AT THE BEGINNING...

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ref. http://math.huji.ac.il/~piz/documents/ShD-lect-ATB.pdf

This lecture presents the prerequisites for the study of diffeology. As we shall see, a good understanding of indefinitely differentialble maps on Eucldean domains, and a few fundamental theorems of calculus, will be a good start.

Let us introduce a little bit of vocabulary. First of all, we call *Euclidean space* any real vector space $\mathbf{R}^n$ for some n.

The Euclidean structure of $\mathbf{R}^n$, defined by the inner product

$$x \cdot y = \sum_{i=1}^{n} x^i y^i \quad \text{with} \quad \|x\| = \sqrt{\sum_{i=1}^{n} (x^i)^2},$$

is used to define its (standard) topology: an open subset $\mathcal{O}$ in $\mathbf{R}^n$ is any *union of open balls*, like:

$$\mathcal{O} = \bigcup_{i \in \mathcal{I}} \mathcal{B}(r_i, \varepsilon_i),$$

where $\mathcal{I}$ is any set of indices, r_i is any point in $\mathbf{R}^n$ and ε_i is any (strictly) positive number.

Therefore, a subset $\mathrm{U} \in \mathbf{R}^n$ is open for the standard topology if (and only if): for each $r \in \mathrm{U}$ there exists an open ball $\mathcal{B}(r, \varepsilon)$ included in U, $\mathcal{B}(r, \varepsilon) \subset \mathrm{U}$.

Next, we call *Euclidean domain* any open subset of an Euclidean space. We generally denote them by some big letter $\mathrm{U}, \mathrm{V}, \mathrm{W}$ etc. and also by cursive letters $\mathcal{O}$ etc.

We denote also by Domains($\mathbf{R}^n$) the set of all Euclidean domains in $\mathbf{R}^n$. We call them simply *n-domains*.

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Note that this is just the topology of $\mathbf{R}^n$ sometimes denoted by $\mathrm{Top}(\mathbf{R}^n)$.

A domain in $\mathbf{R}^n$ is said to be of dimension n, or be an n-domain.

Now, about *continuity*. A map $F\colon U \to V$, where U and V are Euclidean domains, is said to be *continuous* if (and only if): the pullback $f^{-1}(\mathcal{O})$ of any open subset $\mathcal{O} \subset V$ is an open subset in U.

That is equivalent to say that for any open ball $\mathcal{B} \subset V$, there exists a family of open balls $\mathcal{B}_i \subset U$ such that $f^{-1}(\mathcal{B}) = \cup_i \mathcal{B}_i$, for some family of indices $\mathcal{I}$.

1. Differentiable and Smooth Paths

Consider a *path*

$$\gamma\colon]a, b[\to \mathbf{R}^n.$$

Assume that γ is continuous. Consider the map

$$\Delta\gamma\colon (t, t') \mapsto \frac{\gamma(t') - \gamma(t)}{t' - t},$$

defined on

$$]a, b[^2 - \{(t, t)\}_{a<t<b},$$

with values in $\mathbf{R}^n$. The closed subset $\{(t, t)\}_{a<t<b}$ is called the *diagonal*.

The function $\Delta\gamma$ is continuous, but:

1. Definition *If $\Delta\gamma$ extends continuously on the diagonal, we say that γ is continuously differentiable or of class C^1.*

We denote by

$$\dot\gamma(t) \quad \text{or} \quad \frac{d\gamma(t)}{dt} \quad \text{the value} \quad \Delta\gamma(t, t) = \lim_{t' \to t} \Delta\gamma(t', t).$$

The function $\dot\gamma\colon]a, b[\to \mathbf{R}^n$ is then called the *derivative* of γ. The value $\dot\gamma(t)$ is the derivative of γ at the point t, or *at time* t.

