DIFFEOLOGY AND NON-COMMUTATIVE GEOMETRY

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ref. http://math.huji.ac.il/~piz/documents/ShD-lect-DNCG.pdf

In this lecture we show how we can build a bridge between some diffeological spaces and noncommutative geometry, such that diffeomorphic spaces give Morita equivalent $\mathbf{C}^*$-algebras. These spaces are orbifolds, generalized by quasifolds, but regarded as diffeological spaces.

This lecture is based on two papers: "Noncommutative Geometry and Diffeology: The Case of Orbifolds" [IZL17], and "Quasifolds, Diffeology and Noncommutative Geometry" [IZP20]. The basic concepts have been introduced in the first paper and its results extended in the second one.

The second paper is a generalization of the first one. The concept of orbifold has been introduced by Ishiro Stake [IS56, IS57], the notion of quasifold is a generalization introduced by Elisa Prato. In the paper "Orbifolds as Diffeologies" [IKZ10] we include the Satake original definition of V-manifold in the category {Diffeology}. In the second paper we give a diffeological definition of "Quasifolds" that fits correctly the E. Prato original definition [EP01]. Thus, quasifolds are included also in the category {Diffeology}. By this inclusive diffeological approach it is obvious that the Elisa Prato quasifolds are a generalization of the Satake V-Manifolds, renamed by Thurston as Orbifolds, which is not necessarily obvious with the specific definitions. We have then this particular series of full

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subcategories:

$$\{\text{Manifolds}\} \preceq \{\text{Orbifolds}\} \preceq \{\text{Quasifolds}\} \preceq \{\text{Diffeology}\}$$

Then, we associate a $\mathbf{C}^*$-algebra to the orbifold, or quasifold, such that diffeomorphic spaces give Morita equivalent $\mathbf{C}^*$-algebra, which is the minimum required.

I insist on the fact that the process of associating a $\mathbf{C}^*$-algebra to these categories of spaces is not tautological, as it can be with a direct algebraic approach which contains already in the definition of the category, by groupoids or stacks, this particular fact that equivalent structures (groupoids or stacks) give Morita equivalent $\mathbf{C}^*$-algebras. We start here a floor below, with the geometry of the space, that is, its diffeology.

Now, the plan of the talk:

(1) We define orbifolds and quasifolds as subcategories of diffeological spaces.
(2) We introduce charts and atlases that define the structure.
(3) We associate to every atlas a strict generating family and its nebula.
(4) We associate a groupoid over the space with the nebula of each atlas, that captures the local structure point by point.
(5) We show that two different atlases give to equivalent groupoids, in the algebraic sense, which is the minimum required: the groupoid [its class] is a diffeological invariant of the space.
(6) We show that theses groupoids are etale and Hausdorff.
(7) According to Jean Renault's construction, we associate a $*$-algebra, and a $\mathbf{C}^*$-algebra by completion, to each of these groupoids.
(8) Using a central theorem from Muhly-Renault-Willian, we show that two different atlases give Morita equivalent $\mathbf{C}^*$-algebras, which is the minimum expected.
(9) And finally we give two examples: the $\mathbf{C}^*$-algebra associated with the irrational torus T_α, and the $\mathbf{C}^*$-algebra associated with the quotient $\mathbf{R}/\mathbf{Q}$.

I would like to finish this introduction by recalling that the development of diffeology, starting in 1983 with the example of the irrational torus [PDPI83], was deeply motivated by the introduction of noncommutative geometry and the treatment of the quasiperiodic potential in quantum mechanics. It was time to close the loop, at least for now.

What are Orbifolds and Quasifolds

1. Definition [The Orbifolds] An orbifold is a diffeological space that is locally diffeomorphic to some quotient $\mathbf{R}^n/\Gamma$, at each point, where Γ is a finite subgroup of $\mathrm{GL}(n, \mathbf{R})$. The group Γ may change from point to point.

Example 1. The quotient space $\mathcal{Q}_m = \mathbf{C}/\mathcal{U}_m$, with the group of roots of unity $\mathcal{U}_m = \{\exp(2i\pi k/m) \mid k = 1 \dots m\}$, is a *cone-orbifold*.

Example 2. The product $[\mathbf{R}/\{\pm 1\}]^n$ is a *corner-orbifold*.

Example 3. The *waterdrop* is the sphere $\mathrm{S}^2 \subset \mathbf{R}^3 \simeq \mathbf{C}\times\mathbf{R}$, equipped with the following diffeology:

Let $\mathrm{N} = (0, 1)$ be the North pole. The following set of parametrizations ζ defines an orbifold diffeology on S^2 with all points regular, except the north pole whose structure group is the cyclic group $\mathcal{U}_m$. This construction is summarized by the Figure 1.

Let U be an Euclidean domain,

$$\zeta\colon \mathrm{U} \to \mathrm{S}^2 \quad \text{with} \quad \zeta(r) = \begin{pmatrix} z(r) \\ t(r) \end{pmatrix}, \quad \text{and} \quad |z(r)|^2 + t(r)^2 = 1,$$

such that, for all $r_0 \in \mathrm{U}$,

(1) if $\zeta(r_0) \neq \mathrm{N}$, then there exists a small ball $\mathcal{B}$ centered at r_0 such that $\zeta \restriction \mathcal{B}$ is smooth.
(2) If $\zeta(r_0) = \mathrm{N}$, then there exist a small ball $\mathcal{B}$ centered at r_0 and a smooth parametrization z in $\mathbf{C}$ defined on $\mathcal{B}$ such that, for all $r \in \mathcal{B}$,

$$\zeta(r) = \frac{1}{\sqrt{1 + |z(r)|^{2m}}} \begin{pmatrix} z(r)^m \\ 1 \end{pmatrix}.$$

2. Definition [The Quasifolds] A quasifold is a diffeological space that is locally diffeomorphic to $\mathbf{R}^n/\Gamma$, where Γ is a countable subgroup of the affine group $\mathrm{Aff}(\mathbf{R}^n)$, $x \mapsto \mathrm{A}x + \mathrm{B}$, with $\mathrm{A} \in \mathrm{GL}(n, \mathbf{R})$ and $\mathrm{B} \in \mathbf{R}^n$. We see clearly that diffeological quasifolds are a generalization of orbifolds.

