REFRACTION AND REFLEXION ACCORDING TO THE PRINCIPLE OF GENERAL COVARIANCE

PATRICK IGLESIAS-ZEMMOUR

Abstract. We show that the principle of general covariance introduced by Souriau in smoothly uniform contexts, can be extended to singular situations, considering the group of diffeomorphisms preserving the singular locus. As a proof of concept, we show how to get this way the well known laws of reflection and refraction of light in geometric optics, applying an extended general covariance principle to Riemannian metrics, singular along an hypersurface.

Introduction

In his paper "Modèle de particule à spin dans le champ électromagnétique et gravitationnel" published in 1974 [Sou74], Jean-Marie Souriau proposes a precise mathematical interpretation of the principle of general relativity. He names it the *Principle of General Covariance*. Considering only gravitation field[1], he claimed that any material presence in the universe is characterized by a covector defined on the quotient of the set of the Pseudo-Riemannian metrics on space-time, by the group of diffeomorphims. This principle being, according to Souriau, the correct statement of the Einsteins's principle of invariance with respect to any change of coordinates. Actually, the Souriau's general covariance principle can be regarded as the active version of Einstein's invariance statement, where change of coordinates are interpreted from the active point of view as the action of the group of diffeomorphisms.

Now, for reasons relative to the behavior at infinity and results requirement, the group of diffeomorphisms of space-time is reduced to the subgroup of compact supported diffeomorphisms.

In consequence, after some identifications, Souriau interprets a material presence for a Pseudo-Riemannian metric g on the space-time M, responding to this principle of covariance, as a *tensor distribution* $\mathcal{T}$ whose test functions are compacted supported covariant 2-tensors δg defined on M, which vanishes along the infinitesimal action

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[1]In his paper Souriau extends also his consideration to electromagnetic field.

of the compact supported diffeomorphisms group $\mathrm{Diff}_{\mathrm{cpct}}(\mathrm{M})$. That is, in Souriau's notations:

$$\mathcal{T}(\delta_{\mathrm{L}} g) = 0,$$

where $\delta_{\mathrm{L}} g$ is the Lie derivative of the metric g by a compact supported vector field δx. In a more standard notation, denoting by ξ the compact supported vector field on M, $\delta_{\mathrm{L}} g = \pounds_{\xi}(g)$. A tensor distribution satisfying that last equation has been called by Souriau, an *Eulerian distribution*.

There are two main examples of application of this principle. The first one is a continuous distribution of matter described by a C^1-smooth symmetric contravariant 2-tensor $\mathrm{T}^{\mu\nu}$,

$$\mathcal{T}(\delta g) = \frac{1}{2} \int_{\mathrm{M}} \mathrm{T}^{\mu\nu} \delta g_{\mu\nu} \,\mathrm{vol},$$

where vol is the Riemannian volume associated with g, and we use the Einstein's convention on repeated indices. We recall that the Lie derivative of a metric g by a vector field ξ is given, in terms of coordinates, by

$$\delta_{\mathrm{L}} g_{\mu,\nu} = \hat{\partial}_{\mu} \xi_{\nu} + \hat{\partial}_{\nu} \xi_{\mu},$$

where $\hat{\partial}$ denotes the covariant derivative with respect with the Levi-Civita connection relative to g. Then, The Eulerian condition above writes now:

$$\frac{1}{2} \int_{\mathrm{M}} \mathrm{T}^{\mu\nu} (\hat{\partial}_{\mu} \xi_{\nu} + \hat{\partial}_{\nu} \xi_{\mu}) \,\mathrm{vol} = 0,$$

where the $\hat{\partial}$ symbol denotes the covariant derivative (also denoted sometimes by ∇). By reason of symmetries and integrating by parts, recalling that ξ is compact supported, on gets:

$$\int_{\mathrm{M}} \hat{\partial}_{\mu}(\mathrm{T}^{\mu\nu}) \xi_{\nu} \,\mathrm{vol} = 0,$$

for all compact supported vector field ξ. Therefore the Eulerian condition writes:

$$\hat{\partial}_{\mu}(\mathrm{T}^{\mu\nu}) = 0, \quad \text{or} \quad \hat{\mathrm{div}}(\mathrm{T}) = 0,$$

where $\hat{\mathrm{div}}(\mathrm{T})$ is the Riemannian divergence of the tensor T. Hence, the Eulerian character of the distribution $\mathcal{T}$ translates simply in the Euler conservation equations of the energy-momentum tensor. Which is the first principle in general relativity. And that justifies *a posteriori* the vocabulary chosen.