Now, if γ is of class C^1, then $\dot\gamma$, which is defined on the same interval, is continuous. On can ask if $\dot\gamma$ is also C^1? If it is the case, we generally denote by $\ddot\gamma$ the derivative of $\dot\gamma$. We say that γ is of class C^2.

That leads to the definition:

2. Definition *We say that* γ *is of class* C^k, $k > 1$, *if (and only if):*

$$\gamma \text{ is of class } C^{k-1} \text{ and } \frac{d^{k-1}\gamma(t)}{dt^{k-1}} \text{ is of class } C^1.$$

And we say that γ *is smooth, or of class* C^∞, *if* γ *is of class* C^k *for all* $k \in \mathbf{N}$.

Note that continuous paths are said to be of *class* C^0.

2. Smooth Maps, the Holistic Approach

In diffeology we are interested principally in *smooth maps*, or *infinitely differentiable maps*. Precisely, diffeology consists in extending the concept of smooth maps from Eucliden domains to arbitrary sets. That is why we have the choice in the definitions of smooth maps between Euclidean domains. We will present them here, not in their historic order but according with what we think is more speaking to our mind. The first definition is actually a theorem.

3. Definition-Theorem [Boman 1967] *Let* $F: U \to V$ *be a continuous map between two Euclidean domains. The map* F *is infinitely differentiable, or smooth, if (and only if) the composite* $F \circ \gamma$, *where* γ *is any smooth path in* U, *is a smooth path in* V.

As we have seen, smooth paths have been defined previously, independently.

3. Differentiable Maps, the Pedestrian Approach

Now, the classical definition of differentiable map begins with the definition of *tangent maps*.

4. Definition *Let* f *and* g *be two maps defined on a same Euclidean domain* U *to some* $\mathbf{R}^m$. *Let* r_0 *be some point in* U. *We say that* f *and* g *are tangent in* r_0 *if:*

$$\lim_{r \to r_0} \frac{f(r) - g(r)}{\|r - r_0\|} = 0.$$

Of course, that implies in particular that $f(r_0) = g(r_0)$.

5. Definition *Let* f *be a map defined on some domain* $U \in \mathbf{R}^n$ *with values in* $\mathbf{R}^m$. *We say that* f *is differentiable, or derivable,*[1] *at the*

[1] The words *differentiable* and *derivable* are here completely equivalent.

point $r_0 \in \mathrm{U}$ *if it is tangent, at this point, to an affine map*

$$r \mapsto f(r_0) + \mathrm{M}(r - r_0),$$

where $\mathrm{M} \in \mathrm{L}(\mathbf{R}^n, \mathbf{R}^m)$ *is a linear map from* $\mathbf{R}^n$ *to* $\mathbf{R}^m$*. That condition means precisely that,*

$$\lim_{r \to r_0} \frac{f(r) - f(r_0) - \mathrm{M}(r - r_0)}{\|r - r_0\|} = 0. \qquad (\clubsuit)$$

We say that f is differentiable on U*, or simply differentiable, without more precision, if f is tangent to an affine map everywhere.*

<u>**6. Remark**</u> The condition ($\clubsuit$) implies that $f(r)$ converges to $f(r_0)$ when r tends to r_0. That is f is continuous in r_0: to be tangent to an affine map implies to be continuous: <u>a differentiable map is necessarily continuous</u>.