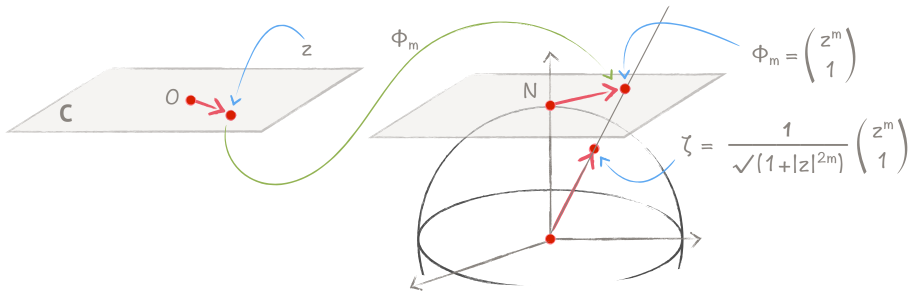

Figure 1. The Waterdrop as a diffeological orbifold.

Example 1. The first example of quasifold is the *irrational torus*, the first special diffeological space studied for itself in 1983 [PDPI83], which is at the source of the development of diffeology:

$$\mathrm{T}_\alpha = \mathrm{T}^2/\Delta_\alpha \simeq \mathbf{R}/\mathbf{Z} + \alpha\mathbf{Z},$$

where $\alpha \in \mathbf{R} - \mathbf{Q}$, $\Delta_\alpha \subset \mathrm{T}^2$ is the projecttion of the line $y = \alpha x$, and $\mathrm{T}^2 = [\mathbf{R}/\mathbf{Z}]^2$.

Example 2. The second example $\mathfrak{G}$ (for Geodesics) is inspired by the first one. The lines of slope α are the geodesic trajectories on the torus T^2 of slope α. The set of all geodesic trajectories of the torus T^2 are bundled over S^1, they are the projections on T^2 of all the affine lines in $\mathbf{R}^2$ directed by a unit vector $u \in \mathrm{S}^1$. Over the vector u we have $\mathfrak{G}_u$, the torus T_u which is rational or irrational depending if the line $\mathbf{R}u$ cut or not the lattice $\mathbf{Z}^2$ elsewhere than in 0.

The set $\mathfrak{G}$ of the geodesic trajectories of the torus T^2 is the quotient of the space of geodesic trajectories of the plane $\mathbf{R}^2$ by the action of $\mathbf{Z}^2$. The space of geodesic trajectories of the plane is equivalent to the cylinder

$$\mathrm{TS}^1 = \{(u, r) \in \mathrm{S}^1 \times \mathbf{R}^2 \mid \langle u, r\rangle = 0\}.$$

The mapping

$$(u, r) \mapsto \big(u, \rho = \langle r, \mathrm{J}u\rangle\big) \quad \text{with} \quad \mathrm{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

identifies

$$\mathrm{TS}^1 \simeq \mathrm{S}^1 \times \mathbf{R}.$$

The action of $\mathbf{Z}^2$ on TS^1 is given by

$$\begin{pmatrix} m \\ n \end{pmatrix} : (u, r) \mapsto \left(u, r + [\mathbf{1} - u\bar{u}] \begin{pmatrix} m \\ n \end{pmatrix}\right).$$

Translated on (u, ρ) that gives:

$$\begin{pmatrix} m \\ n \end{pmatrix} : (u, \rho) \mapsto \left(u, \rho + \left\langle \begin{pmatrix} m \\ n \end{pmatrix}, \mathrm{J}u \right\rangle\right).$$

That is,

$$\begin{pmatrix} m \\ n \end{pmatrix} : (u, \rho) \mapsto (u, \rho + nu_x - mu_y) \quad \text{with} \quad u = \begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$

In other words, $\mathfrak{G}$ is diffeomorphic to the quotient of $\mathbf{R} \times \mathbf{R}$ by the relation

$$(t, \rho) \sim \big(t + \ell, \rho + n\cos(2\pi t) - m\sin(2\pi t)\big) \quad \text{with} \quad \ell, n, m \in \mathbf{Z}.$$

3. Definitions [Charts, Atlases and Strict Generating Families] Since orbifolds are a full subcategory of quasifolds in diffeology, what is defined for orbifolds in the following applies immediately for quasifolds. So, paraphrasing the definition, a diffeological space X is a quasifold if, for all $x \in \mathrm{X}$, there exist a countable subgroup $\Gamma \subset \mathrm{Aff}(\mathbf{R}^n)$, and a local diffeomorphism φ from $\mathbf{R}^n/\Gamma$ to X, defined on some open subset $\mathrm{U} \subset \mathbf{R}^n/\Gamma$, such that $x \in \varphi(\mathrm{U})$. The subset U is open for the D-topology, that is in this case, the quotient topology by the projection map $\mathrm{class}\colon \mathbf{R}^n \to \mathbf{R}^n/\Gamma$.

Definition: *Any such diffeomorphism is called a chart. A set of charts $\mathcal{A}$, covering* X, *is called an atlas.*

Let $f\colon \mathrm{U} \to \mathrm{X}$ be a chart, then U is an open subset of some $\mathbf{R}^n/\Gamma$ for the D-Topology. Thus $\tilde{\mathrm{U}} = \mathrm{class}^{-1}(\mathrm{U})$ is a Γ-invariant open subset in $\mathbf{R}^n$. Hence, $\mathrm{F} = f \circ \mathrm{class}$ is a plot of X. We shall call it the *strict lifting* of f.

Definition: *Let $\mathcal{F}$ be the set of strict liftings* $\mathrm{F} = f \circ \mathrm{class}$, *where* $f\colon \mathrm{U} \to \mathrm{X}$ *runs over the charts in* $\mathcal{A}$. *Then,* $\mathcal{F}$ *is a generating family of* X. *We shall say that* $\mathcal{F}$ *is the strict generating family associated with* $\mathcal{A}$.

Structure Groupoids

4. Lemma [Lifting the identity] Let $\mathcal{Q} = \mathbf{R}^n/\Gamma$ where Γ is a countable subgroup of $\mathrm{Aff}(\mathbf{R}^n)$. Consider a local smooth map F from $\mathbf{R}^n$ to itself, such that

$$\mathrm{class} \circ \mathrm{F} = \mathrm{class}\,.$$

In other words, F is a local lifting of the identity on $\mathcal{Q}$. Then,

Theorem. F *is locally equal to some group action*

$$\mathrm{F}(r) =_{\mathrm{loc}} \gamma \cdot r = \mathrm{A}r + b,$$

where $\gamma = (\mathrm{A}, b) \in \Gamma$, *for some* $\mathrm{A} \in \mathrm{GL}(\mathbf{R}^n)$ *and* $b \in \mathbf{R}^n$.