The second example is a preamble to the subject of our paper. It concerns the construction of geodesics as Eulerian distributions supported by a curve. So, let $t \mapsto \mathrm{T}^{\mu\nu}$ be a C^2-smooth curve in the fiber bundle of symmetric contravariant 2-tensor over M, over a curve $\gamma : t \mapsto x$. Let then,

$$\mathcal{T}(\delta g) = \frac{1}{2} \int_{-\infty}^{+\infty} \mathrm{T}^{\mu\nu} \delta g_{\mu\nu} \, dt.$$

After some astute manipulation [Sou74], Souriau shows that this distribution is Eulerian if and only if the supporting curve γ is a geodesic, that is,

$$\frac{\hat{d}}{dt}\frac{dx}{dt} = 0.$$

And then:

$$\mathcal{T}(\delta g) = \frac{k}{2}\int_{-\infty}^{+\infty} \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\delta g_{\mu\nu}\, dt,$$

where k can be any real constant. In general k is interpreted as the mass m of the particle moving along the geodesic γ with velocity $\mathbf{v} = dx/dt$. But for classical geometric optics k will be interpreted as the color coefficient $h\nu/c^2$, where h is the Plank constant, ν is the frequency of the ray and c is the speed of light in vacuum.

Thus, the condition to be geodesic is interpreted through this approach as an immediate consequence of the principle of general relativity, which is quite satisfactory.

This Souriau's construction has been also commented in a great way, and applied unexpectedly by Shlomo Sternbeg, to derive the Schrödinger's equation from the same covariance principle [Ste06].

In this paper, we shall see how, this general covariance principle can also be extended to the case of an interface. That is, a singular situation where the metric is not anymore smooth, but has a singularity along an hypersurface. This is interpreted as a refraction/reflexion problem, as we shall see. And we shall get then, as an application of this extended general covariance principle, the Snell-Descartes law of reflection and refraction of light.

The General Covariance Principle with interface

In the following we shall consider the following situation, described by Fig. 1. A manifold M is splited in two parts M_1 and M_2 by an embedded hypersurface Σ. We can consider that M_1 and M_2 are two manifolds with a shared boundary Σ, but their union is the smooth manifold without boundary M:

$$M_1 \cap M_2 = \Sigma \quad \text{and} \quad M_1 \cup M_2 = M.$$

Actually since what we are going to investigate is semi-local, global subtleties here are irrelevant.

Next, on each part M_1 and M_2 we define two smooth Riemannian (or Pseudo-Riemannian) metrics g_1 and g_2. That means precisely that g_1 is the restriction to M_1 of a metric defined on a small open neighborood of M_1, idem for g_2 with M_2. The two metrics have then a limit on Σ which may not coincide. We shall denote by $g = (g_1, g_2)$ this metric on M, having a discontinuity on Σ.

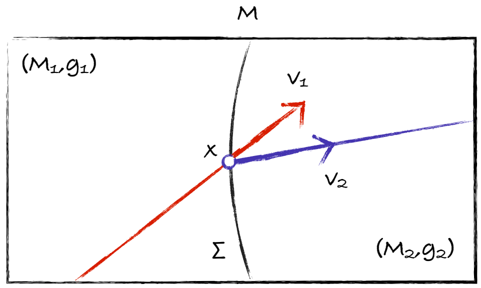

FIGURE 1. Refraction of Ligh Rays

This situation is represented by the figure Fig. 1. Next, we introduce the subgroup of compact supported diffeomorphisms of M preserving the hypersurface Σ,

$$\mathrm{Diff}_{\mathrm{cpct}}(\mathrm{M},\Sigma)=\{\varphi\in\mathrm{Diff}_{\mathrm{cpct}}(\mathrm{M})\mid\varphi(\Sigma)=\Sigma\}.$$

Actually we want φ to preserve separately the two parts M_1 and M_2. That is, $\varphi\in\mathrm{Diff}_{\mathrm{cpct}}(\mathrm{M})$, $\varphi\restriction\mathrm{M}_1\in\mathrm{Diff}_{\mathrm{cpct}}(\mathrm{M}_1)$, $\varphi\restriction\mathrm{M}_2\in\mathrm{Diff}_{\mathrm{cpct}}(\mathrm{M}_2)$, and of course $\varphi\restriction\Sigma\in\mathrm{Diff}_{\mathrm{cpct}}(\Sigma)$. To say that φ is connected to the identity would be sufficient to get these properties.

