DIFFERENTIAL FORMS ON MANIFOLDS WITH BOUNDARY AND CORNERS

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Abstract. We prove that, for the subset diffeology, a differential form defined on a corner, or a submanifold with boundary, extends to a smooth form on an open neighbourhood. We illustrate these results with the decomposition of invariant symplectic forms.

Introduction

In diffeology, the definition of manifold with corners[1] follows our natural intuition:

Definition. — *A* n-manifold with corners *is a diffeological space locally diffeomorphic to the corner* K^n *at each point.*

But, the introduction of manifolds with corners goes back to J. Cerf and A. Douady in [Cer61, Dou62], and has been since adapted or refined by many authors, for example [ADLH73, GP74, Lee06, Joy10, etc.]. Fortunately, the two approaches define the same category (art. 3), thanks to the following theorem (art. 2).

Theorem. — *Every map from* K^n *to* $\mathbf{R}$ *which is smooth when composed with a smooth parametrization*$\mathrm{P}\colon \mathrm{U} \to \mathbf{R}^n$ *taking values in* K^n *is the restriction of a smooth map defined on some open neighborhood of* K^n.

It is interesting to notice that the heuristic defining the smooth real maps from a corner, as restrictions of smooth maps defined on open neighbourhoods, is a property of this subset diffeology. And there is no need to modify the axioms of diffeology to get this behavior, as it was suggested by some author. In particular, it is not necessary to go back to Chen's axiomatic [Che77].

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[1] We denote $\mathrm{K}^n \subset \mathbf{R}^n$ the n-corner defined by $x_i \geq 0$, $i = 1, \ldots, n$.

Next, after the 0-forms, the question on the behavior of general k-forms on manifolds with corners is natural. In (art. 6) we prove first:

THEOREM. — *Let* M *be a manifold with boundary imbedded in some manifold* N *as a* pièce à bord. *If* ∂M *is compact, then every differential k-form on* M *extends on a neighborhood of* M *in* N.

Considering corners specifically, we prove in (art. 8) the following extension theorem:

THEOREM. *Every differential form on the corner* K^n *extends to a smooth form on an open neighborhood of* K^n *in* $\mathbf{R}^n$. *Thus, a differential form defined on a a* pièce à coins *extends locally to a smooth form on some open neighbourhood.*

The notion of «pièce à bord/coins» is borrowed from [Dou62].

In (art. 9) we give a variation of this theorem for other corners, made with powers of various half-lines. In (art. 10) we show an application of these considerations to the construction of $\mathrm{SO}(2)^n$-invariant parasymplectic forms on $\mathbf{R}^{2n}$.

DIFFEOLOGY OF MANIFOLDS WITH CORNERS

In this section, we recall the notions of diffeology and the few constructions we use in the following. The reader is referred to the book Diffeology for the details [PIZ13].

1. DIFFEOLOGY AND DIFFEOLOGICAL SPACES. A *diffeology* on a set X is the choice of a set $\mathcal{D}$ of parametrizations[2] in X that satisfies the following axioms.

(1) COVERING : $\mathcal{D}$ contains the constant parametrizations.
(2) LOCALITY : Let P be a parametrization in X. If for all $r \in \mathrm{dom}(\mathrm{P})$ there is an open neighbourhood V of r such that $\mathrm{P} \restriction \mathrm{V} \in \mathcal{D}$, then $\mathrm{P} \in \mathcal{D}$.
(3) SMOOTH COMPATIBILITY : For all $\mathrm{P} \in \mathcal{D}$, for all $\mathrm{F} \in \mathrm{C}^\infty(\mathrm{V}, \mathrm{dom}(\mathrm{P}))$, where V is a Euclidean domain, $\mathrm{P} \circ \mathrm{F} \in \mathcal{D}$.

A set X equipped with a diffeology is a *diffeological space*. The elements of $\mathcal{D}$ are then called plots of the diffeological space.

SMOOTH MAPS. — *A map* $f : \mathrm{X} \to \mathrm{X}'$ *is said to be* smooth *if for any plot* P *in* X, $f \circ \mathrm{P}$ *is a plot in* X′. *If* f *is smooth, bijective, and if its inverse* f^{-1} *is smooth, then* f *is said to be a* diffeomorphism.

Diffeological spaces and smooth maps constitute the category {Diffeology} whose isomorphisms are diffeomorphisms.

SUBSET DIFFEOLOGY, SUBSPACES. — *Let* A *be a subset of a diffeological space* X. *The plots in* X *which take their values in* A *are a diffeology called* subset diffeology. *Equipped with this diffeology,* A *is said to be a* subspace *of* X.