Proof. Let us assume first that F is defined on an open ball $\mathcal{B}$. Then, for all r in the ball, there exists a $\gamma \in \Gamma$ such that $\mathrm{F}(r) = \gamma \cdot r$. Next, for every $\gamma \in \Gamma$, let

$$\mathrm{F}_\gamma \colon \mathcal{B} \to \mathbf{R}^n \times \mathbf{R}^n \quad \text{with} \quad \mathrm{F}_\gamma(r) = (\mathrm{F}(r), \gamma \cdot r).$$

Let $\Delta \subset \mathbf{R}^n \times \mathbf{R}^n$ be the diagonal and let us consider

$$\Delta_\gamma = \mathrm{F}_\gamma^{-1}(\Delta) = \{r \in \mathcal{B} \mid \mathrm{F}(r) = \gamma \cdot r\}.$$

Lemma 1. There exist at least one $\gamma \in \Gamma$ such that the interior $\mathring{\Delta}_\gamma$ is non-empty.

◀ Indeed, since F_γ is smooth (thus continuous), the preimage Δ_γ by F_γ of the diagonal is closed in $\mathcal{B}$. However, the union of all the preimages $\mathrm{F}_\gamma^{-1}(\Delta)$ — when γ runs over Γ — is the ball $\mathcal{B}$. Then, $\mathcal{B}$ is a countable union of closed subsets. According to Baire's theorem, there is at least one γ such that the interior $\mathring{\Delta}_\gamma$ is not empty. ▶

Lemma 2. The union $\mathring{\Delta}_\Gamma = \cup_{\gamma\in\Gamma}\mathring{\Delta}_\gamma$ is an open dense subset of $\mathcal{B}$.

◀ Indeed, let $\mathcal{B}' \subset \mathcal{B}$ be an open ball. Let us denote with a prime the sets defined above but for $\mathcal{B}'$. Then, $\Delta'_\gamma = (\mathrm{F}_\gamma \restriction \mathcal{B}')^{-1}(\Delta) = \Delta_\gamma \cap \mathcal{B}'$, and then $\mathring{\Delta}'_\gamma = \mathring{\Delta}_\gamma \cap \mathcal{B}'$. Thus, $\mathcal{B}' \cap \mathring{\Delta}_\Gamma = \mathcal{B}' \cap (\cup_{\gamma\in\Gamma}\mathring{\Delta}_\gamma) = \cup_{\gamma\in\Gamma}\mathring{\Delta}'_\gamma$, which is not empty for the same reason that $\cup_{\gamma\in\Gamma}\mathring{\Delta}_\gamma$ is not empty. Therefore, $\mathring{\Delta}_\Gamma$ is dense. ▶

Hence, there exists a subset of Γ, indexed by a family $\mathcal{J}$, for which $\mathcal{O}_i = \mathring{\Delta}_{\gamma_i} \subset \mathcal{B}$ is open and non-empty, $\cup_{i\in\mathcal{J}}\mathcal{O}_i$ is an open dense subset of $\mathcal{B}$, and $\mathrm{F} \restriction \mathcal{O}_i \colon r \mapsto \mathrm{A}_i r + b_i$, where $(\mathrm{A}_i, b_i) \in \mathrm{Aff}(\mathbf{R}^n)$. Since F is smooth, the first derivative $\mathrm{D}(\mathrm{F})$ restricted to $\mathcal{O}_i$ is equal to A_i, and then the second derivative $\mathrm{D}^2(\mathrm{F}) \restriction \mathcal{O}_i = 0$, for all $i \in \mathcal{J}$.

Then, since $D^2(F) = 0$ on an open dense subset of $\mathcal{B}$, $D^2(F) = 0$ on $\mathcal{B}$, that is $D(F)(r) = A$ for all $r \in \mathcal{B}$, with $A \in \mathrm{GL}(n, \mathbf{R})$. Now, the map $r \mapsto F(r) - Ar$, defined on $\mathcal{B}$, is smooth. But, restricted on $\mathcal{O}_i$ it is equal to b_i. Its derivative vanishes on the open dense subset $\cup_{i \in \mathcal{J}} \mathcal{O}_i$ and thus vanishes on $\mathcal{B}$. Therefore, $F(r) - Ar = b$ on the whole $\mathcal{B}$, for $b \in \mathbf{R}^n$ and $F(r) = Ar + b$ on $\mathcal{B}$, with $\gamma = (A, b) \in \Gamma$. $\square$

5. Construction [Building the groupoid of a quasifold.] Let X be a quasifold, let $\mathcal{A}$ be an atlas and let $\mathcal{F}$ be the strict generating family over $\mathcal{A}$. We denote by $\mathcal{N}$ the nebula of $\mathcal{F}$, that is, the sum of the domains of its elements:

$$\mathcal{N} = \coprod_{F \in \mathcal{F}} \mathrm{dom}(F) = \{(F, r) \mid F \in \mathcal{F} \text{ and } r \in \mathrm{dom}(F)\}.$$

The *evaluation map* is the natural subduction

$$\mathrm{ev} \colon \mathcal{N} \to X \quad \text{with} \quad \mathrm{ev}(F, r) = F(r).$$

The *structure groupoid* of the quasifold X, associated with the atlas $\mathcal{A}$, is defined as the subgroupoid $\mathbf{G}$ of germs of local diffeomorphisms of $\mathcal{N}$ that project to the identity of X along ev. That is,

$$\begin{cases} \mathrm{Obj}(\mathbf{G}) &=& \mathcal{N}, \\ \mathrm{Mor}(\mathbf{G}) &=& \{ \mathrm{germ}(\Phi)_\nu \mid \Phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathcal{N}) \text{ and } \mathrm{ev} \circ \Phi = \mathrm{ev} \restriction \mathrm{dom}(\Phi)\}. \end{cases}$$