Now, we introduce the test tensors for the distribution tensors relative to this configuration. Let $g=(g_1,g_2)$ be a metric on $\mathrm{M}=(\mathrm{M}_1,\mathrm{M}_2)$ as described above and let $s\mapsto g_s=(g_{1,s},g_{2,s})$ two smooth paths in the set of metrics, pointed at g, that is, $g_0=g$. We shall denote $\delta g=(\delta g_1,\delta g_2)$ the variation of g_s at $s=0$,

$$\delta g_1=\left.\frac{\partial g_{1,s}}{\partial s}\right|_{s=0}\quad\text{and}\quad\delta g_2=\left.\frac{\partial g_{2,s}}{\partial s}\right|_{s=0}.$$

Therefore, an infinitesimal diffeomorphism which preserve the figure gives a variation of g which is a Lie derivative:

$$\delta_{\mathrm{L}}g=(\delta_{\mathrm{L}}g_1,\delta_{\mathrm{L}}g_2)\quad\text{with}\quad\delta_{\mathrm{L}}g_i=\left.\frac{\partial(e^{s\xi})^*(g_i)}{\partial s}\right|_{s=0},$$

where ξ is a smooth vector field on M such that $\xi(x)\in\mathrm{T}_x\Sigma$ for all $x\in\Sigma$. Its exponential $e^s\xi$ belongs to $\mathrm{Diff}_{\mathrm{cpct}}(\Sigma)$. We have then

$$(\delta_{\mathrm{L}}g_i)_{\mu\nu}=\hat{\partial}_{i,\mu}\xi_\nu+\hat{\partial}_{i,\nu}\xi_\mu,$$

where $\hat{\partial}_i$ is the covariant derivative with respect to g_i. Let us come back now to the tensor distribution, according to what was said until now, any tensor distribution $\mathcal{T}$ relative to this configuration splits in the sum of two tensor, each relative to a part of M, that is,

$$\mathcal{T}(\delta g)=\mathcal{T}_1(\delta g_1)+\mathcal{T}_2(\delta g_2).$$

The tensor distribution $\mathscr{T}_1$ is equal to $\mathscr{T}(\delta g)$ for $\delta g_2 = 0$, as well for $\mathscr{T}_2$ with $\delta g_1 = 0$. Therefore, for $\mathscr{T}$ to be Eulerian means:

$$\mathscr{T}(\delta_{\rm L} g) = 0 \quad \Leftrightarrow \quad \mathscr{T}_1(\delta_{\rm L} g_1) + \mathscr{T}_2(\delta_{\rm L} g_2) = 0.$$

The Broken Geodesics

We consider now a curve γ in M that cuts Σ at some point x for $t = 0$, and let γ_1 and γ_2 be the two pieces of γ,

$$\gamma_1 \colon]-\infty, 0] \to \mathrm{M}_1 \quad \text{and} \quad \gamma_2 \colon [0, +\infty[\to \mathrm{M}_2, \quad \text{with} \quad \gamma_1(0) = \gamma_2(0) = x.$$

Let $\mathscr{T}$ be a tensor distribution supported by γ,

$$\mathscr{T}(\delta g) = \frac{1}{2}\int_{-\infty}^{0} \mathrm{T}_1^{\mu\nu} \delta g_{1,\mu\nu}\, dt + \frac{1}{2}\int_{0}^{+\infty} \mathrm{T}_2^{\mu\nu} \delta g_{2,\mu\nu}\, dt.$$

Solving the Eulerian condition $\mathscr{T}(\delta_{\rm L} g) = 0$, by considering a vector field ξ whose supports lies first anywhere in M_1, then in M_2 we get, thanks to Souriau's computation, that γ_1 and γ_2 are geodesic for the respective metric and also that:

$$\mathscr{T}(\delta g) = \frac{k_1}{2}\int_{-\infty}^{0} \frac{dx^\mu}{dt}\frac{dx^\nu}{dt} \delta g_{\mu\nu}\, dt + \frac{k_2}{2}\int_{0}^{+\infty} \frac{dx^\mu}{dt}\frac{dx^\nu}{dt} \delta g_{\mu\nu}\, dt.$$

