General Relativity, or The Principle of General Covariance

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Preamble

In his famous article *Modèle de particule à spin dans le champ électromagnétique et gravitationnel* published in 1974, J.-M. Souriau explains the principle of general relativity as a general covariance principle, involving the (infinite dimensional) space of all pseudo-Riemannian metrics on space-time modulo the action of the (infinite dimensional) group of compactly supported diffeomorphisms. Specifically: how to find the equations of passive matter in all its forms (particle, continuous medium, fluid, string, with spin or without etc.) in general relativity using a unique covariance principle. In this first talk, I will follow his paper in its heuristic approach. In a second talk, later in the year, I will show how this heuristic construction can be given a rigorous diffeological foundation.

Bibliographical References

This presentation is based on the following article:

- Jean-Marie Souriau. *Modèle de particule à spin dans le champ électromagnétique et gravitationnel*. Ann. Inst. Henri Poincaré, XX A, 1974.

With a special mention to

- Jean-Marie Souriau. *Du bon usage des élastiques*. In Journées Relativistes (Clermont-Ferrand, 6-8 Apr. 1973). Publ. Dépt Math. Univ. de Clermont, 1974.

Bibliographical References

I also recommend the following three papers:

- Shlomo Sternberg. *General Covariance and the Passive Equations of Physics*. Albert Einstein Memorial Lecture, ISBN 965-208-173-6. The Israel Academy of Sciences and Humanities, Jerusalem 2006.
- Patrick Iglesias-Zemmour. *Refraction and reflexion according to the principle of general covariance*. Journal of Geometry and Physics, 142, pp. 1–8, 2019.
- Thibault Damour. *Editorial note to: On the motion of spinning particles in general relativity by Jean-Marie Souriau*. General Relativity and Gravitation, 56(10):127, 2024. Co-author P.I-Z for the Souriau short biography.

The basic framework of General Relativity is defined by:

1. The space-time: a 4-dimensional manifold M. A point x is identified by its coordinates $(x^\mu)_{\mu=1}^4$ in a chart.
2. A Poincaré-Lorentz pseudo-metric g, with components $(g_{\mu\nu})_{\mu,\nu=1}^4$, of signature $(+\ -\ -\ -)$.

It is also traditional to write the coordinates (t, x, y, z) and the pseudo-metric $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$.

Fields Equations

The theory of general relativity is most often considered as a theory of gravitation. The fields equations (Einstein's equations), describe the relationship between the gravitational field g and the distribution of matter characterized by a 4-Stress Tensor T,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}.$$

Where $R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature and Λ is a (cosmological) constant.

The tensor T take various form depending on the nature of the matter: dust, fluid of continuous medium.

Passive Matter

In contrast to active matter that satisfies Einstein's equation, we have the passive matter evolving in a gravitational field, submitted to it without interfering with it.

It can be, for examples:

- A particle.
- A string.
- A veil.
- A continuous medium.

Their motions in the space-time is described by specific equations. We shall see a few examples.

Continuous Medium

A 3-dimensional continuous medium is described by a continuous symmetric covariant 2-tensor $T_{\mu\nu}$.

The evolution of the 3-dimensional continuous medium, submitted to the gravitational field $g_{\mu\nu}$, is discribed by the Euler's continuity equations, for all $\nu = 1 \dots 4$ and covariant derivatives:

$$\hat{\mathrm{d}}\mathrm{iv}\ T = 0, \quad \text{that is,} \quad \sum_{\mu=1}^{4} \frac{\hat{\partial} T^{\mu\nu}}{\partial x^{\mu}} = 0 \quad \text{or} \quad \hat{\partial}_{\mu} T^{\mu\nu} = 0,$$

with Einstein's conventions.[1]

[1] For some reason, I use the symbol $\hat{\partial}$ for the covariant derivative, in general denoted by ∇.

Particles

A particle submitted to the gravitational fields g satisfies the so-called Geodesic Principle: its motion is a geodesic trajectory in space-time. It satisfies the geodesic equations:

$$\frac{dx}{dt} = v \quad \text{and} \quad \frac{\hat{d}v}{dt} = 0.$$

That is,

$$v^\mu = \frac{dx^\mu}{dt} \quad \text{and} \quad \frac{\hat{d}v^\mu}{dt} = \frac{dv^\mu}{dt} + \Gamma^\mu_{\sigma\rho} v^\sigma \frac{dx^\rho}{dt} = 0,$$

with Einstein summation notations.