LOCAL SMOOTH MAPS. — *The finest topology on* X *such that the plots are continuous is called D-Topology. A map* $f : \mathrm{A} \to \mathrm{X}'$, *where* A *is a subset of* X, *is said to be* local

[2]A parametrization is a map defined on an open subset of an Euclidean space.

smooth[3] *if* A *is a D-open subset of* X, *and* f *is smooth for the subset diffeology*[4]. *We denoted by* $\mathrm{C}^\infty_{\mathrm{loc}}(\mathrm{X},\mathrm{X}')$ *the set of local smooth maps from* X *to* X′.

Local Diffeomorphisms. — *We say that* $f\colon \mathrm{A}\to\mathrm{X}'$ *is a local diffeomorphism if it is a local smooth injective map, as well as its inverse* $f^{-1}\colon f(\mathrm{A})\to\mathrm{X}$. *We denoted by* $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{X},\mathrm{X}')$ *the set of local diffeomorphisms from* X *to* X′.

2. Corners as Diffeological Spaces. We denote by K^n the *corner*

$$\mathrm{K}^n = \{(x_i)_{i=1}^n \in \mathbf{R}^n \mid x_i \geq 0,\ i=1,\dots,n\},$$

and we equip it with the subset diffeology[5]. A plot in K^n is just a smooth parametrization to $\mathbf{R}^n$ but taking its values in K^n.

The corner K^n is *embedded* in $\mathbf{R}^n$, and closed. That is, the D-topology of the induction $\mathrm{K}^n\subset\mathbf{R}^n$ coincides with the induced topology[6] of $\mathbf{R}^n$.

The natural filtration of K^n, $\mathrm{X}_0=\{0\}\subset\mathrm{X}_1\subset\cdots\subset\mathrm{X}_n=\mathrm{K}^n$, is defined by

$$\mathrm{X}_j = \{(x_i)_{i=1}^n \in \mathrm{K}^n \mid \text{there exist } i_1<\cdots<i_{n-j} \text{ such that } x_{i_\ell}=0\}.$$

The *stratum* $\mathrm{S}_j=\mathrm{X}_j-\mathrm{X}_{j-1}$ is the subset of points in $\mathbf{R}^n$ that have j and only j coordinates strictly positive and the rest zero.

$$\mathrm{S}_j = \left\{ (x_i)_{i=1}^n \in \mathbf{R}^n \;\middle|\; \begin{array}{l} \text{There exist } i_1<\cdots<i_j \text{ such that } x_{i_\ell}>0, \\ \text{and } x_m = 0 \text{ for all } m\notin\{i_1,\dots,i_j\} \end{array} \right\}.$$

The stratum S_j is D-open in X_j, $j\geq 1$. It is the sum of $\binom{n}{j}$ connected components (also called strata) indexed by a string of j ones and $n-j$ zeros.

Theorem 1. — *Let* $f\colon\mathrm{K}^n\to\mathbf{R}^k$ *be some map. If for all smooth parametrizations* $\mathrm{P}\colon\mathrm{U}\to\mathbf{R}^n$ *such that* $\mathrm{P}(\mathrm{U})\subset\mathrm{K}^n$, $f\circ\mathrm{P}\in\mathrm{C}^\infty(\mathrm{U},\mathbf{R}^k)$, *then there exists an open neighborhood* $\mathcal{O}$ *of* K^n *and* $\mathrm{F}\in\mathrm{C}^\infty(\mathcal{O},\mathbf{R}^k)$ *such that* $f=\mathrm{F}\restriction\mathrm{K}^n$.

Theorem 2. — *Let* $f\in\mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n)$. *Then,* f *respects the natural stratification, i.e. if* $x\in\mathrm{S}_j$, *then* $f(x)\in\mathrm{S}_j$. *Moreover,* f *is the restriction of an étale map defined on an open neighbourhood of its domain of definition.*

In clear, this theorem says that *the map* f *is smooth for the subset diffeology if and only if it extends to a local smooth maps on an open neighbourhood of* K^n.

Proof. For the Theorem 1, consider the smooth parametrization $\mathrm{sq}\colon(t_1,\dots,t_n)\mapsto(t_1^2,\dots,t_n^2)$, defined on $\mathbf{R}^n$ with values in K^n. By hypothesis, the composite $f\circ\mathrm{sq}$ is smooth and it is an even function in each parameter t_i. Thanks to Whitney

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[3]Actually f is local smooth if and only if, for all plot P in X, the composite $f\circ\mathrm{P}\colon\mathrm{P}^{-1}(\mathrm{A})\to\mathrm{X}'$ is a plot. That implies in particular that A is D-open, by definition of the D-topology.