Where k_1 and k_2 are *a priori* two arbitrary real constants. Applied to $\delta_{\rm L} g$ for a compact supported vector field ξ tangent to Σ, one gets, by regroupings terms, using the fact that the curves are geodesic, and integrating by parts:

$$\begin{aligned}
0 &= k_1 \int_{-\infty}^{0} \frac{dx^\mu}{dt}\frac{dx^\nu}{dt} \hat{\partial}_{1,\nu}\xi_{\mu}, dt + k_2 \int_{0}^{+\infty} \frac{dx^\mu}{dt}\frac{dx^\nu}{dt} \hat{\partial}_{2,\nu}\xi_{\mu}, dt \\
&= k_1 \int_{-\infty}^{0} \frac{\hat{\partial}_1 \xi_\mu}{\partial x^\nu}\frac{dx^\nu}{dt}\frac{dx^\mu}{dt}\, dt + k_2 \int_{0}^{+\infty} \frac{\hat{\partial}_2 \xi_\mu}{\partial x^\nu}\frac{dx^\nu}{dt}\frac{dx^\mu}{dt}\, dt \\
&= k_1 \int_{-\infty}^{0} \frac{\hat{d}_1 \xi_\mu}{dt}\frac{dx^\mu}{dt}\, dt + k_2 \int_{0}^{+\infty} \frac{\hat{d}_2 \xi_\mu}{dt}\frac{dx^\mu}{dt}\, dt \\
&= k_1 \int_{-\infty}^{0} \frac{\hat{d}_1}{dt}\left[\xi_\mu \frac{dx^\mu}{dt}\right] dt + k_2 \int_{0}^{+\infty} \frac{\hat{d}_2}{dt}\left[\xi_\mu \frac{dx^\mu}{dt}\right] dt \\
&= k_1 \xi(x)_\mu \lim_{t\to 0^-} \frac{dx^\mu}{dt} - k_2 \xi(x)_\mu \lim_{t\to 0^+} \frac{dx^\mu}{dt}.
\end{aligned}$$

Then, denoting by $\mathbf{v}_1$ and $\mathbf{v}_2$ the limits of the velocities

$$\mathbf{v}_1 = \lim_{t\to 0^-} \dot{\gamma}_1(t) \quad \text{and} \quad \mathbf{v}_2 = \lim_{t\to 0^+} \dot{\gamma}_2(t),$$

we get the Eulerian condition for the tensor distribution $\mathscr{T}$ at the point $x \in \Sigma$ which writes

$$k_1 g_1(\mathbf{v}_1, \delta x) = k_2 g_2(\mathbf{v}_2, \delta x) \quad \text{for all} \quad \delta x \in \mathrm{T}_x(\Sigma). \tag{$\spadesuit$}$$

We get then a general condition which must be satisfied by the two geodesics γ_1 and γ_2 at the interface, to define an Eulerian distribution. But the system is underdetermined due to the arbitrary choice of constants k_1 and k_2. We can diminish the arbirary by considering the symplectic structure of the spaces of geodesics.

Remark. — The case above simplifies in the case where there is no part 2 and just the curve γ_1. In that situation the equation (♠) becomes $g_1(\mathbf{v}_1, \delta x) = 0$ for all $\delta x \in \mathrm{T}_x(\Sigma)$, that is,

$$\mathbf{v}_1 \perp \Sigma. \qquad (♠')$$

This situation has been observed for light rays in a saline solution for which the concentration tends to infinity near the border. It is regarded sometimes as a model for the Poincaré's half-plane.

The Symplectic Diffusion

We shall use now the symplectic structure of the space of geodesics and we shall request the distribution $\mathscr{T}$ to be a symplectic diffusion $\mathscr{T}_1 \to \mathscr{T}_2$, in order to lower the degree of indetermination in the equation (♠).