The set $\mathrm{Mor}(\mathbf{G})$ is equipped with the functional diffeology inherited by the full groupoid of germs of local diffeomorphisms.[1] Note that, given $\Phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathcal{N})$ and $\nu \in \mathrm{dom}(\Phi)$, there exist always two plots F and F′ in $\mathcal{F}$ such that $\nu = (F, r)$, with $r \in \mathrm{dom}(F)$, and a local diffeomorphism φ of $\mathbf{R}^n$, defined on an open ball centered in r, such that $\mathrm{dom}(\varphi) \subset \mathrm{dom}(F)$, $\varphi = \Phi \restriction \{F\} \times \mathrm{dom}(F)$ and $F' \circ \varphi = F \restriction \mathrm{dom}(\varphi)$. That is summarized by the diagram:

$$\begin{tikzcd} \mathrm{dom}(F) \supset \mathrm{dom}(\varphi) \arrow[rr, "\varphi"] \arrow[dr, "F"'] & & \mathrm{dom}(F') \arrow[dl, "F'"] \\ & X & \end{tikzcd}$$

[1] That is defined precisely in the paper on Orbifolds and $\mathbf{C}^*$-algebras [IZL17, §2 & 3]

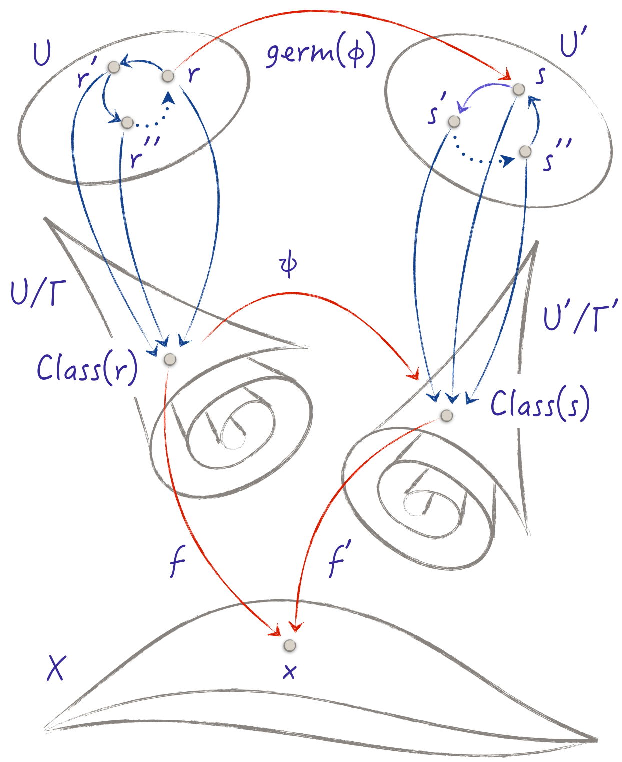

Figure 2. The three levels of a quasifold.

Note. According to the theorem in §4, the local diffeomorphisms, defined on the domain of a generating plot, and lifting the identity of the quasifold, are just the elements of the structure group associated with the plot.

We can legitimately wonder what is the point of involving general germs of local diffeomorphisms, if we merely end up with the structure group we could have began with? The reason is that: *The structure groups connect the points of the nebula that project on a same point of the quasifold, only when they are inside the same domain*. They cannot connect the points of the nebula that project on the same point of the quasifold but belonging to different domains, with maybe different structure groups. This is the reason why we cannot avoid the use of germs of local diffeomorphisms in the nebula, to begin with. That situation is illustrated in Figure 2.

6. Theorem [Lifting local diffeomorphisms] Let $\mathcal{Q} = \mathbf{R}^n/\Gamma$ and $\mathcal{Q}' = \mathbf{R}^{n'}/\Gamma'$, where $\Gamma \subset \mathrm{Aff}(\mathbf{R}^n)$ and $\Gamma' \subset \mathrm{Aff}(\mathbf{R}^{n'})$ are countable subgroups. Then,

Theorem. *Every local smooth lifting $\tilde{f}$ of any local diffeomorphism f of $\mathcal{Q}$ is necessarily a local diffeomorphism. In particular $n = n'$. Moreover, let $x \in \mathrm{dom}(f)$, $x' = f(x)$, $r, r' \in \mathbf{R}^n$ be such that $\mathrm{class}(r) = x$ and $\mathrm{class}(r') = x'$. Then, the local lifting $\tilde{f}$ can be chosen such that $\tilde{f}(r) = r'$.*

Note that n is also the diffeological dimension of $\mathbf{R}^n/\Gamma$.

Proof. This theorem is a consequence of the previous theorem on the lifting of the identity ☐

7. Theorem [Equivalence of structure groupoids] Let us recall that a functor $\mathrm{S}\colon \mathrm{A} \to \mathrm{C}$ is an equivalence of categories if and only if, S is full and faithful, and each object c in C is isomorphic to $\mathrm{S}(a)$ for some object a in A [SML78, Chap. 4 § 4 Thm. 1]. If A and C are groupoids, the last condition means that, for each object c of C, there exist an object a of A and an arrow from $\mathrm{S}(a)$ to c.

In other words: let the *transitivity-components* of a groupoid be the maximal full subgroupoids such that each object is connected to any other object by an arrow.

Thus: The functor S is an equivalence of groupoids if it is full and faithful, and surjectively projected on the set of transitivity-components.

Now, consider an n-quasifold X. Let $\mathcal{A}$ be an atlas, let $\mathcal{F}$ be the associated strict generating family, let $\mathcal{N}$ be the nebula of $\mathcal{F}$ and let $\mathbf{G}$ the associated structure groupoid. Let us first describe the *morphology* of the groupoid.

Proposition. *The fibers of the subduction $\mathrm{ev}\colon \mathrm{Obj}(\mathbf{G}) \to \mathrm{X}$ are exactly the transitivity-components of $\mathbf{G}$. In other words, the space of transitivity components of the groupoid $\mathbf{G}$ associated with any atlas of the quasifold X, equipped with the quotient diffeology, is the quasifold itself.*

Theorem. *Different atlases of X give equivalent structure groupoids. The structure groupoids associated with diffeomorphic quasifolds are equivalent.*

In other words, the equivalence class of the structure groupoids of a quasifold is a diffeological invariant.