Strings I

The evolution of a string (rubber band) in space-time is described by a surface $V \subset M$, normal-hyperbolic for the induced pseudo-Riemmanian metric. The material structure of the string is characterized by a symmetric covariant 2-tensor Θ, defined on the surface with (normed) orthogonal proper vectors:

- A timelike vector I with proper value ρ, the specific mass.
- A space-like vector J with proper value θ, the tension.

We denote the associated derivation by δ_I and δ_J

Strings II

The evolution of the string satisfies the system:

$$\begin{cases} \delta_I \rho - [\rho - \theta]\langle I, C\rangle = 0 \\ \delta_J \theta - [\rho - \theta]\langle J, \Gamma\rangle = 0 \\ \rho\Gamma - \theta C \text{ tangent to } V \\ \theta = 0 \text{ at } \partial V. \end{cases}$$

With

$$\Gamma = \hat{\delta}_I I \quad \text{and} \quad C = \hat{\delta}_J J$$

The integral curves of the vector I are the worldlines of the molecules of the string, Γ is the geodesic acceleration of the molecules. The integral curves of the vector J are successives spacelike sections (positions) of the string.

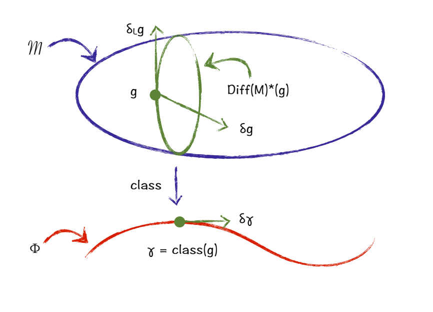
The Dispute

Einstein presented a principle of general relativity as the (vague) statement that “The shape of the equation of physics are invariant by change of coordinates”. As we have seen in the previous examples, the “shape of an equation” is not easy to define, to say the least, in a concise statement.

Rather than trying to resolve this difficulty, many physicists, like Vladimir Fock, for example, prefer to abandon the principle of general relativity and reduce the theory to a relativistic theory of gravitation, of which we have seen an example of the essential equations: field and passive matter.

On the contrary, Jean-Marie Souriau has produced a precise geometric statement that embodies the Einstein intention. He called it, the Principle of General Covariance.

The Principle of General Covariance I



- $\mathcal{M}$ is the space of all pseudo-Riemannian metrics on M.
- Φ is the Physis: the quotient of $\mathcal{M}$ by $\mathrm{Diff}_c(M)$, acting by pushforward $\varphi : g \mapsto \varphi_*(g)$.
- g is a metric, γ is its class modulo $\mathrm{Diff}_c(M)$.
- δg is a variation of g, $\delta\gamma$ is its projection on Φ.
- $\delta_L g$ is a vertical variation of g, the kernel of $D(\mathrm{class})$.

The Principle of General Covariance II

J.-M. Souriau proposed the following replacement of the too vague principle of general relativity:

> Principle of General Covariance (J.M. Souriau) *Any passive matter, in presence of a gravitational field* $\mathsf{g} \in \mathcal{M}$, *is represented by covector at* $\gamma = \mathrm{class}(\mathsf{g}) \in \Phi$.

That is, a passive matter will be represented by a tensor distribution $\mathcal{T}$, in the algebraic dual of the space of compact supported variations $\delta\mathsf{g}$. Coming from Φ, it must satisfy the Eulerian condition, and conversely.

$$\boxed{\mathcal{T}(\delta_L \mathsf{g}) = 0 \quad \text{for all vertical variation } \delta_L \mathsf{g}.}$$

Let's recall that "vertical variations" are Lie derivatives.