[4]To underline the fact that the ambient space X is involved, we denote $f\colon\mathrm{X}\supset\mathrm{A}\to\mathrm{X}'$.

[5]The corner K^n is the diffeological n-power of the half-line $\mathrm{K}=[0,\infty[\,\subset\mathbf{R}$, equipped with the subset diffeology.

[6]The standard topology of $\mathbf{R}^n$ is the D-topology of its standard smooth structure.

theorem [Whi43, Remark p. 160], there exists a smooth map $F\colon \mathbf{R}^n \to \mathbf{R}$ such that $f(t_1^2,\ldots,t_n^2) = F(t_1^2,\ldots,t_n^2)$. Thus $f = F \restriction K^n$.

For Theorem 2, let P be a parametrization in an Euclidean domain. We denote by $\mathrm{rk}(P)_x$ the rank of P at the point x, that is, the dimension of the image of the tangent linear map $D(P)(x)$.

Lemma. — *Let* $P\colon U \to K^n$ *be a plot. If* $P(r) \in S_j$, *then* $\mathrm{rk}(P)_r \leq j$.

◀ *Proof.* Let $P\colon U \to K^n$ be a plot and assume that $P(r') \in S_j$. Then, $P(r') = (P_1(r'), P_2(r'), \ldots, P_n(r'))$ where there exist exactly $i_1 < \cdots < i_{n-j}$ indices such that $P_{i_k}(r') = 0$. Since $P_{i_k}(r) \geq 0$ for all $r \in U$ and $P_{i_k}(r') = 0$, then $D(P_{i_k})(r') = 0$. That is, $\mathrm{rk}(P)_r \leq j$. ▶

Now, let us come back to the local diffeomorphism f, and assume first that f is defined on all K^n. Let $x \in S_j$ and $x' = f(x) \in S_k$ and $k \neq j$. We can choose $k > j$. There exists a smooth map F defined on an open neighborhood $\mathcal{O} \supset K^n$, such that f and F coincide on K^n, $f = F \restriction K^n$. And also, there exists a smooth map G defined on an open neighborhood $\mathcal{O}' \supset K^n$, such that f^{-1} and G coincide on K^n, $f^{-1} = G \restriction K^n$. The restriction of G on S_k is a plot of K^n, and $G \restriction S_k \colon x' \mapsto x \in S_j$. By the lemma, $\mathrm{rk}(G \restriction S_k)_{x'} \leq j$. But $G \restriction S_k = G \circ j_k$, where $j_k : S_k \hookrightarrow K^n$ is identified with a plot. And we know that $(F \circ G \restriction S_k)(t) = F \circ G \circ j_k(t) = F \circ G(j_k(t))$. But j_k takes values in ∂K^n (the border of K^n). Now, since f is a homeomorphism of K^n for the D-topology, it maps the border into the border, and G and f^{-1} coincide on the border. So we have $F \circ G(j_k(t)) = F \circ f^{-1}(j_k(t))$. As well, F and f coincide on the border, and $F \circ G(j_k(t)) = f \circ f^{-1}(j_k(t)) = j_k(t)$. Thus, $\mathrm{rk}(F \circ G \restriction S_k)_{x'} = \mathrm{rk}(j_k)_{x'} = k \leq \mathrm{rk}(G \restriction S_k)_{x'} \leq j$. But, we assumed that $k > j$ which is a contradiction, and $k = j$.

Now, consider the smooth parametrization $G \circ F\colon \mathcal{U} \to \mathbf{R}^n$, with $\mathcal{U} = F^{-1}(\mathcal{O}')$. Then, $\mathcal{U} \supset K^n$ and $G \circ F \restriction K^n = \mathbf{1}_{K^n}$. Hence, for all $x \in \mathring{K}^n$, $D(G \circ F)(x) = D(f^{-1} \circ f)(x) = D(\mathbf{1}_{\mathring{K}^n}) = \mathbf{1}_{\mathbf{R}^n}$, and by continuity, for all $x \in K^n$, $D(G \circ F)(x) = \mathbf{1}_{\mathbf{R}^n}$. Therefore, for all $x \in K^n$, $\mathrm{rk}(F)_x$ is maximum and equal to n. Hence, F is étale at each point of K^n. And obviously, the same for G. The situation where f is only local is not more complicated. □

3. Manifolds with Corners as Diffeological Spaces. The concept of manifolds with corners goes back to Cerf [Cer61, Chap. 1 §1.2], and Douady [Dou62, §4] (as *variétés à bords anguleux*). Over time the various descriptions of manifolds with boundary or corners evolved to a commonly accepted definition, see for example Lee in [Lee06, pp. 251-252] or more recently Joyce in [Joy10, Chap. 2], from which we extract the following definition.

