let $r, r' \in \mathrm{U}$ and $\gamma\colon t \mapsto \mathrm{P}(tr' + (1-t)r)$, defined on a small open neighbourhood of $[0,1]$. Then, γ is a plot in $\mathbf{R}$ and therefore continuous. Let $q = \gamma(0) = \mathrm{P}(r)$ and $q' = \gamma(1) = \mathrm{P}(r')$, by hypothesis $q, q' \in \mathbf{Q}$. Since γ is continuous, according to the *intermediate values theorem*,[3] γ takes every values between q and q'. if $q \neq q'$, there exists always an irrational number in between, which cannot be because P takes its values only in $\mathbf{Q}$. Therefore $\gamma(0) = \gamma(1)$, that is, $\mathrm{P}(r) = \mathrm{P}(r')$. The plot P is then locally constant since it will be constant on every small ball around every $r \in \mathrm{U}$. ☐

19. Example [The Circle] Let S^1 be the circle, defined by

$$\mathrm{S}^1 = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}.$$

It is equivalent the the set $\mathrm{U}(1)$ of complex numbers of modulus 1, $z = x + iy \in \mathrm{U}(1)$ means[4] $\bar{z}z = x^2 + y^2 = 1$. The plots of S^1, as a diffeological subspace of $\mathbf{R}^2$, are just the pairs $r \mapsto (x(r), y(r))$ of smooth real functions defined on some Euclidean domain U, such that for all r in U, $x(r)^2 + y(r)^2 = 1$. In particular, for $r = \theta \in \mathbf{R}$:

Proposition. *The projection, from* $\mathbf{R}$ *to* S^1*,*

$$\pi\colon \theta \mapsto (\cos(\theta), \sin(\theta))$$

is a subduction.

Indeed, For any θ, one of the derivative $x'(\theta) = -\sin(\theta)$ or $y'(\theta) = \cos(\theta)$ does not vanishes, since $x'(\theta)^2 + y'(\theta)^2 = 1$. Assume that $x'(\theta_0) \neq 0$, then according to the inverse function theorem,[5] there exists a small interval $\mathcal{I}$ centered at θ_0 such that $\varphi = \cos \restriction \mathcal{I}$ is a diffeomorphism onto its image, an open intervall that we denote by $\mathcal{J} = \cos(\mathcal{I})$. So, let $r \mapsto (x(r), y(r))$ be a plot in S^1 and assume that $x(r_0) = \cos(\theta_0)$ and $\sin(\theta_0) \neq 0$. The preimage $\mathcal{O} = x^{-1}(\mathcal{J})$

[3] https://www.wikiwand.com/en/Intermediate_value_theorem

[4] $\bar{z}$ or z^* denote the conjugate $x - iy$ of $z = x + iy$.

[5] https://www.wikiwand.com/en/Inverse_function_theorem

is an open subset in U, since $r \mapsto x(r)$ is smooth. Then, let

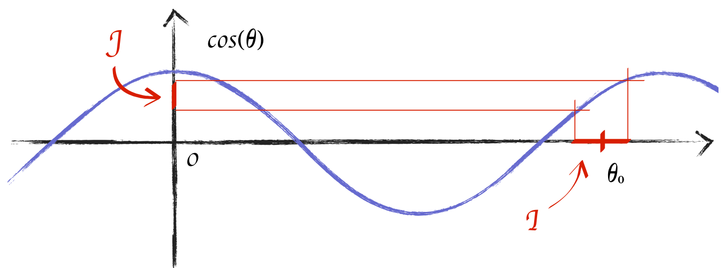

Figure 2. Function Cosinus

$\theta(r) = \varphi^{-1}(x(r))$, defined on $\mathcal{O}$, and for all $r \in \mathcal{O}$, $x(r) = \cos(\theta(r))$ and $y(r) = \sin(\theta(r))$. The map $r \mapsto \theta(r)$ is a *local lifting*, along the projection π, of the plot $r \mapsto (x(r), y(r))$.

There exists then a bijection $f\colon \mathbf{R}/2\pi\mathbf{Z} \to S^1$

$$f(\mathrm{class}(x))) = (\cos(x), \sin(x)),$$

and since $\pi\colon x \mapsto (\cos(x), \sin(x))$ is a subduction, f is a diffeomorphisme satisfying $f \circ \mathrm{class} = \pi$, and therefore S^1 is a smooth repreesentant of $\mathbf{R}/2\pi\mathbf{Z}$.