Application to Continuous Medium

A continuous medium is defined wih a continuous symmetric tensor $T^{\mu\nu}$ by:

$$\mathcal{T}(\delta g) = \frac{1}{2}\int_M T^{\mu\nu}\delta g_{\mu\nu}\text{vol}$$

For all compact supported vector fied $\delta x = \xi$:

$$\mathcal{T}(\delta_L g) = 0 \Leftrightarrow \frac{1}{2}\int_M T^{\mu\nu}(\hat{\partial}_\mu \xi_\nu + \hat{\partial}_\nu \xi_\mu)\text{vol} = \int_M T^{\mu\nu}\hat{\partial}_\mu \xi_\nu \text{vol} = 0$$

$$\text{i.e.} \quad \int_M \hat{\partial}_\mu (T^{\mu\nu}\xi_\nu)\text{vol} - \int_M (\hat{\partial}_\mu T^{\mu\nu})\xi_\nu \text{vol} = 0.$$

But $\hat{\partial}_\mu(T^{\mu\nu}\xi_\nu) = \hat{\text{div}}(\theta)$, with θ a compact supported vector field. Thus $\int_M \hat{\partial}_\mu(T^{\mu\nu}\xi_\nu)\text{vol} = 0$ and therefore for all compact supported vector field ξ

$$\int_M (\hat{\partial}_\mu T^{\mu\nu})\xi_\nu \text{vol} = 0 \;\Rightarrow\; \boxed{\hat{\partial}_\mu T^{\mu\nu} = 0.}$$

Application to Geodesics I

Consider the worldline c of a particle and a symmetric tensor $T^{\mu\nu}$ with support the line c emerging from any compact.

$$\mathcal{T}(\delta g) = \frac{1}{2}\int_c T^{\mu\nu}\delta g_{\mu\nu}\,ds$$

We multiply ξ by a function α equal to 0 on c.

$$\begin{aligned} 0 = \int_c T^{\mu\nu}\hat{\partial}_\mu(\alpha\xi_\nu)\,ds &= \int_c T^{\mu\nu}\xi_\nu\partial_\mu\alpha\,ds \\ + \int_c T^{\mu\nu}\underbrace{\alpha}_{=\,0}\hat{\partial}_\mu\xi_\nu\,ds &= \int_c T^\nu_\mu N^\mu\xi_\nu\,ds \quad \text{with } N^\mu = g^{\mu\nu}\partial_\nu\alpha. \end{aligned}$$

Application to Geodesics II

Then,

$$T^{\mu}_{\nu} N^{\nu} = 0 \quad \text{for all} \quad N \perp \frac{dx}{ds}.$$

Thus, there is a vector P at each point of the curve such that

$$T^{\mu\nu} = P^{\mu} \frac{dx^{\nu}}{ds}.$$

And by symmetry of the tensor T

$$P \propto \frac{dx}{ds}.$$

Thus,

$$0 = \int_c P^{\nu} \frac{dx^{\mu}}{ds} \hat{\partial}_{\mu} \xi_{\mu} \, ds = \int_c P^{\nu} \frac{\hat{\partial} \xi_{\mu}}{\partial x^{\nu}} \frac{dx^{\mu}}{ds} \, ds = \int_c P^{\nu} \frac{\hat{d} \xi_{\nu}}{ds} \, ds$$

$$= \underbrace{\int_c \frac{d}{ds} (P^{\nu} \xi_{\nu}) \, ds}_{=\,0} - \int_c \frac{\hat{d} P^{\nu}}{ds} \xi_{\nu} \, ds$$

Application to Geodesics III

We get finally

$$\frac{\hat{d}P}{ds} = 0,$$

and therefore, with the arc length of the curve for s,

$$\frac{d(P_\mu P^\mu)}{ds} = 0 \;\Rightarrow\; P = m\frac{dx}{ds} \quad \text{with} \quad \begin{cases} m = \text{cst} \\ \dfrac{\hat{d}}{ds}\left(\dfrac{dx}{ds}\right) = 0. \end{cases}$$

We recognize the equation of a geodesic of “mass” m.

Conclusion Geodesics are also covectors on the Physis, as well as continuous medium.

Srings, Veils etc

The evolution of strings in a gravitational field described by the equations above can be obtained also by considering the Eulerian distributions of the form

$$\mathcal{T}(\delta g) = \int_V T^{\mu\nu} \delta g_{\mu\nu} \ \mathrm{surf},$$

where T is a symmetric covariant 2-tensor with support the world-surface V, leaving any compact. And then, make the strings objects of the same type as continuous medium or geodesics, that is, covectors on the Physis.

The veils can be described the same way even if that has sill not be written down specifically.