Let us recall now that the space $\mathfrak{Z}$ of geodesics curves in a Riemannian (or Pseudo-Riemannian) manifold (M, g) has a canonical symplectic structure. It is the projection of the exterior derivative $d\varpi$ of the Cartan 1-form ϖ defined on $\mathrm{Y} = \mathbf{R} \times (\mathrm{TM} - \mathrm{M})$ by

$$\varpi(\delta y) = k\, g(\mathbf{v}, \delta x) - \frac{k}{2} g(\mathbf{v}, \mathbf{v}) \delta t,$$

where $y = (t, x, \mathbf{v}) \in \mathrm{Y}$ and $\delta y \in \mathrm{T}_y\mathrm{Y}$. Let $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ be the space of geodesics in (M_1, g_1) and (M_2, g_2). In the two cases a parametrized geodesic that cut (transversaly) the interface Σ is completely determined by $(t, x, \mathbf{v})$, where $x \in \Sigma$ is the intersection of the geodesic with the interface, $t \in \mathbf{R}$ is the time of the impact and $\mathbf{v} \in \mathrm{T}_x\mathrm{M}$ is the velocity of the geodesic at the point x. This defined two open subsets $\mathfrak{Z}_1^\star \subset \mathfrak{Z}_1$ and $\mathfrak{Z}_2^\star \subset \mathfrak{Z}_2$. These two subspaces are equipped naturally with the restriction of their Cartan form ϖ_1 and ϖ_2, as stated above.

Because of the indetermination we have no clear map $\mathscr{T}_1 \to \mathscr{T}_2$, but a relation $\mathscr{R}$ that we can describe by its graph in $\mathfrak{Z}_1^\star \times \mathfrak{Z}_2^\star$. Let $y_1 = (t_1, x_1, \mathbf{v}_1)$ and $y_2 = (t_2, x_2, \mathbf{v}_2)$ represent two geodesics in each of these spaces:

$$y_1 \mathscr{R} y_2 \quad \Leftrightarrow \quad \begin{cases} t_1 = t_2 = t, \quad x_1 = x_2 = x \\ \text{and} \\ k_1 g_1(\mathbf{v}_1, \delta x) = k_2 g_2(\mathbf{v}_2, \delta x) \quad \text{for all } \delta x \in \mathrm{T}_x\Sigma \end{cases}$$

Actually, we shall ask $\mathscr{R}$ to to preserve the Cartan form. It is a stronger request but usual in this kind of problems. Let us precise that, for the relation $\mathscr{R}$, to preserve the

Cartan form means that the form $\varpi_1 \ominus \varpi_2$, defined on $\mathfrak{Z}_1^\star \times \mathfrak{Z}_2^\star$, vanishes on $\mathscr{R}$. That is, for all $y \in \mathscr{R}$, if $\delta y \in \mathrm{T}_y\mathscr{R}$ then $\varpi_1 \ominus \varpi_2(\delta y) = 0$. But

$$\begin{aligned} \varpi_1 \ominus \varpi_2(\delta y) &= k_1 g_1(\mathbf{v}_1, \delta x) - \frac{k_1}{2} g_1(\mathbf{v}_1, \mathbf{v}_1)\delta t \\ &- k_2 g_2(\mathbf{v}_2, \delta x) + \frac{k_2}{2} g_2(\mathbf{v}_2, \mathbf{v}_2)\delta t. \end{aligned}$$

Therefore, $\mathscr{R}$ preserves the Cartan form between the two spaces of geodesics if and only if

$$k_1 g_1(\mathbf{v}_1, \mathbf{v}_1) = k_2 g_2(\mathbf{v}_2, \mathbf{v}_2). \qquad (\clubsuit)$$

We notice here that $k_1 g_1(\mathbf{v}_1, \mathbf{v}_1)/2$ is the moment of the symmetry group of time translation $(t, x, \mathbf{v}) \mapsto (t + e, x, \mathbf{v})$, for all $e \in \mathbf{R}$. And preserving the Cartan form is equivalent to the preservation of the energy-moment when crossing the interface (which is not surprising). It describes a conservative — *i.e.* non-dissipative — process, which is the rule in symplectic mechanics.

Thus, the symplectic framework decreases the level of indetermination in the scatering process, but it remains the arbitrary of the ratio k_2/k_1 which maybe lifted by a state function proper to the system considered.

We shall see that, for geometric optics, the physical context lifts indeed the indetermination.

Note. — This construction iron out also the question of *reflexion*, when γ_2 lies in M_1 too. In this case $k_1 = k_2$ and the symplectic condition states that $\mathbf{v}_1$ and $\mathbf{v}_2$ have the same norm relatively to g_1, which solves the question without other considerations.