Proof. The idea is to consider the groupoids $\mathbf{G}$ and $\mathbf{G}'$ associated to the atlases $\mathcal{A}$ and $\mathcal{A}'$, then the groupoid $\mathbf{G}''$ associated to the atlas $\mathcal{A}'' = \mathcal{A} \coprod \mathcal{A}'$. Then the groupoids $\mathbf{G}$ and $\mathbf{G}'$ are two full subgroupoids of $\mathbf{G}''$ and to show that, thanks to the theorem on lifting local diffeomorphisms, all three have the same space of transitivity components, that is X. ☐

The $\mathbf{C}^*$-Algebra

We use the construction of the $\mathbf{C}^*$-Algebra associated with an arbitrary locally compact groupoid $\mathbf{G}$, equipped with a Haar system, introduced and described by Jean Renault in [JR80, Part II, §1]. Note that, for this construction, only the topology of the groupoid is involved, and diffeological groupoids, when regarded as topological groupoids, are equipped with the D-topology[2].

We will denote by $\mathcal{C}(\mathbf{G})$ the completion of the compactly supported continuous complex functions on $\mathrm{Mor}(\mathbf{G})$, for the uniform norm. And we still consider, as is done for orbifolds, the particular case where the Haar system is given by the *counting measure*. Let f and g be two compactly supported complex functions, the convolution and the involution are defined by

$$f * g(\gamma) = \sum_{\beta \in \mathbf{G}^x} f(\beta \cdot \gamma) g(\beta^{-1}) \quad \text{and} \quad f^*(\gamma) = f(\gamma^{-1})^*.$$

The sums involved are supposed to converge. Here, $\gamma \in \mathrm{Mor}(\mathbf{G})$, $x = \mathrm{src}(\gamma)$ and $\mathbf{G}^x = \mathrm{trg}^{-1}(x)$ is the subset of arrows with target x. The star in z^* denotes the conjugate of the complex number z. By definition, the vector space $\mathcal{C}(\mathbf{G})$, equipped with these two operations, is the $\mathbf{C}^*$-algebra associated with the groupoid $\mathbf{G}$.

8. Theorem [The structure groupoid is étale and Hausdorff.] Let $\mathcal{A}$ be an atlas of a quasifold X. The structure groupoid $\mathbf{G}$ associated with the generating family of the atlas $\mathcal{A}$ is étale, namely: the projection $\mathrm{src}\colon \mathrm{Mor}(\mathbf{G}) \to \mathrm{Obj}(\mathbf{G})$ is an étale smooth map, that is, a local diffeomorphism at each point [PIZ13, §2.5].

Proposition 1. *For all $g \in \mathrm{Mor}(\mathbf{G})$, there exists a D-open superset $\mathcal{O}$ of g such that* src *restricted to $\mathcal{O}$ is a local diffeomorphism.*

Proposition 2. *The groupoid $\mathbf{G}$ is locally compact and Hausdorff.*

[2]Since smooth maps are D-continuous and diffeomorphism are D-homeomorphisms.

Note. Since the atlas $\mathcal{A}$ is assumed to be locally finite, the preimages of the objects of $\mathbf{G}$ by the source map, or the target map, are countable.

9. Theorem [MRW-equivalence of structure groupoids.] We consider a quasifold X and two atlases $\mathcal{A}$ and $\mathcal{A}'$, with associated strict generating families $\mathcal{F}$ and $\mathcal{F}'$. We shall show in this section that the associated groupoids are equivalent in the sense of Muhly-Renault-Williams [MRW87, 2.1]; this will later give Morita-equivalent $\mathbf{C}^*$-algebras.

This section follows [IZL17, §8]; we just check that the fact that the structure groups are countable and not just finite, does not change the result.

Let us recall what is an MRW-equivalence of groupoids. Let $\mathbf{G}$ and $\mathbf{G}'$ be two locally compact groupoids. We say that a locally compact space Z is a $(\mathbf{G}, \mathbf{G}')$-equivalence if

(i) Z is a left principal $\mathbf{G}$-space.
(ii) Z is a right principal $\mathbf{G}'$-space.
(iii) The $\mathbf{G}$ and $\mathbf{G}'$ actions commute.
(iv) The action of $\mathbf{G}$ on Z induces a bijection of $\mathrm{Z}/\mathbf{G}$ onto $\mathrm{Obj}(\mathbf{G}')$.
(v) The action of $\mathbf{G}'$ on Z induces a bijection of $\mathrm{Z}/\mathbf{G}'$ onto $\mathrm{Obj}(\mathbf{G})$.

Let $\mathrm{src}\colon \mathrm{Z} \to \mathrm{Obj}(\mathbf{G})$ and $\mathrm{trg}\colon \mathrm{Z} \to \mathrm{Obj}(\mathbf{G}')$ be the maps defining the composable pairs associated with the actions of $\mathbf{G}$ and $\mathbf{G}'$. That is, a pair (g, z) is composable if $\mathrm{trg}(g) = \mathrm{src}(z)$, and the composite is denoted by $g \cdot z$. Moreover, a pair (g', z) is composable if $\mathrm{src}(g') = \mathrm{trg}(z)$, and the composite is denoted by $z \cdot g'$.

Let us also recall that an action is principal in the sense of Muhly-Renault-Williams, if it is free: $g \cdot z = z$ only if g is a unit, and the *action map* $(g, z) \mapsto (g \cdot z, z)$, defined on the composable pairs, is proper [MRW87, §2].

Now, using the hypothesis and notations of the previous pragraphs, let us define Z to be the space of germs of local diffeomorphisms, from the nebula of the family $\mathcal{F}$ to the nebula of the family $\mathcal{F}'$, that project on the identity by the evaluation map. That is,

$$\mathrm{Z} = \left\{ \mathrm{germ}(f)_r \;\middle|\; \begin{array}{l} f \in \mathrm{Diff}_{\mathrm{loc}}(\mathrm{dom}(\mathrm{F}), \mathrm{dom}(\mathrm{F}'), r \in \mathrm{dom}(\mathrm{F}), \\ \mathrm{F} \in \mathcal{F}, \mathrm{F}' \in \mathcal{F}' \text{ and } \mathrm{F}' \circ f = \mathrm{F} \restriction \mathrm{dom}(f). \end{array} \right\}.$$

Let[3]

$$\mathrm{src}(\mathrm{germ}(f)_r) = r \quad \text{and} \quad \mathrm{trg}(\mathrm{germ}(f)_r) = f(r).$$