Classical Definition. — *Let* M *be a paracompact Hausdorff topological space. A n-chart with corners for* M *is a pair* (U, φ), *where* U *is an open subset of* K^n, *and* φ *is a homeomorphism from* U *in* M. *Two charts with corners* (U,φ) *and* (V,ψ) *are said to be smoothly compatible if the composite map* $\psi^{-1} \circ \varphi\colon \varphi^{-1}(\psi(V)) \to \psi^{-1}(\varphi(U))$ *is a* *diffeomorphism in the sense that it admits a smooth extension to an open set in* $\mathbf{R}^n$*. An* n*-atlas with corners for* M *is a pairwise compatible family of* n*-charts with corners covering* M*. A maximal atlas is an atlas which is not a proper subset of any other atlas. An* n*-manifold with corners is a paracompact Hausdorff topological space* M *equipped with a maximal* n*-atlas with corners.*

Now naturally, from the point of view of diffeology:

Diffeological Definition. — *A* n*-manifold with corners is a diffeological space* X *which is locally diffeomorphic to the corner* K^n *at each point.*

The {Manifolds with Corners} form a subcategory of {Diffeology}. The smooth maps between manifolds with corners are just the smooth maps between diffeological spaces. Theorems 1 and 2 of the previous article insure that morphisms between manifolds with corners preserve mechanically their natural stratifications.

Theorem. — *Let* $(\mathrm{M}, \mathscr{A})$ *be a* n*-manifold with corners according to the classical framework,* $\mathscr{A}$ *being the maximal atlas of* M*. The finest diffeology* $\mathscr{D}$ *on* M *such that the charts* $\mathrm{F} \in \mathscr{A}$ *are smooth, is a diffeology of manifold with corners for which* $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M}) = \mathscr{A}$*, the D-topology of* $(\mathrm{M}, \mathscr{D})$ *coinciding with the given topology of* $(\mathrm{M}, \mathscr{A})$*. We shall denote* $\Phi\colon (\mathrm{M}, \mathscr{A}) \mapsto (\mathrm{M}, \mathscr{D})$ *this association. Conversely, let* $(\mathrm{M}, \mathscr{D})$ *be a diffeological* n*-manifold with corners. Equip* M *with its D-topology, then* $\mathscr{A} = \mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M})$ *is a maximal atlas equipping* M *with a usual structure of manifold with corners. Let* $\Psi\colon (\mathrm{M}, \mathscr{D}) \mapsto (\mathrm{M}, \mathscr{A})$ *be this association. Then* Φ *and* Ψ *are inverse of each other.*

Hence, the category of manifolds with corners, defined by adjunction of the heuristic about *what does mean to be smooth map on a closed Euclidean domain*, fits fully and faithfully in the category {Diffeology}, without needing to adapt any axioms, but just by specifying the local model to be a corner equipped with its subset diffeology.

Note 1. — As ordinary manifolds, the category {Manifolds with corners} is closed for products and sums but is not closed for the other usual set theoretic constructions. As members of the category {Diffeology}, manifolds with corners inherit of all the diffeological constructions: fiber bundles, homotopy, differential calculus, homology, cohomology, etc.

Note 2. — Since for a corner, coordinates are smooths, manifolds with corners are *reflexive diffeological spaces* [PIZ13, Exercise 79]. Also, the set of smooth real maps from a manifold with corner is a *differential structure* in the sense of Sikorski [Sik72]. Indeed, real smooth maps from K^n to $\mathbf{R}^m$ are still the restrictions of smooth maps on open neighbourhoods of K^n.

Note 3. — The diffeology framework gives a new perspective on the definition of strata of a n-manifold with corners M. Indeed, following Theorem 2 of (art. 2), one can define the different strata of M as the connected components of the orbits of the pseudogroup of local diffeomorphisms $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{M})$. That is,

$$\mathrm{Strat}(\mathrm{M}) = \{\mathscr{O}_i \in \pi_0(\mathscr{O}) \mid \mathscr{O} \in \mathrm{M}/\mathrm{Diff}_{\mathrm{loc}}(\mathrm{M})\}.$$

Moreover, Strat(M) does not capture only the decomposition of M is strata, but equipped with the quotient diffeology of M, it captures also its (transversal) smooth structure. Note also that the *regular part* of M, that is, the principal orbit of $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{M})$, is the union of strata of dimension n. It is a regular n-submanifold and an open dense subset of M as it must be. Actually, M has a (geometrical) structure of locally fibered stratified space [GIZ17].