◀ *proof* Since $\lim_{r \to r_0} \frac{f(r)-f(r_0)-\mathrm{M}(r-r_0)}{\|r-r_0\|} = 0$, $\lim_{r \to r_0} f(r) - f(r_0) - \mathrm{M}(r - r_0) = 0$. Thus, since $\lim_{r \to r_0} \mathrm{M}(r - r_0) = 0$, $\lim_{r \to r_0} f(r) - f(r_0) = 0$. That is $f(r) \xrightarrow[r \to r_0]{} f(r_0)$, and f is continuous at r_0. That is the logic of the situation. More formally, since $\lim_{r \to r_0} f(r) - f(r_0) - \mathrm{M}(r - r_0) = 0$, for all $\varepsilon > 0$, there exists $\eta > 0$ such that $\|r - r_0\| < \eta$ implies $\|f(r) - f(r_0) - \mathrm{M}(r - r_0)\| < \varepsilon$. On the other hand, let m be $\|\mathrm{M}\| = \sup_{\|u\|=1} \|\mathrm{M}(u)\|$ (remember, the sphere S^{n-1} is compact). Thus, for all r, $\|\mathrm{M}(r - r_0)\| \leq m\|r - r_0\|$. That is, for all $\varepsilon > 0$ there is $\eta' > 0$ such that $\|r - r_0\| < \eta$ implies $\|\mathrm{M}(r - r_0)\| < \varepsilon$, actually η' can be chosen equal to ε/m. Now, the inequality on a triangle $u + v = w$ says that $\|w\| \leq \|u\| + \|v\|$, or $\|w\| - \|u\| \leq \|v\|$. Applied to $u = \mathrm{M}(r - r_0)$, $w = f(r) - f(r_0)$ and $v = w - u = f(r) - f(r_0) - \mathrm{M}(r - r_0)$, that gives: $\|f(r) - f(r_0)\| - \|\mathrm{M}(r - r_0)\| \leq \|f(r) - f(r_0) - \mathrm{M}(r - r_0)\| \leq \varepsilon$. Then, $\|f(r) - f(r_0)\| \leq \varepsilon + \|\mathrm{M}(r - r_0)\|$. Now, chosing $\eta'' < \inf(\eta, \eta')$, we get $\|f(r) - f(r_0)\| \leq \varepsilon + \varepsilon$. Changing ε to $\varepsilon/2$, we get: for all $\varepsilon > 0$, there exists $\eta'' > 0$ such that $\|r - r_0\| < \eta''$ implies $\|f(r) - f(r_0)\| \leq \varepsilon$. Therefore, f is continuous at r_0. ▶

Now let describe more precisely the linear part of the affine tangent map. Pick a vector $v \in \mathbf{R}^n$, $v \neq 0$, and let $r = r_0 + tv$, where $t > 0$ is small enough for r to be in the domain U. The last condition writes:

$$\lim_{t \to 0} \frac{f(r_0 + tv) - f(r_0) - \mathrm{M}(tv)}{t\|v\|} = 0.$$

And then,

$$M(v) = \lim_{t \to 0} \frac{f(r_0 + tv) - f(r_0)}{t}.$$

The linear map M is then called the *tangent linear map* and denoted by

$$Df_{r_0}(v) \text{ or } Df(r_0)(v) \text{ or } D(f)(r_0)(v) = \lim_{t \to 0} \frac{f(r_0 + tv) - f(r_0)}{t}.$$

So, f is differentiable on U if it admits a tangent linear map at every point in U,

$$f' \text{ or } Df \text{ or } D(F) : U \to L(\mathbf{R}^n, \mathbf{R}^m).$$

The *affine tangent map* at the point r_0 writes then

$$r \mapsto f(r_0) + D(f)(r_0)(r - r_0).$$

Note that There is another way to express the approximation of the function f by its affine tangent map around r_0:

$$f(r) = f(r_0) + D(f)(r_0)(r - r_0) + o(\|r - r_0\|),$$

where Landau's *Little-O* notation $o(x)$ means $\lim_{x \to 0} o(x)/x = 0$.