Charged particles I

The principle of general covariance applies also to the case of charged particles. The geometry is then described by the pair $(g, A) \in \mathcal{M}'$, where A is the vector potential of the electromagnetic field.

The group of invariance must be changed to the semi-direct product of compact supported diffeomorphisms of space-time $\mathrm{Diff}_c(M)$ by the compact supported gauge transformations $C_c^\infty(M, \mathbf{R})$, with the action law:

$$\varphi : (g, A) \mapsto (\varphi_*(g), \varphi_*(A)) \quad \text{and} \quad f : (g, A) \mapsto (g, A + df).$$

Charged particles II

A charged continuous medium is described by a continuous tensor field on $\mathcal{M}'$, where J is the electrical current.

$$\mathcal{T}(\delta g, \delta A) = \int_M \left(\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + J^\rho \delta A_\rho\right) \mathrm{vol}.$$

The Eulerian distribution are defined by the infinitesimal covariant condition

$$\mathcal{T}(\delta_L g, \delta_L A + df) = 0.$$

That is,

$$\int_M [T^{\mu\nu} \hat{\partial}_\mu \xi_\nu + J^\rho (\underbrace{\xi^\mu \hat{\partial}_\mu A_\rho + A_\mu \hat{\partial}_\rho \xi^\mu}_{(\delta_L A)_\rho} + \partial_\rho f)] \mathrm{vol} = 0.$$

The solution of this equation is then:

$$a)\ \hat{\partial}_\mu J^\mu = 0 \quad b)\ \hat{\partial}_\mu T^\mu_\nu + F_{\mu\nu} J^\mu = 0 \text{ with } F = dA.$$

Condensed States I

A charged particle in presence of gravitation and an electro-magnetic field is described by a tensor distribution supported by a line in M.

$$\mathcal{T}(\delta g, \delta A) = \int_c \left(\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + J^\rho \delta A_\rho \right) ds.$$

The Eulerian condition above reduces to,

$$\mathcal{T}(\delta g, \delta A) = \int_c \left\{ \frac{1}{2} P^\mu \delta g_{\mu\nu} + q \delta A_\nu \right\} \frac{dx^\nu}{ds} ds,$$

where $q \in \mathbf{R}$ is the electric charge.

Condensed States II

The vector P and the charge q satisfy then the system:

$$\begin{cases} P^{\mu}\frac{dx^{\nu}}{ds} - P^{\nu}\frac{dx^{\mu}}{ds} = 0 & \text{(P is tangent to } c\text{)} \\ \frac{dq}{ds} = 0 & (q \text{ is constant}) \\ \frac{\hat{d}}{ds}P^{\mu} + qF_{\mu\nu}\frac{dx^{\nu}}{ds} = 0 & \end{cases}$$

These are the equations of motion of an electrical charged particle in presence of gravitation and electromagnetic field, in general relativity. Once again, it is an object of the "cotangent space" of the Physis, quotient of the space of geometries modulo diffeomorphisms and gauge transformations.

Spin and Dipole Moments I

Until now we have just considered measure tensors, depending only on the values of the fields. The spin of the particles appears when we consider the dipole moments, that is, when we consider not only the values of the fields but also their first derivatives.

$$\mathcal{T}(\delta g, \delta A) = \int_c \left\{ \frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + \frac{1}{2} \Phi^{\rho\mu\nu} \hat{\partial}_\rho \delta g_{\mu\nu} \right. \\ \left. + J^\mu \delta A_\mu + \Omega^{\mu\nu} \hat{\partial}_\mu A_\nu \right\} ds$$

The Eulerian condition introduces first: the spin tensor S, the magnetic field B and the magnetic moment $\mathcal{M}$

$$\Phi^{\mu\nu\rho} = \frac{dx^\nu}{ds} A^{\nu\rho} + \frac{1}{2} \left[S^{\mu\nu} \frac{dx^\rho}{ds} + S^{\mu\rho} \frac{dx^\nu}{ds} \right] \ \& \ \Omega^{\mu\nu} = \frac{dx^\mu}{ds} B^\nu + \mathcal{M}^{\mu\nu}$$

Spin and Dipole Moments II

Reintroducing these decomposition into the Eulerian condition, we get the equations of motions of the spin electric particle in presence of ravitation and electromagnetic field.