Proof. Let us begin by a manifold with corners, according to the usual definition. The finest diffeology $\mathscr{D}$ making the charts $\mathrm{F} \in \mathscr{A}$ smooth is the set of parametrizations $\mathrm{P} \colon \mathrm{U} \to \mathrm{M}$ that satify the following: there is a covering of U by a family of open sets U_i, and for each index i a chart $\mathrm{F}_i \in \mathscr{A}$ and a smooth maps $\mathrm{Q}_i \colon \mathrm{U}_i \to \mathrm{K}^n$ such that $\mathrm{P} \restriction \mathrm{U}_i = \mathrm{F}_i \circ \mathrm{Q}_i$. We write $\mathrm{P} = \sup \mathrm{F}_i \circ \mathrm{Q}_i$.

Now, the charts $\mathrm{F} \in \mathscr{A}$ are smooth, by construction, and injective. Their domains are open for the induced topology of K^n, which is also the D-topology of K^n, according to above.

Let us show now that the topology of M and its D-topology coincide. Let first $\mathscr{O} \subset \mathrm{M}$ be an open subset of M. Let P be a plot of M, then $\mathrm{P} = \sup_i \mathrm{F}_i \circ \mathrm{Q}_i$ for some family of indices, with the F_i in $\mathscr{A}$ and the Q_i smooth parametrizations in K^n. Then, $\mathrm{P}^{-1}(\mathscr{O}) = (\sup \mathrm{F}_i \circ \mathrm{Q}_i)^{-1}(\mathscr{O}) = \cup_i \mathrm{Q}_i^{-1}(\mathrm{F}_i^{-1}(\mathscr{O}))$. And since the F_i and the Q_i are continuous, $\mathrm{P}^{-1}(\mathscr{O})$ is open. Thus, $\mathscr{O}$ is open for the D-topology. Conversely, let $\mathscr{O}$ be open for the D-topology. For all $x \in \mathscr{O}$, there exists $\mathrm{F}_x \in \mathscr{A}$ such that $x \in \mathrm{val}(\mathrm{F}_x)$. Since F_x is a plot for $\mathscr{D}$, $\mathrm{F}_x^{-1}(\mathscr{O})$ is open in K^n, and since F_x is a local homeomorphism from K^n to M, $\mathrm{F}_x \restriction \mathrm{F}_x^{-1}(\mathscr{O})$ is still a local homeomophism. Then, $\mathrm{val}(\mathrm{F}_x \restriction \mathrm{F}_x^{-1}(\mathscr{O}))$ is open in M. But $\mathrm{val}(\mathrm{F}_x \restriction \mathrm{F}_x^{-1}(\mathscr{O})) = \mathscr{O} \cap \mathrm{val}(\mathrm{F}_x)$, thus $\mathscr{O} = \cup_x \mathscr{O} \cap \mathrm{val}(\mathrm{F}_x)$ is a union of open subsets, then open in M. Therefore the topologies coincide.

Let us prove now that $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M}) = \mathscr{A}$. Let $\Phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M})$. Since the two topologies coincide, we know already that Φ is a local homeomorphism from K^n to M. Now let $\mathrm{F} \in \mathscr{A}$, thus $\mathrm{F}^{-1} \circ \Phi = \mathrm{F}^{-1} \circ (\sup \mathrm{F}_i \circ \mathrm{Q}_i)$, where $\Phi = \sup \mathrm{F}_i \circ \mathrm{Q}_i$, as previously. Hence, $\mathrm{F}^{-1} \circ \Phi = \sup(\mathrm{F}^{-1} \circ \mathrm{F}_i) \circ \mathrm{Q}_i$. But the $\mathrm{F}^{-1} \circ \mathrm{F}_i$ and the Q_i are smooth, and moreover local diffeomorphisms, thus $\mathrm{F}^{-1} \circ \Phi$ is a local diffeomorphism, and then also $\Phi^{-1} \circ \mathrm{F}$. Hence, since $\mathscr{A}$ is maximal, $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M}) \subset \mathscr{A}$. Next let $\mathrm{F} \in \mathscr{A}$. We now already that F is smooth, and a local homeomorphism for both topologies. Let us show that $\mathrm{F}^{-1} \colon \mathrm{val}(\mathrm{F}) \to \mathrm{K}^n$ is smooth. Let P be a plot in $\mathrm{val}(\mathrm{F}) \subset \mathrm{M}$, then $\mathrm{P} = \sup \mathrm{F}_i \circ \mathrm{Q}_i$. Hence, $\mathrm{F}^{-1} \circ \mathrm{P} = \sup(\mathrm{F}^{-1} \circ \mathrm{F}_i) \circ \mathrm{Q}_i$. Thus, F^{-1} is smooth and F is a local diffeomorphism. Therefore, $\mathscr{A} \subset \mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M})$, and then $(\mathrm{M}, \mathscr{D})$ is a diffeological manifold with corners such that $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M}) = \mathscr{A}$.