As we know the set of linear maps $L(\mathbf{R}^n, \mathbf{R}^m)$ is a real vector space of dimension $n \times m$, equivalent to $\mathbf{R}^{n \times m}$, as such:

7. Definition *We say that $f : U \to \mathbf{R}^m$ is of class C^1 if f is differentiable on U, and if the map that associates its tangent linear map with each point in U is continuous. That is, if the map $r \mapsto Df(r)$, defined on U with values in $L(\mathbf{R}^n, \mathbf{R}^m)$ is continuous.*

8. Remark An affine map from $\mathbf{R}^n$ to $\mathbf{R}^m$ writes $x \mapsto Ax + b$, where $A \in L(\mathbf{R}^n, \mathbf{R}^m)$ and $b \in \mathbf{R}^m$. The set of these affine maps is denoted by $\mathrm{Aff}(\mathbf{R}^n, \mathbf{R}^m)$, it is naturally a vector space of dimension $m + n \times m$, equivalent to the space of matrices

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \quad \text{such that} \quad \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + b \\ 1 \end{pmatrix},$$

and inherits the standard topology of $\mathbf{R}^{m+n \times m}$. The affine tangent map writes, for all $r \in U$:

$$\begin{pmatrix} Df_r & f(r) - Df_r(r) \\ 0 & 1 \end{pmatrix}.$$

We can notice that, to be C^1, it is equivalent to request that the maps that associate the linear tangent map in $L(\mathbf{R}^n, \mathbf{R}^m)$, or the affine tangent map in $\mathrm{Aff}(\mathbf{R}^n, \mathbf{R}^m)$, with every point r in U, are continuous. However, on a conceptual level: to be differentiable implies to be continuous, then, having a *first local affine approximation*, — which is exactly what the affine tangent map is — continuous everywhere is a natural request. And that is the meaning of being C^1.

4. Higher Order Derivatives and Smooth Maps

Now, we can look for *higher order derivatives*. Let us begin by the second order. Assume that f is derivable, then its derivative D(F) is defined on U with values in $\mathrm{L}(\mathbf{R}^n, \mathbf{R}^m)$. The second derivative of f is then defined by

$$\mathrm{D}^2(f)(r) = \mathrm{D}\big(r \mapsto \mathrm{D}(f)(r)\big)(r) \in \mathrm{L}\big(\mathbf{R}^n, \mathrm{L}(\mathbf{R}^n, \mathbf{R}^m)\big).$$

Note that if f is derivable and if its derivative $\mathrm{D}f$ is derivable, that implies that $\mathrm{D}f$ is continuous and f is C^1.

So, we get a recursive definiton of higher derivative of f and then a recursive definitoin of class C^k:

9. Definition *The kth-derivative of f, if it exists, is the derivative denoted and defined recursively by*

$$\mathrm{D}^k(f) = \mathrm{D}\big(r \mapsto \mathrm{D}^{k-1}(f)(r)\big).$$

Note that, if f is differentiable until order k, then $\mathrm{D}^\ell(f)$ is continuous until $\ell = k - 1$, thanks to Remark 6. Therefore, if f admits a kth-derivative, then f is C^{k-1}, with the definition:

10. Definition , *A function f is C^k, or of class C^k, if f is derivable till order k and its k-derivative is continuous.*

We say that f is infinitely differentiable, or infinitely derivable, or smooth, if f is of class C^k for all integer k.

11. Remark Boman theorem cited previously says exactly that definition 9 and Definition-Theorem 3 are coherent. Beware, it is a subtle and non-trivial theorem.

5. The Tangent Linear Map

We consider a differentiable map $f\colon \mathrm{U} \to \mathbf{R}^m$, with $\mathrm{U} \subset \mathbf{R}^n$ an open subset.

Let us denote by x and y the source and target variables involved in f, that is, $f\colon x \mapsto y$ with $x = (x_1, \dots, x_n) \in \mathrm{U}$ and $y = (y_1, \dots, y_m) \in \mathbf{R}^m$. The tangent linear map can be written indifferently

$$\mathrm{D}(f)(x) \text{ or } \mathrm{D}(x \mapsto y)(x),$$

depending what we want to focus on. For example, if you denote the square root function by sqrt, you can write it derivative as $\mathrm{D}(\mathrm{sqrt})(x)$, or $\mathrm{D}(x \mapsto \sqrt{x})(x)$.