$$\begin{cases} \frac{\hat{d}P_\sigma}{ds} = qF_{\sigma\rho}\frac{dx^\rho}{ds} + \frac{1}{2}\mathcal{M}^{\mu\nu}\hat{\partial}_\sigma F_{\mu\nu} - \frac{1}{2}R_{\mu\nu,\rho\sigma}S^{\mu\nu}\frac{dx^\rho}{ds} \\ \frac{\hat{d}S^{\mu\nu}}{ds} = P^\mu\frac{dx^\nu}{ds} - P^\nu\frac{dx^\mu}{ds} - \mathcal{M}^{\mu\rho}F^\nu_\rho + \mathcal{M}^{\nu\rho}F^\mu_\rho. \end{cases},$$

where $R_{\mu\nu,\rho\sigma}$ is the Riemann curvature tensor of the metric g.

J.-M. Souriau writes: *The model we have just constructed by purely mathematical means is a candidate for describing all "point particle systems": an electron, a molecule or a star — insofar as we can neglect the quadrupole moments.*

I understand that Thibault Damour has used this model to study recently discovered gravitational waves.

Let $\mathcal{C}$ be a space, we shall call the *space of fields*. Let $\mathcal{M}$ be a space, we shall call the *space of geometries*. Here $\mathcal{C}$ and $\mathcal{M}$ are manifolds. Let G be a Lie group acting on the fields and the geometries. Let $c \in \mathcal{C}$, $g \in \mathcal{M}$ and $\phi \in G$. Let $\phi_*(c)$ and $\phi_*(g)$ be the action of ϕ on c and g. Assume that the action of G on $\mathcal{C}$ is *transitive*. Let $\mathcal{A}\colon \mathcal{C} \times \mathcal{M} \to \mathbf{R}$ be a smooth map we call the *action function*. Assume that the action is invariant under the diagonal action of G on $\mathcal{C} \times \mathcal{M}$. That is

$$\mathcal{A}(\phi_*(c), \phi_*(g)) = \mathcal{A}(c, g) \quad \text{for all} \quad \phi \in G.$$

Consider a 1-parameter subgroup $s \mapsto \phi_s$ in G and put

$$\delta_L c = \left.\frac{\partial \phi_{s*}(c)}{\partial s}\right|_{s=0} \quad \text{and} \quad \delta_L g = \left.\frac{\partial \phi_s^*(g)}{\partial s}\right|_{s=0}.$$

Behind the scene II

Now, taking the derivative of the above invariance identity of the action, we get that, for all infinitesimal action δ_L of the group G:

$$\mathcal{J}(\delta_L c) - \mathcal{T}(\delta_L g) = 0 \quad \text{with} \quad \mathcal{J} = \frac{\partial \mathcal{A}}{\partial c} \quad \text{and} \quad \mathcal{T} = \frac{\partial \mathcal{A}}{\partial g}.$$

Thus,

$$\underbrace{\mathcal{T}(\delta_L g) = 0}_{\text{Eulerian condition}} \quad \Leftrightarrow \quad \underbrace{\mathcal{J}(\delta_L c) = 0}_{\text{Critical point}},$$

1. If c is a critical point for $\mathcal{A}_g : c \mapsto \mathcal{A}(c, g)$, then $\mathcal{T}(\delta_L g) = 0$ for all $\delta_L g$. That is, $\mathcal{T}$ is Eulerian.
2. If $\mathcal{T}$ is Eulerian, then $\mathcal{J}(\delta_L c) = 0$ for all $\delta_L c$. But since G is transitive on $\mathcal{C}$, we get $\mathcal{J}(\delta c) = 0$ for any variation δc. Thus, c is a critical point.

Behind the scene II

In conclusion, The Principle of General Covariance is a (semi) generalisation of Maupertuis' Principle of Least Action. The philosophical Principle of Finality embodied by the principle of least action is advantageously replaced by a principle of covariance, also known as the Principle of General Relativity, stating that things do not depend on their representations, which is satisfactory.

Note A semi-generalisation because if we do not need anymore to find, or invent, a Lagrangian function for the system we study: Not all Lagrangian system have its equivalents Eulerian distribution. Indeed, an irrational geodesic of the 2-torus T^2 has no equivalent because the stress-tensor $\mathcal{T}$ do not converge: the geodesic "fills" any compact.