Conversely, let $(\mathrm{M}, \mathscr{D})$ be a diffeological manifold with corners. Equip M with its D-topology. Since local diffeomorphisms are local homeomorphisms for the D-topology, and since local diffeomorphisms of K^n admit smooth extensions on $\mathbf{R}^n$, then $\mathscr{A} =$

$\mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M})$ gives M a structure of manifold with corners in the usual sense. The atlas $\mathscr{A}$ is obviously maximal.

Now, because in the two directions $\mathscr{A} = \mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M})$, the fact that Φ and Ψ are inverse of each other is pretty obvious. □

Extension of Differential Forms on Manifolds with Boundary

We already showed that any k-forms on half-spaces H^n can be extended to a neighborhood of $\mathrm{H}^n \subset \mathbf{R}^n$ [GIZ16]. Here we extend this result to manifolds with boundary.

Let us just recall that a differential k-form on a diffeological space X is a mapping α that associates with each plot P in X, a smooth k-form $\alpha(\mathrm{P})$ on dom(P), such that the smooth compatibility condition $\alpha(\mathrm{F} \circ \mathrm{P}) = \mathrm{F}^*(\alpha(\mathrm{P}))$ is satisfied, where F is any smooth parametrization in dom(P).

4. Taylor's Series with Parameters. Let $[x \mapsto f_x] \in \mathrm{C}^\infty(\mathrm{X}, \mathrm{C}^\infty(\mathbf{R}, \mathbf{R}))$, with X any diffeological space and $\mathrm{C}^\infty(\mathbf{R}, \mathbf{R})$ equipped with the functional diffeology. Then, for all positive integer n there exists $a_n \in \mathrm{C}^\infty(\mathrm{X}, \mathbf{R})$ and there exists $[x \mapsto \varphi_{n,x}] \in \mathrm{C}^\infty(\mathrm{X}, \mathrm{C}^\infty(\mathbf{R}, \mathbf{R}))$, such that:

$$f_x(t) = a_0(x) + t a_1(x) + \cdots + t^{n-1} a_{n-1}(x) + t^n \varphi_{n,x}(t).$$

This is the Taylor's series of smooth real functions, with parameters in a diffeological space. It is based on a parameter version of Hadamard's Lemma, which is the order 1 of the series.

Proof. For all $x \in \mathrm{X}$ and all $t \in \mathbf{R}$, one has

$$f_x(t) = f_x(0) + t\, g_x(t) \quad \text{with} \quad g_x(t) = \int_0^1 f_x'(st)\, ds.$$

Since $(x, t) \mapsto \int_0^1 f_x'(st)\, ds$ is smooth, $[x \mapsto g_x]$ belongs to $\mathrm{C}^\infty(\mathrm{X}, \mathrm{C}^\infty(\mathbf{R}, \mathbf{R}))$. The Taylor's series is then built by iteration. □

5. Whitney Theorem on Even Functions with Parameters. Let X be a diffeological space, let $[x \mapsto f_x] \in \mathrm{C}^\infty(\mathrm{X}, \mathrm{C}^\infty(\mathbf{R}, \mathbf{R}))$. If, for all $t \in \mathbf{R}$ and all $x \in \mathrm{X}$, $f_x(t) = f_x(-t)$, then there exists $[x \mapsto g_x] \in \mathrm{C}^\infty(\mathrm{X}, \mathrm{C}^\infty(\mathbf{R}, \mathbf{R}))$ such that $f_x(t) = g_x(t^2)$.