The Case of Light

Let us consider now the special case of light rays propagating in two homogeneous media with different indices n_1 and n_2, separated by the hypersurface Σ. We shall see that the particular context of geometric optics will achieve to lift the indetermination in the rule ($\spadesuit$), with the condition of symplectic diffusion ($\clubsuit$). Let v_1 and v_2 be the speed of light in the two different media. Then,

$$g_1 = n_1^2 g_0 \quad \text{and} \quad g_2 = n_2^2 g_0 \quad \text{with} \quad n_1 = \frac{c}{v_1} \quad \text{and} \quad n_2 = \frac{c}{v_2},$$

where g_0 is an ambiant smooth metric on M, the vacuum metric. Assuming $\mathrm{M} \subset \mathbf{R}^3$, we chose g_0 to be the standard scalar product, and we note as usual $g_0(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$ and $g_0(\mathbf{v}, \mathbf{v}) = ||\mathbf{v}||^2$. Let us recall that we defined the color coefficients as

$$k_1 = \frac{h\nu_1}{c^2} \quad \text{and} \quad k_2 = \frac{h\nu_2}{c^2},$$

and let,

$$\mathbf{v}_1 = v_1 \mathbf{u}_1 \quad \text{and} \quad \mathbf{v}_2 = v_1 \mathbf{u}_2 \quad \text{with} \quad ||\mathbf{u}_1||^2 = ||\mathbf{u}_2||^2 = 1.$$

Next, the symplectic condition (♣) writes

$$k_1 n_1^2 \|\mathbf{v}_1\|^2 = k_2 n_2^2 \|\mathbf{v}_2\|^2, \quad \text{that is,} \quad k_1 \frac{c^2}{v_1^2} v_1^2 = k_2 \frac{c^2}{v_2^2} v_2^2.$$

Hence,

$$k_1 = k_2 \quad \textit{i.e.} \quad \nu_1 = \nu_2.$$

The frequency of the light ray is preserved when crossing the interface. Now, the covariant condition (♠) writes, for all $\delta x \in \mathrm{T}_x \Sigma$,

$$n_1^2 \mathbf{v}_1 \cdot \delta x = n_2^2 \mathbf{v}_2 \cdot \delta x \Leftrightarrow \frac{c^2}{v_1^2} v_1 \mathbf{u}_1 \cdot \delta x = \frac{c^2}{v_2^2} v_2 \mathbf{u}_2 \cdot \delta x \Rightarrow \frac{c}{v_1} \mathbf{u}_1 \cdot \delta x = \frac{c}{v_2} \mathbf{u}_2 \cdot \delta x.$$

That is, $(n_1 \mathbf{u}_1 - n_2 \mathbf{u}_2) \cdot \delta x = 0$ for all $\delta x \in \mathrm{T}_x \Sigma$. And then,

$$n_2 \mathbf{u}_2 = n_1 \mathbf{u}_1 + \alpha \mathbf{n},$$

where $\mathbf{n}$ is the unit normal vector to Σ at x, and α is a solution of the second degree equation

$$\alpha^2 + 2 n_1 \cos(\theta_1) \alpha + n_1^2 - n_2^2 = 0,$$

where the θ_i are the angles between the $\mathbf{u}_i$ and the normal $\mathbf{n}$. We notice that, in this case, the distribution tensor $\mathscr{T}$ defines a difusion map $\mathscr{T}_1 \to \mathscr{T}_2$, from the inward geodesic γ_1 to the outward geodesic γ_2.

Finally, from $n_2 \mathbf{u}_2 - n_1 \mathbf{u}_1 \propto \mathbf{n}$ we deduce the classical Snell-Descartes law of refraction of light,

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2), \qquad (\heartsuit)$$

which becomes a consequence of the general covariance principle, conjugated with the symplectic structure of the space of light rays. The case of reflexion is also solved and gives as usual $\sin(\theta_1) = \sin(\theta_2)$.

Behind the Scenes

It is very intriguing that the evolution of material media, that are generally described by a principle of least action, can be found again by a principle of invariance (actually covariance) by diffeomorphisms. The medium can be elastic, or a particle, or even a light ray refracted by an interface.

There is always some kind of mystery behind the principle of least action, sort of God willing principle. Or the Nature which — by a hidden will — decides at each instant what is the best for the system in its immediate future. Every student who has been there once, knows this feeling of some magic that governs the world of matter.