Then, the action of $g \in \mathrm{Mor}(\mathbf{G})$ on $\mathrm{germ}(f)_r$ is defined by composition if $\mathrm{trg}(g) = r$, that is, $g \cdot \mathrm{germ}(f)_r = \mathrm{germ}(f \circ \varphi)_s$, where $g = \mathrm{germ}(\varphi)_s$, $\varphi \in \mathrm{Diff}_{\mathrm{loc}}(\mathcal{N})$ and $\varphi(s) = r$. Symmetrically, the action of $g' \in \mathrm{Mor}(\mathbf{G}')$ on $\mathrm{germ}(f)_r$ is defined if $\mathrm{src}(g') = f(r)$ by $z \cdot g' = \mathrm{germ}(\varphi' \circ f)_r$, where $g' = \mathrm{germ}(\varphi')_{f(r)}$. Then, we have:

Theorem. *The actions of* $\mathbf{G}$ *and* $\mathbf{G}'$ *on* Z *are principal, and* Z *is a* $(\mathbf{G}, \mathbf{G}')$*-equivalence in the sense of Muhly-Renault-Williams.*

Proof. First of all, let us point out that Z is a subspace of the morphisms of the groupoid $\mathbf{G}''$ built previously by adjunction of $\mathbf{G}$ and $\mathbf{G}'$, and is equipped with the subset diffeology. All these groupoids are locally compact and Hausdorff as we seen previously.

Let us check that the action of $\mathbf{G}$ on Z is free. In our case, $z = \mathrm{germ}(f)_r$ and $g = \mathrm{germ}(\varphi)_s$, where f and φ are local diffeomorphisms. If $g \cdot z = z$, then obviously $g = \mathrm{germ}(\mathbf{1})_r$.

Next, let us denote by ρ the action of $\mathbf{G}$ on Z, defined on

$$\mathbf{G} \star \mathrm{Z} = \{(g, z) \in \mathrm{Mor}(\mathbf{G}) \times \mathrm{Z} \mid \mathrm{trg}(g) = \mathrm{src}(z)\} \quad \text{by} \quad \rho(g, z) = g \cdot z.$$

This action is smooth because the composition of local diffeomorphisms is smooth, and passes onto the quotient groupoid in a smooth operation, see [IZL17, §3]. Moreover, this action is invertible, its inverse being defined on

$$\mathrm{Z} \star \mathrm{Z} = \{(z', z) \in \mathrm{Z} \times \mathrm{Z} \mid \mathrm{trg}(z') = \mathrm{trg}(z)\}$$

by

$$\rho^{-1}(z', z) = (g = z' \cdot z^{-1}, z).$$

In detail, $\rho^{-1}(\mathrm{germ}(h)_s, \mathrm{germ}(f)_r) = (\mathrm{germ}(f^{-1} \circ h)_s, \mathrm{germ}(f)_r)$, with $f(r) = h(s)$. Now, the inverse is also smooth, when $\mathrm{Z} \star \mathrm{Z} \subset \mathrm{Z} \times \mathrm{Z}$ is equipped with the subset diffeology. In other words, ρ is an induction, that is, a diffeomorphism from $\mathbf{G} \star \mathrm{Z}$ to $\mathrm{Z} \star \mathrm{Z}$. However,

[3] For the sake of simplicity, we make an abuse of notation: in reality one should write, more precisely, $\mathrm{src}(\mathrm{germ}(f)_r) = (\mathrm{F}, r)$ and $\mathrm{trg}(\mathrm{germ}(f)_r) = (\mathrm{F}', f(r))$.

since $\mathbf{G} \star \mathrm{Z}$ and $\mathrm{Z} \star \mathrm{Z}$ are defined by closed relations, and $\mathbf{G}$ and Z are Hausdorff, $\mathbf{G} \star \mathrm{Z}$ and $\mathrm{Z} \star \mathrm{Z}$ are closed into their ambient spaces. Thus, the intersection of a compact subset in $\mathrm{Z} \times \mathrm{Z}$ with $\mathrm{Z} \star \mathrm{Z}$ is compact, and its preimage by the induction ρ is compact. Therefore, ρ is proper. We notice that the fact that the structure groups are no longer finite but just countable does not play a role here.

It remains to check that the action of $\mathbf{G}$ on Z induces a bijection of $\mathrm{Z}/\mathbf{G}$ onto $\mathrm{Obj}(\mathbf{G}')$. Let us consider the map class: $\mathrm{Z} \to \mathrm{Obj}(\mathbf{G}')$ defined by $\mathrm{class}(\mathrm{germ}(f)_r) = f(r)$. Then, let $\mathrm{class}(z) = \mathrm{class}(z')$, with $z = \mathrm{germ}(f)_r$ and $z' = \mathrm{germ}(f')_{r'}$, that is, $f(r) = f'(r')$. However, since f and f' are local diffeomorphisms, $\varphi = f'^{-1} \circ f$ is a local diffeomorphism with $\varphi(r') = r$. Let $g = \mathrm{germ}(\varphi)_{r'}$, then $g \in \mathrm{Mor}(\mathbf{G})$ and $z' = g \cdot z$. Hence, the map class projects onto an injection from $\mathrm{Z}/\mathbf{G}$ to $\mathrm{Obj}(\mathbf{G}')$. Now, let $(\mathrm{F}', r') \in \mathrm{Obj}(\mathbf{G}')$, and let $x = \mathrm{F}'(r') \in \mathrm{X}$. Since $\mathcal{F}$ is a generating family, there exist $(\mathrm{F}, r) \in \mathrm{Obj}(\mathbf{G})$ such that $\mathrm{F}(r) = x$. Let ψ and ψ' be the charts of X defined by factorization: $\mathrm{F} = \psi \circ \mathrm{class}$ and $\mathrm{F}' = \psi' \circ \mathrm{class}'$, where class: $\mathbf{R}^n \to \mathbf{R}^n/\Gamma$ and $\mathrm{class}' \colon \mathbf{R}^n \to \mathbf{R}^n/\Gamma'$. Let $\xi = \mathrm{class}(r)$ and $\xi' = \mathrm{class}'(r')$. Since $\psi(\xi) = \psi'(\xi') = x$, $\Psi =_{\mathrm{loc}} \psi'^{-1} \circ \psi$ is a local diffeomorphism from $\mathbf{R}^n/\Gamma$ to $\mathbf{R}^n/\Gamma'$ mapping ξ to ξ'. Hence, according to the lifting of local diffeomorphisms, there exists a local diffeomorphism f from $\mathrm{dom}(\mathrm{F})$ to $\mathrm{dom}(\mathrm{F}')$, such that $\mathrm{class}' \circ f = \Psi \circ \mathrm{class}$ and $f(r) = r'$. Thus, $z = \mathrm{germ}(f)_r$ belongs to Z and $\mathrm{class}(z) = r'$ (precisely the element (F', r') of the nebula of $\mathcal{F}'$). Therefore, the injective map class from $\mathrm{Z}/\mathbf{G}$ to $\mathrm{Obj}(\mathbf{G}')$ is also surjective, and identifies the two spaces. Obviously, what has been said for the side $\mathbf{G}$ can be translated to the side $\mathbf{G}'$; the construction is completely symmetric. In conclusion, Z satisfies the conditions of a $(\mathbf{G}, \mathbf{G}')$-equivalence, in the sense of Muhly-Renault-Williams. □