Proof. This is a direct adaptation of Whitney's original proof in [Whi43], a game of substitution. Thanks to Taylor's Series with Parameters (art. 4) we have $f_x(t) = a_0(x) + t a_1(x) + \cdots + t^{n-1} a_{n-1}(x) + t^n \varphi_{n,x}(t)$. But because f_x is even, the odd terms vanish, and we rewrite

$$f_x(t) = a_0(x) + t^2 a_1(x) + \cdots + t^{2n-2} a_{n-1}(x) + t^{2n} \varphi_{2n,x}(t).$$

Following Whitney, we put $\psi_{n,x}(u) = \psi_{n,x}(-u) = \varphi_{2n,x}(\sqrt{u})$ and

$$g_x(u) = a_0(x) + ua_1(x) + \cdots + u^{n-1}a_{n-1}(x) + u^n\psi_{n,x}(u).$$

According to Whitney, for every $x \in \mathrm{X}$, the function g_x is smooth. Let us check that $[x \mapsto g_x]$ is smooth. That is, for all plot $r \mapsto x_r$ in X, the parametrization $(r,t) \mapsto g_{x_r}(t)$ is smooth. Let then $(r,t) \mapsto \mathrm{F}(r,t) = f_{x_r}(t)$. The function F is even in t. According to Whitney (*op. cit.*), $\mathrm{F}(r,t) = \mathrm{G}(r,t^2)$, with $(r,u) \mapsto \mathrm{G}(r,u)$ smooth. Let us check then that $\mathrm{G}(r,u) = g_{x_r}(u)$. On the one hand, we have:

$$\mathrm{F}(r,t) = \alpha_0(r) + t^2\alpha_1(r) + \cdots + t^{2n}\Phi_{2n}(r,t).$$

We put $\Psi_n(r,u) = \Psi_n(r,-u) = \Phi_{2n}(r,\sqrt{u})$ and, according to Whithney, we have:

$$\mathrm{G}(r,u) = \alpha_0(r) + u\alpha_1(r) + \cdots + u^n\Psi_n(r,u).$$

On the other hand, we have:

$$g_{x_r}(u) = a_0(x_r) + ua_1(x_r) + \cdots + u^n\psi_{n,x_r}(u).$$

Now, since $\mathrm{F}(r,t) = f_{x_r}(t)$ and $\partial^k[\mathrm{F}(r,t) - f_{x_r}(t)]/\partial t^k\ |_{t=0} = 0$ for all $k < 2n$, $a_i(x_r) = \alpha_i(r)$ for all i and r, and $\varphi_{2n,x_r}(t) = \Phi_{2n}(r,t)$. But $\varphi_{2n,x_r}(\sqrt{u}) = \psi_{n,x_r}(u)$ and $\Phi_{2n}(r,\sqrt{u}) = \Psi_n(r,u)$, hence $\psi_{n,x_r}(u) = \Psi_n(r,u)$. Thus, $g_{x_r}(u) = \mathrm{G}(r,u)$. Therefore $(r,u) \mapsto g_{x_r}(u)]$ is smooth, that is, $[x \mapsto g_x] \in \mathrm{C}^\infty(\mathrm{X}, \mathrm{C}^\infty(\mathbf{R},\mathbf{R}))$ and $f_x(t) = g_x(t^2)$. ☐

6. Differential Forms on Manifolds with Boundary. Let M be a n-manifold, and consider the $(n+1)$-manifold with boundary $\mathrm{M} \times [0,1[$. Let $\alpha \in \Omega^k(\mathrm{M} \times [0,1[)$, be a differential k-form [PIZ13, §6.28]. Then α extends to a k-form $\underline{\alpha}$ on an open neighborhood of $\mathrm{M} \times [0,1[$ in $\mathrm{M} \times]-1,+1[$.

Corollary. — Let M be a manifold with boundary. and $\mathrm{M} \hookrightarrow \mathrm{N}$ be an embedding as a *pièce à bord* [ADLH73]. If the boundary $\partial\mathrm{M}$ is compact[7], then every differential k-form on M extends to an open neighborhood of M in N.

Proof. First of all, consider $\alpha \restriction \mathrm{M} \times]0,1[$. Then, there exist two smooth parametrizations $t \mapsto a_t \in \Omega^{k-1}(\mathrm{M})$ and $t \mapsto b_t \in \Omega^k(\mathrm{M})$, defined on $]0,1[$, with the space of forms equipped with the functional diffeology [PIZ13, §6.29], such that:

$$\alpha \restriction \mathrm{M} \times]0,1[= dt \wedge a_t + b_t,$$

with

$$dt \wedge a_{t,x}\begin{pmatrix}\delta_1 x\\ \delta_1 t\end{pmatrix}\cdots\begin{pmatrix}\delta_k x\\ \delta_k t\end{pmatrix} = \sum_{i=1}^{k}(-1)^{i-1}\delta_i t \times a_{t,x}(\delta_1 x, \ldots, \widehat{\delta_i x}, \ldots, \delta_k x),$$