And suddenly, we have the same behavior described just by an invariance principle, where all the magic has disappeared: that *material presence and its evolution do not depend on the way they are described*, they have an objective nature. And it is this objectivity which is captured by this principle of general covariance: the laws of nature are described by a covector on the space of geometries modulo diffeomorphisms. How is that possible?

Actually, these two principles are not as independant as they seem. *The principle of least action and the principle of general covariance are indeed the two faces of the same coin*. And to bring that to light, we shall consider the following toy model.

Let $\mathscr{C}$ be a space, we shall call the *space of fields*. Let $\mathscr{M}$ be a space, we shall call the *space of geometries*. Here $\mathscr{C}$ and $\mathscr{M}$ will be manifolds. Let G be a Lie group acting on the fields and the geometries. Let $c \in \mathscr{C}$, $g \in \mathscr{M}$ and $\varphi \in \mathrm{G}$. Let $\varphi_*(c)$ and $\varphi_*(g)$ be the action of φ on c and g. Assume that the action of G on $\mathscr{C}$ is *transitive*. Let $\mathscr{A} : \mathscr{C} \times \mathscr{M} \to \mathbf{R}$ be a smooth map we call the *action function*. Assume that the action is invariant under the diagonal action of G on $\mathscr{C} \times \mathscr{M}$. That is

$$\mathscr{A}(\varphi_*(c), \varphi_*(g)) = \mathscr{A}(c, g) \quad \text{for all} \quad \varphi \in \mathrm{G}.$$

Next, consider a 1-parameter subgroup $s \mapsto \varphi_s$ in G and let us put

$$\delta_{\mathrm{L}} c = \left.\frac{\partial \varphi_{s*}(c)}{\partial s}\right|_{s=0} \quad \text{and} \quad \delta_{\mathrm{L}} g = \left.\frac{\partial \varphi_s^*(g)}{\partial s}\right|_{s=0},$$

where $\varphi^* = \varphi^{-1}{}_*$. Therefore, taking the derivative of the invariance identity of the action above, we get that for all infinitesimal action δ_{L} of the group G we have

$$\mathscr{J}(\delta_{\mathrm{L}} c) - \mathscr{T}(\delta_{\mathrm{L}} g) = 0 \quad \text{with} \quad \mathscr{J} = \frac{\partial \mathscr{A}}{\partial c} \quad \text{and} \quad \mathscr{T} = \frac{\partial \mathscr{A}}{\partial g}.$$

Thus, obviously, if $\mathscr{T}(\delta_{\mathrm{L}} g) = 0$, then $\mathscr{J}(\delta_{\mathrm{L}} c)$, and conversely. Hence:

(1) If c is a critical point for $\mathscr{A}_g : c \mapsto \mathscr{A}(c, g)$, then $\mathscr{T}(\delta_{\mathrm{L}} g) = 0$ for all $\delta_{\mathrm{L}} g$. That is, $\mathscr{T}$ is Eulerian.

(2) If $\mathscr{T}$ is Eulerian, then $\mathscr{J}(\delta_{\mathrm{L}} c) = 0$ for all $\delta_{\mathrm{L}} c$. But since G is transitive on $\mathscr{C}$, we get $\mathscr{J}(\delta c) = 0$ for any variation δc. Which means that

$$\delta \mathscr{A}_g = \left.\frac{\partial \mathscr{A}_g(c_s)}{\partial s}\right|_{s=0} = 0 \quad \text{with} \quad \mathscr{A}_g : c \mapsto \mathscr{A}(c, g),$$

for all smooth path $s \mapsto c_s$. Therefore, c is a critical point of the action $\mathscr{A}_g$, and a solution of the least action principle for this action.

Of course, for the cases treated in general the space of fields and the space of geometries are not manifolds but infinite dimmensional spaces. There is certainly a right way to treat these situations rigourously, from the point of view of diffeology for example. But that is not the point here. The idea here is just to see how the principle of general covariance withdraws the magic from the principle of least action, by giving the main role to the laws of symmetries and the principle of objectivity/relativity which definitely deserve it. At the same time the principle of general covariance is a well founded *a priori* generalization of the least action principle.

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Patrick Iglesias-Zemmour — Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France — & — Einstein Institute of Mathematics Edmond J. Safra Campus, The Hebrew University of Jerusalem Givat Ram. Jerusalem, 9190401, Israel.

E-mail address: piz@math.huji.ac.il