<u>**10. Theorem**</u> [The $\mathbf{C}^*$-algebra of a quasifold.] Let X be a quasifold, let $\mathcal{A}$ be an atlas and let $\mathbf{G}$ be the structure groupoid associated with $\mathcal{A}$. Since the atlas $\mathcal{A}$ is locally finite, the convolution defined above is well defined. Indeed, in this case:

<u>Proposition.</u> *For every compactly supported complex function f on $\mathbf{G}$, for all $\nu = (\mathrm{F}, r) \in \mathcal{N} = \mathrm{Obj}(\mathbf{G})$, the set of arrows $g \in \mathbf{G}^\nu$ such that $f(g) \neq 0$ is finite. That is, $\# \,\mathrm{Supp}(f \restriction \mathbf{G}^\nu) < \infty$. The convolution is then well defined on $\mathcal{C}(\mathbf{G})$.*

Then, for each atlas $\mathcal{A}$ of the quasifold X, we get the $\mathbf{C}^*$-algebra $\mathfrak{A} = (\mathcal{C}(\mathbf{G}), *)$. The dependence of the $\mathbf{C}^*$-algebra on the atlas is given by the following theorem, which is a generalization of [IZL17, §9].

Theorem. *Different atlases give Morita-equivalent $\mathbf{C}^*$-algebras. Diffeomorphic quasifolds have Morita-equivalent $\mathbf{C}^*$-algebras.*

In other words, we have defined a functor from the subcategory of isomorphic {Quasifolds} in diffeology, to the category of Morita-equivalent {$\mathbf{C}^*$-Algebras}.

11. Example [The $\mathbf{C}^*$-Algebra of the irrational torus.] The first and most famous example is the so-called Denjoy-Poincaré torus, or irrational torus, or noncommutative torus, or, more recently, quasitorus. It is, according to its first definition, the quotient set of the 2-torus T^2 by the irrational flow of slope $\alpha \in \mathbf{R} - \mathbf{Q}$. We denote it by $\mathrm{T}_\alpha = \mathrm{T}^2/\Delta_\alpha$, where Δ_α is the image of the line $y = \alpha x$ by the projection $\mathbf{R}^2 \to \mathrm{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. This space has been the first example studied with the tools of diffeology, in [PDPI83], where many non trivial properties have been highlighted.[4] Diffeologically speaking,

$$\mathrm{T}_\alpha \simeq \mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z}).$$

The composite

$$\mathbf{R} \xrightarrow{\text{class}} \mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z}) \xrightarrow{f} \mathrm{T}_\alpha, \quad \text{with } \mathrm{F} = f \circ \text{class},$$

summarizes the situation where $\mathcal{A} = \{f : \mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z}) \to \mathrm{T}_\alpha\}$ is the canonical atlas of T_α, containing the only chart f, and $\mathcal{F} = \{\mathrm{F} = f \circ \text{class}\}$ is the associated canonical strict generating family. According to lifting the identity, the groupoid $\mathbf{G}_\alpha$ associated with the atlas $\mathcal{A}$ is simply

$$\mathrm{Obj}(\mathbf{G}_\alpha) = \mathbf{R} \quad \text{and} \quad \mathrm{Mor}(\mathbf{G}_\alpha) = \{(x, t_{n+\alpha m}) \mid x \in \mathbf{R} \text{ and } n, m \in \mathbf{Z}\}.$$

However, we can also identify T_α with $(\mathbf{R}/\mathbf{Z})/[(\mathbf{Z} + \alpha\mathbf{Z})/\mathbf{Z}]$, that is

$$\mathrm{T}_\alpha \simeq \mathrm{S}^1/\mathbf{Z}, \text{ with } \quad \underline{m}(z) = e^{2i\pi\alpha m} z,$$

for all $m \in \mathbf{Z}$ and $z \in \mathrm{S}^1$. Moreover, the groupoid $\mathbf{S}$ of this action of $\mathbf{Z}$ on $\mathrm{S}^1 \subset \mathbf{C}$ is simply

$$\mathrm{Obj}(\mathbf{S}_\alpha) = \mathrm{S}^1 \quad \text{and} \quad \mathrm{Mor}(\mathbf{S}_\alpha) = \{(z, e^{2i\pi\alpha m}) \mid z \in \mathrm{S}^1 \text{ and } m \in \mathbf{Z}\}.$$

[4] See for example Exercise 4 and §8.39 in [PIZ13].

The groupoids $\mathbf{G}_\alpha$ and $\mathbf{S}_\alpha$ are equivalent, thanks to the functor Φ from the first to the second:

$$\Phi_{\mathrm{Obj}}(x) = e^{2i\pi x} \quad \text{and} \quad \Phi_{\mathrm{Mor}}(x, t_{n+\alpha m}) = (e^{2i\pi x}, e^{2i\pi\alpha m}).$$

Moreover, they are also MRW-equivalent, by considering the set of germs of local diffeomorphisms $x \mapsto e^{2i\pi x}$, everywhere from $\mathbf{R}$ to S^1. Therefore, their associated $\mathbf{C}^*$-algebras are Morita equivalent. The algebra associated with $\mathbf{S}_\alpha$ has been computed numerous times and it is called *irrational rotation algebra* [MR81]. It is the universal $\mathbf{C}^*$-algebra generated by two unitary elements U and V, satisfying the relation

$$\mathrm{VU} = e^{2i\pi\alpha}\mathrm{UV}.$$

Remark 1. Thanks to the theorem on equivalence of diffeomorphic qusifolds, and because two irrational tori T_α and T_β are diffeomorphic if and only if α and β are conjugate modulo $\mathrm{GL}(2, \mathbf{Z})$ [PDPI83], we get the corollary that, if α and β are conjugate modulo $\mathrm{GL}(2, \mathbf{Z})$, then $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$ are Morita equivalent. Which is the direct sense of Rieffel's theorem [MR81, Thm 4].