Now, decompose x, v and y on the canonical basis,

$$x = \sum_{i=1}^{n} x^i \boldsymbol{e}_i, \quad v = \sum_{j=1}^{m} v^i \boldsymbol{e}_i \quad \text{and} \quad y = \sum_{j=1}^{m} y^j \boldsymbol{e}_j \quad \text{with} \quad y^j = f^j(x).$$

Now, the tangent linear map writes

$$\begin{aligned}
\mathrm{D}(x \mapsto y)(x)(v) &= \mathrm{D}(x \mapsto y)(x)\left(\sum_{i=1}^{n} v^i \boldsymbol{e}_i\right) \\
&= \sum_{i=1}^{n} v^i \mathrm{D}(x \mapsto y)(x)(\boldsymbol{e}_i) \\
&= \sum_{i=1}^{n} v^i \mathrm{D}\left(x \mapsto \sum_{j=1}^{m} y^j \boldsymbol{e}_j\right)(x)(\boldsymbol{e}_i) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} v^i \mathrm{D}\left(x \mapsto y^j\right)(x)(\boldsymbol{e}_i) \cdot \boldsymbol{e}_j
\end{aligned}$$

In other words, the tangent linear map $\mathrm{D}(x \mapsto y)(x)$ is represented by the matrix $(\mathrm{D}_i^j)_{i=1}^{n}{}_{j=1}^{m}$, where

$$\mathrm{D}_i^j = \mathrm{D}\left(x \mapsto y^j\right)(x)(\boldsymbol{e}_i).$$

Now, what does exactly represent $\mathrm{D}\left(x \mapsto y^j\right)(x)(\boldsymbol{e}_i)$?

$$\mathrm{D}\left(x \mapsto y^j\right)(x)(\boldsymbol{e}_i) = \lim_{t \to 0} \frac{f^j(x_1, \dots, x_i + t, \dots, x_n) - f^j(x_1, \dots, x_i, \dots, x_n)}{t},$$

which is, by definition, the partial derivative

$$\mathrm{D}_i^j = \frac{\partial y^j}{\partial x^i} \quad \text{also denoted by} \quad \partial_i y^j.$$

Notations 8. *By commodity we will denote also*

$$\mathrm{D}(x\mapsto y)(x) \quad \textit{by} \quad \frac{\partial y}{\partial x}, \textit{ the derivative};$$

and

$$\mathrm{D}(x\mapsto y)(a) \quad \textit{by} \quad \left.\frac{\partial y}{\partial x}\right|_{x=a}, \textit{ the value of the derivative.}$$

Eventually, the tangent linear map can by written as the matrix of partial derivatives

$$\mathrm{D}(x\mapsto y)(x) = \mathrm{D}\left(\begin{pmatrix} x^1\\ \vdots \\ x^n\end{pmatrix} \mapsto \begin{pmatrix} y^1\\ \vdots \\ y^m\end{pmatrix}\right)(x) = \begin{pmatrix} \partial_1 y^1 & \cdots & \partial_n y^1\\ \vdots & \ddots & \vdots \\ \partial_1 y^m & \cdots & \partial_n y^m\end{pmatrix}$$

such that

$$\mathrm{D}(x\mapsto y)(x)(v) = \begin{pmatrix} \partial_1 y^1 & \cdots & \partial_n y^1\\ \vdots & \ddots & \vdots \\ \partial_1 y^m & \cdots & \partial_n y^m\end{pmatrix}\begin{pmatrix} v^1\\ \vdots \\ v^n\end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n v^i\partial_i y^1\\ \vdots \\ \sum_{i=1}^n v^i\partial_i y^m\end{pmatrix}.$$