On the other hand, we can define also

$$\sigma\colon x \mapsto \frac{x}{2\pi} - \left[\frac{x}{2\pi}\right],$$

where the bracket denotes the integer part. Then $\sigma(x) = \sigma(x')$ if and only if $x' = x + 2\pi k$, with $k \in \mathbf{Z}$. Set theoretically, the interval $[0, 2\pi[= \sigma(\mathbf{R}) \subset \mathbf{R}$ represent $\mathbf{R}/2\pi\mathbf{Z}$, but equipped with the subset diffeology, σ is discontinuous and it cannot be a smooth representation of the quotient $\mathbf{R}/2\pi\mathbf{Z}$. Of course we can push forward the

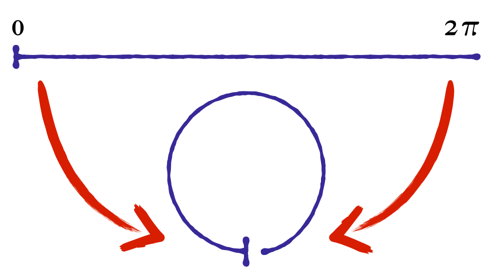

Figure 3. Closing the circle by push forward

smooth diffeology of $\mathbf{R}$ onto the segment $[0, 2\pi[$, but that would consist to glue the end of the interval near 2π to the origin 0, and so to reconstruct the circle as shown in Figure 3.

<u>20. Definition</u> [Strict Maps] *The last example of the projection* $\pi\colon \theta \mapsto (\cos(\theta), \sin(\theta))$, *from* $\mathbf{R}$ *to* $S^1 \subset \mathbf{R}^2$ *suggest the definition of a new kind of map, the strict maps.*

Every map $f\colon X \to Y$. *defines the following commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "class"'] & X' \\ X/f \arrow[r, "\varphi"'] & f(X) \arrow[u, "j"'] \end{tikzcd}$$

where

(1) X/f *denotes the quotien by the relation* $f(x) = f(x')$.
(2) *The map* $j\colon f(X) \to X'$ *is the inclusion.*
(3) *The map* $\varphi\colon X/f \to f(X)$ *is defined by* $\varphi(\mathrm{class}(x)) = f(x)$.

Let now X *and* X' *be two diffeological spaces:*

We say that f *is strict if* φ *is a diffeomorphism when* X/f *is equipped with the quotient diffeology and* $f(X)$ *with the subset diffeology.*

In particular, the map π *above is strict. Strict maps realize quotient as subset of other diffeological spaces.*

<u>21. Definition</u> [Direct Sum Diffeology] *Consider a family* $(X_i)_{i\in\mathcal{J}}$ *of diffeological spaces, for any family of indices. The direct sum, or simply the sum of the (elements of) family is defined by*

$$\coprod_{i\in\mathcal{J}} X_i = \{(i,x) \mid i \in \mathcal{J} \text{ and } x \in X_i\}.$$

Proposition. *There exists on the sum* $X = \coprod_{i\in\mathcal{J}} X_i$ *a finest diffeology such that every injections*

$$j_i\colon X_i \to X \text{ defined by } j_i(x) = (i,x)$$

is smooth. The plots of this diffeology are the parametrizations $r \mapsto (i(r), x(r))$ *such that* $r \mapsto i(r)$ *is locally constant. In other words, a plot is locally with values in only one component of the sum.*

Actually, the injections j_i *are inductions. The space* X *is called the diffeological sum of the* X_i.

Now, the category {Diffeology} is also <u>*closed by sum*</u>.

22. Examples [Some Diffeological Sums] Consider

$$\mathcal{E} = \coprod_{x \in \mathbf{R}} \mathbf{R}.$$

Every element of E is a pair $(x, y) \in \mathbf{R}^2$, but of course the diffeology of the sum is not the smooth diffeology of $\mathbf{R}^2$. Indeed, a plot in E write always locally $r \mapsto (x, y(r))$, where x is constant and $r \mapsto y(r)$ is smooth. We could call this diffeology the "comb diffeology" of $\mathbf{R}^2$.

Another example, bigger: let $n \in \mathbf{N}$, let $x \in \mathbf{R}^n$ and $\varepsilon \in]0, \infty[$, let $\mathcal{B}(x, \varepsilon)$ be the open ball in $\mathbf{R}^n$ centered in x with radius ε. The *world of balls* would be the sum

$$\mathcal{X} = \coprod_{n \in \mathbf{N}} \coprod_{\substack{x \in \mathbf{R}^n \\ \varepsilon \in]0,\infty[}} \mathcal{B}(x, \varepsilon).$$

In diffeology, don't be afraid to think big :-)

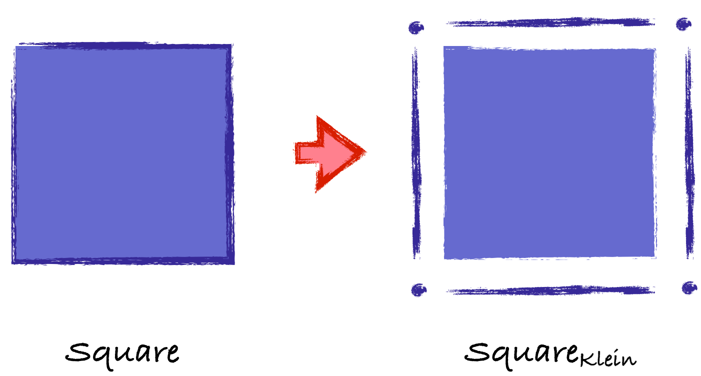

Figure 4. Klein's exploded view of the square

One can also access an aspect of the structure of a diffeological space by means of sum of parts: consider the group of diffeomorphisms Diff(X) a a diffeological space X. It decomposes the space into a set of orbits $\mathcal{O} \in \mathrm{X}/\operatorname{Diff}(\mathrm{X})$. Then we reconstruct a finer diffeological space by considering the sum of the orbits, a exploded view of the space reavealing its singular structure. We can call it the Klein's exploded view.