Remark 2. The converse of Rieffel's theorem is a different matter altogether. We should recover a diffeological groupoid $\mathbf{G}_\alpha$ from the algebra $\mathfrak{A}_\alpha$. Then, the space of transitive components would be the required quasifold, as stated by the proposition on equivalence of structure groupoids. In the case of the irrational torus, it is not very difficult. The spectrum of the unitary operator V is the circle S^1 and the adjoint action by the operator U gives $\mathrm{UVU}^{-1} = e^{2i\pi\alpha}\mathrm{V}$, which translates on the spectrum by the irrational rotation of angle α. In that way, we recover the groupoid of the irrational rotations on the circle, which gives T_α as quasifold.

Remark 3. Of course, the situation of the irrational torus is simple and we do not exactly know how it can be reproduced for an arbitrary quasifold. However, this certainly is the way to follow to recover the quasifold from its algebra: find the groupoid made with the Morita invariant of the algebra, which will give the space of transitivity components as the requested quasifold.

12. Example [The example of $\mathbf{R}/\mathbf{Q}$.] The diffeological space $\mathbf{R}/\mathbf{Q}$ is a legitimate quasifold. This is a simple example with a groupoid

$\mathbf{G}$ given by

$$\mathrm{Obj}(\mathbf{G}) = \mathbf{R} \quad \text{and} \quad \mathrm{Mor}(\mathbf{G}) = \{(x, t_r) \mid x \in \mathbf{R} \text{ and } r \in \mathbf{Q}\}.$$

The algebra that is associated with $\mathbf{G}$ is the set $\mathfrak{A}$ of complex compact supported functions on $\mathrm{Mor}(\mathbf{G})$. Let us identify $\mathcal{C}^0(\mathrm{Mor}(\mathbf{G}), \mathbf{C})$ with $\mathrm{Maps}(\mathbf{Q}, \mathcal{C}^0(\mathbf{R}, \mathbf{C}))$ by

$$f = (f_r)_{r \in \mathbf{Q}} \quad \text{with} \quad f_r(x) = f(x, t_r), \quad \text{and let} \quad \mathrm{Supp}(f) = \{r \mid f_r \neq 0\}.$$

Then,

$$\mathfrak{A} = \left\{ f \in \mathrm{Maps}(\mathbf{Q}, \mathcal{C}^0(\mathbf{R}, \mathbf{C})) \mid \# \, \mathrm{Supp}(f) < \infty \right\}.$$

The convolution product and the algebra conjugation are, thus, given by:

$$(f * g)_r(x) = \sum_s f_{r-s}(x+s) g_s(x), \quad \text{and} \quad f_r^*(x) = f_{-r}(x+r)^*.$$

Now, the quotient $\mathbf{R}/\mathbf{Q}$ is also diffeomorphic to the $\mathbf{Q}$-circle

$$S_{\mathbf{Q}} = S^1/\mathcal{U}_{\mathbf{Q}}, \quad \text{where} \quad \mathcal{U}_{\mathbf{Q}} = \{e^{2i\pi r}\}_{r \in \mathbf{Q}}$$

is the subgroup of *rational roots of unity*. As a diffeological subgroup of S^1, $\mathcal{U}_{\mathbf{Q}}$ is discrete. The groupoid $\mathbf{S}$ of the action of $\mathcal{U}_{\mathbf{Q}}$ on S^1 is given by:

$$\mathrm{Obj}(\mathbf{S}) = S^1 \quad \text{and} \quad \mathrm{Mor}(\mathbf{S}) = \left\{ (z, \tau) \;\middle|\; z \in S^1 \text{ and } \tau \in \mathcal{U}_{\mathbf{Q}} \right\}.$$

The exponential $x \mapsto z = e^{2i\pi x}$ realizes a MRW-equivalence between the two groupoids $\mathbf{G}$ and $\mathbf{S}$. Their associated algebras are Morita-equivalent. The algebra $\mathfrak{S}$ associated with $\mathbf{S}$ is made of families of continuous complex functions indexed by rational roots of unity, in the same way as before:

$$\mathfrak{S} = \left\{ (f_\tau)_{\tau \in \mathcal{U}_{\mathbf{Q}}} \;\middle|\; f_\tau \in \mathcal{C}^0(S^1, \mathbf{C}) \text{ and } \# \, \mathrm{Supp}(f) < \infty \right\}.$$

The convolution product and the algebra conjugation are, then, given by:

$$(f * g)_\tau(z) = \sum_\sigma f_{\bar\sigma\tau}(\sigma z) g_\sigma(z) \quad \text{and} \quad f_\tau^*(z) = f_{\bar\tau}(\tau z)^*,$$

where $\bar\tau = 1/\tau = \tau^*$, the complex conjugate.

Now, consider f and let $\mathcal{U}_p$ be the subgroup in $\mathcal{U}_{\mathbf{Q}}$ generated by $\mathrm{Supp}(f)$; this is the group of some root of unity ε of some order p.

Let $\mathrm{M}_p(\mathbf{C})$ be the space of $p \times p$ matrices with complex coefficients. Define $f \mapsto \mathrm{M}$, from $\mathfrak{S}$ to $\mathrm{M}_p(\mathbf{C}) \otimes \mathcal{C}^0(\mathrm{S}^1, \mathbf{C})$, by

$$\mathrm{M}(z)^\sigma_\tau = f_{\bar{\sigma}\tau}(\sigma z), \quad \text{for all } z \in \mathrm{S}^1 \text{ and } \sigma, \tau \in \mathcal{U}_p.$$

That gives a representation of $\mathfrak{S}$ in the tensor product of the space of endomorphisms of the infinite-dimensional $\mathbf{C}$-vector space $\mathrm{Maps}(\mathcal{U}_{\mathbf{Q}}, \mathbf{C})$ by $\mathcal{C}^0(\mathrm{S}^1, \mathbf{C})$, with finite support.

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