12. Proposition *Let $f\colon \mathrm{U}\to\mathrm{V}$ and $g\colon \mathrm{V}\to\mathrm{W}$ be two C^1 maps, with $\mathrm{U}\subset\mathbf{R}^n$, $\mathrm{V}\subset\mathbf{R}^m$ and $\mathrm{W}\subset\mathbf{R}^\ell$. The derivative of the composite, which is also C^1, satisfies the so-called chain-rule:*

$$\mathrm{D}(g\circ f)(r) = \mathrm{D}(g)(f(r))\circ\mathrm{D}(f)(r).$$

Note also that the derivation D is linear. If f_1 and f_2 are defined from U to V and $\lambda,\mu\in\mathbf{R}$, then:

$$\mathrm{D}(\lambda f_1 + \mu f_2)) = \lambda\mathrm{D}(f_1) + \mu\mathrm{D}(f_2).$$

13. Note The chain-rule writes in term of partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}\frac{\partial y}{\partial x},$$

and the value at the point $x = a$, with where $b = f(a)$:

$$\left.\frac{\partial z}{\partial x}\right|_{x=a} = \left.\frac{\partial z}{\partial y}\right|_{y=b}\left.\frac{\partial y}{\partial x}\right|_{x=a}.$$

In terms of coordinates:

$$\partial_i z^j = \sum_{k=1}^m \partial_i y^k \partial_k z^j \quad \text{that is} \quad \frac{\partial z^j}{\partial x^i} = \sum_{k=1}^m \frac{\partial z^j}{\partial y^k}\frac{\partial y^k}{\partial x^i},$$

where $i = 1\dots n$, $k = 1\dots m$ and $j = 1\dots\ell$.

14. Note In particular, for $\gamma\colon \mathcal{I} \to U$, a smooth path in U,

$$D(f \circ \gamma)(t)(1) = D(f)(\gamma(t)) \circ D(\gamma)(t)(1),$$

but

$$D(\gamma)(t)(1) = \lim_{\varepsilon \to 0} \frac{\gamma(t + \varepsilon \times 1) - \gamma(t)}{\varepsilon} = \frac{d\gamma(t)}{dt} = \dot{\gamma}(t).$$

Therefore:

$$D(f \circ \gamma)(t)(1) = D(f)(\gamma(t))\big(\dot{\gamma}(t)\big).$$

15. Notation It hapens that a tangent vector $v \in \mathbf{R}^n$ would be denoted by a variational notation δx, meaning that v is regarded as a *variation* of the point x. Write $\gamma\colon t \mapsto x$ and $f\colon x \mapsto y$, then $f \circ \gamma\colon t \mapsto y$, meaning $t \mapsto x \mapsto y$. Using the partial derivative notation introduced previously, the formula above writes:

$$\delta y = \frac{\partial y}{\partial x}(\delta x),$$

with

$$x = \gamma(t),\ \ y = f(x),\ \ \delta x = \frac{dx}{dt},\ \ \delta y = \frac{dy}{dt},$$

to which we can add:

$$\delta x = \frac{\partial x}{\partial t}(\delta t) \quad \text{and} \quad \delta t = 1 \ \Rightarrow\ \delta x = \frac{dx}{dt}.$$

Of course, every vector v can be realised as a variation of the point x by considering $\gamma(t) = x + tv$.

6. Higher Derivatives Components

What have been made for the tangent linear map can be done for higher derivatives. For example, let $f\colon x \mapsto y$. The second derivative

$$D^2(f)(x) = D\big(x \mapsto D(f)(x)\big)(x)$$

is represented itself by the bilinear form:

$$D^2(f)(x)(v)(w) = D\big(x \mapsto D(f)(x)(v)\big)(x)(w),$$

for all $v, w \in \mathbf{R}^n$. Let decompose $v = \sum_{i=1}^n v^i \mathbf{e}_i$ and $w = \sum_{j=1}^n w^j \mathbf{e}_j$. We get:

$$\begin{aligned}
\mathrm{D}^2(f)(x)(v)(w) &= \mathrm{D}\Big(x \mapsto \mathrm{D}(f)(x)\Big(\sum_{i=1}^n v^i \mathbf{e}_i\Big)\Big)(x)\Big(\sum_{j=1}^n w^j \mathbf{e}_j\Big) \\
&= \sum_{j=1}^n w^j \mathrm{D}\big(x \mapsto \sum_{i=1}^n v^i \mathrm{D}(f)(x)(\mathbf{e}_i)\big)(x)(\mathbf{e}_j) \\
&= \sum_{j=1}^n w^j \sum_{i=1}^n v^i \mathrm{D}\big(x \mapsto \mathrm{D}(f)(x)(\mathbf{e}_i)\big)(x)(\mathbf{e}_j) \\
&= \sum_{j,i=1}^n w^j v^i \frac{\partial}{\partial x^j}\left(\frac{\partial y}{\partial x^i}\right).
\end{aligned}$$

And since $y = (y^1, \ldots, y^m)$,

$$\mathrm{D}^2(f)(x)(v)(w) = \sum_{j,i=1}^n w^j v^i \begin{pmatrix} \frac{\partial^2 y^1}{\partial x^j \partial x^i} \\ \vdots \\ \frac{\partial^2 y^m}{\partial x^j \partial x^i} \end{pmatrix} = \sum_{j,i=1}^n w^j v^i \begin{pmatrix} \partial^2_{ji} y^1 \\ \vdots \\ \partial^2_{ji} y^m \end{pmatrix}$$

The partial derivatives $\partial_{ij} y^k$ are the components of the bilinear map $\mathrm{D}^2(f)(x)$.

A main property of continuously differentiable function is the commutativity of the partial derivatives:

<u>**16. Theorem**</u> [Schwarz] *Let $f\colon \mathrm{U} \to \mathbf{R}^m$, with U a n-domain, be a C^2 map. Then, the second derivative $\mathrm{D}^2(f)(x)$ is symetric. In other words, the partial derivatives commutent:*

$$\frac{\partial}{\partial x^i}\left(\frac{\partial y^k}{\partial x^j}\right) = \frac{\partial}{\partial x^j}\left(\frac{\partial y^k}{\partial x^i}\right) \quad \textit{i.e.} \quad \partial^2_{ij} y^k = \partial^2_{ji} y^k.$$

For C^k maps, the k-derivatives $\mathrm{D}^k(f)(x)$ is a k-linear map, with components the partial derivatives:

$$\partial^k_{i_1 \ldots i_k} y^\ell = \partial_{i_1}\big(\partial^{k-1}_{i_2 \ldots i_k} y^\ell\big)$$

And, thanks to the last proposition, this multilinear map is symetric: the order of the indices does not matter.

<u>**17. Theorem**</u> *Let $f\colon \mathrm{U} \to \mathbf{R}^m$ be a map, where U is an n-domain. The map f is smooth if and only if f admits partial derivatives at any order.*

◀ *proof* Indeed, and that is the main point: if all the partial derivatives at any order exist, that implies in particular that they are continuous, thanks to 6, and then the map f is *continuously infinitely differentiable*. ▶

This theorem is a practical criterion to check if a map is smooth.

That is everything we need to know to introduce and study diffeology. That is what makes diffeology a good alternative to the usual teaching of differential geometry. And we shall see in the future how it could be understood formally as a *geometry in the sens of Felix Klein*.

7. The Category of Euclidean Domains

Euclidean domains are the objects of a category for which the arrows are the smooth maps. We denote it by {Euclidean Domains}. The goal of diffeology is to transfert some properties of this category to arbitrary sets.

8. Some Theorems We Should Know

There are a few important theorems of differential calculus in $\mathbf{R}^n$ we will need in future development of diffeology, for example the implicit function theorem or the rectification of vector fields and some others. However, they are not necessary for now to learn diffeology. That is why it is better to introduce them only when they will be used.

References

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