$$\mathrm{X}_{\text{Klein}} = \coprod_{\mathcal{O} \in \mathrm{X}/\operatorname{Diff}(\mathrm{X})} \mathcal{O}.$$

23. Definition [Product Diffeology] *Consider a family* $(\mathrm{X}_i)_{i\in \mathcal{I}}$ *of diffeological spaces, for any family of indices. Let* pr_1 *be the first projection of the direct sum of the* X_i*, that is*

$$\mathrm{pr}_1 \colon \coprod_{i\in\mathcal{I}} \mathrm{X}_i \to \mathcal{I} \quad \textit{with} \quad \mathrm{pr}_1(i,x) = i.$$

The product of the X_i *is defined as the set of section of* pr_1*, that is*

$$\prod_{i\in\mathcal{I}} \mathrm{X}_i = \left\{ \mathrm{x} \colon \mathcal{I} \to \coprod_{i\in\mathcal{I}} \mathrm{X}_i \mid \mathrm{pr}_1 \circ \mathrm{x} = 1_{\mathcal{I}} \right\}$$

Let $\mathrm{X} = \prod_{i\in\mathcal{I}} \mathrm{X}_i$*. An element* $x \in \mathrm{X}$ *can be denoted as* $x = (x_i)_{i\in\mathcal{I}}$*, where* $x_i = x(i)$*. The projection* π_i *on the i-th factor* X_i *is defined by*

$$\pi_i(x) = x_i.$$

Proposition. *There exists on the product* X *a coarsest diffeology such that each projection* π_i *is smooth. Equiped with this diffeology,* X *is called the diffeological product, or simply the product, of the* X_i*.*

Actually, the projections are subductions. A parametrization of the product writes $r \mapsto (x_i(r))_{i\in\mathcal{I}}$*, where the* x_i *are plots of the* X_i*.*

Now, the category {Diffeology} is also closed by product.

24. Examples [Some Diffeological Products] The main example here is the power $\mathbf{R}^n$ which is the product of n copies of $\mathbf{R}$ equiped with the smooth diffeology.

A special and remarkable feature of diffeology is that the set of the smooth maps between diffeological spaces carries a natural diffeology:

25. Definition [Functional Diffeology] *Let* X *and* X′ *be two diffeological spaces. There exists on* $\mathcal{C}^\infty(\mathrm{X},\mathrm{X}')$ *a coarsest diffeology such that the evaluation map*

$$\mathrm{ev} \colon \mathcal{C}^\infty(\mathrm{X},\mathrm{X}') \times \mathrm{X} \to \mathrm{X}' \quad \textit{defined by} \quad \mathrm{ev}(f,x) = f(x),$$

is smooth. Thus diffeology is called the functional diffeology.

The plots of that diffeology are the parametrizations $r \mapsto f_r$*, defined on some domain* U*, such that*

$$(r,x) \mapsto f_r(x) \quad \textit{from} \quad \mathrm{U}\times\mathrm{X} \quad \textit{to} \quad \mathrm{X}'$$

is smooth. That means that for every plot $s \mapsto x_s$ *in* X*, defined on some domain* V*, the parametrization* $(r, s) \mapsto f_r(x_s)$*, defined on* $\mathrm{U} \times \mathrm{V}$ *is a plot of* X'*.*

Now, let X*,* X' *and* X'' *be three diffeological space. Then,*

(1) *The product*

$$\circ(f, g) = g \circ f,$$

defined on $\mathcal{C}^\infty(\mathrm{X}, \mathrm{X}') \times \mathcal{C}^\infty(\mathrm{X}', \mathrm{X}'')$ *to* $\mathcal{C}^\infty(\mathrm{X}, \mathrm{X}'')$*, is smooth.*

(2) *The spaces* $\mathcal{C}^\infty(\mathrm{X}, \mathcal{C}^\infty(\mathrm{X}', \mathrm{X}''))$ *and* $\mathcal{C}^\infty(\mathrm{X} \times \mathrm{X}', \mathrm{X}'')$ *are diffeomorphic. The diffeomorphism* φ *consists in the game of parenthesis, for all* $f \in \mathcal{C}^\infty(\mathrm{X}, \mathcal{C}^\infty(\mathrm{X}', \mathrm{X}''))$

$$\varphi(f) \colon (x, x') \mapsto f(x)(x').$$

We say that the category {Diffeology} is Cartesian closed.

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