[7]The non-compact case seems to need more work, see for example [Bro62] [Con71].

where $\delta_i x \in \mathrm{T}_x(\mathrm{M})$, $\delta_i t \in \mathbf{R}$, and $\widehat{\delta_i x}$ means the vector $\delta_i x$ is omitted. And with an abuse of notation:

$$b_{t,x}\begin{pmatrix}\delta_1 x\\ \delta_1 t\end{pmatrix}\cdots\begin{pmatrix}\delta_k x\\ \delta_k t\end{pmatrix} = b_{t,x}(\delta_1 x)\cdots(\delta_k x).$$

Now, let

$$\mathrm{sq}\colon \mathrm{M}\times\,]-1,+1[\,\to \mathrm{M}\times[0,+1[\quad\text{defined by}\quad \mathrm{sq}(x,t)=(x,t^2).$$

Then, there exist two smooth parametrizations $t\mapsto \mathrm{A}_t \in \Omega^{k-1}(\mathrm{M})$ and $t\mapsto \mathrm{B}_t\in\Omega^k(\mathrm{M})$, defined on $]-1,+1[$, such that

$$\mathrm{sq}^*(\alpha) = dt\wedge \mathrm{A}_t + \mathrm{B}_t.$$

Consider $\varepsilon\colon (x,t)\mapsto(x,-t)$. Then, $\mathrm{sq}\circ\varepsilon = \mathrm{sq}$, thus $\mathrm{sq}^*(\alpha) = \varepsilon^*(\mathrm{sq}^*(\alpha))$, that is,

$$dt\wedge\mathrm{A}_t + \mathrm{B}_t = -dt\wedge \mathrm{A}_{-t} + \mathrm{B}_{-t}.$$

Hence, $t\mapsto \mathrm{A}_t$ is odd, $\mathrm{A}_{-t} = -\mathrm{A}_t$. Thus $\mathrm{A}_0 = 0$, and thanks to (art. 4)[8] there exists a smooth parametrization $t\mapsto \underline{\mathrm{A}}_t$, defined on $]-1,+1[$ into $\Omega^{k-1}(\mathrm{M})$, such that:

$$\mathrm{A}_t = 2t\times\underline{\mathrm{A}}_t.$$

Hence,

$$\mathrm{sq}^*(\alpha) = 2t\times dt\wedge\underline{\mathrm{A}}_t + \mathrm{B}_t.$$

But,

$$\mathrm{sq}^*(\alpha\restriction \mathrm{M}\times\,]0,1[) = \mathrm{sq}^*(\alpha)\restriction \mathrm{M}\times\,]-1,0[\,\cup\,\mathrm{M}\times\,]0,+1[.$$

That is,

$$2t\times dt\wedge a_{t^2} + b_{t^2} = 2t\times dt\wedge\underline{\mathrm{A}}_t + \mathrm{B}_t.$$

Hence,

$$\text{for all } t\neq 0\quad a_{t^2} = \underline{\mathrm{A}}_t\quad\text{and}\quad b_{t^2} = \mathrm{B}_t.$$

Thus, $t\mapsto\underline{\mathrm{A}}_t$ and $t\mapsto\underline{\mathrm{B}}_t$, defined on $]-1,+1[$ are even. Then, according to (art. 5)[9], there exist two smooth parametrizations $t\mapsto\underline{a}_t$ and $t\mapsto\underline{b}_t$, defined on $]-1,+1[$, such that

$$\underline{\mathrm{A}}_t = \underline{a}_{t^2}\quad\text{and}\quad \mathrm{B}_t = \underline{b}_{t^2}.$$

Let us now define $\underline{\alpha}\in\Omega^k(\mathrm{M}\times\,]-1,+1[)$

$$\underline{\alpha} = dt\wedge\underline{a}_t + \underline{b}_t.$$

Then,

$$2t\times dt\wedge\underline{\mathrm{A}}_t + \underline{\mathrm{B}}_t = 2t\times dt\wedge\underline{a}_{t^2} + \underline{b}_{t^2},$$

[8] We apply the Hadamard's Lemma with parameters to the map $y\mapsto[t\mapsto f_y(t)]$, from $\mathrm{T}^k\mathrm{M}$ to $\mathrm{C}^\infty(\mathbf{R},\mathbf{R})$, with $y=(x,v_1,\ldots,v_k)$ and $f_y(t)=\mathrm{A}_t(x)(v_1,\ldots,v_k)$.

[9] As above, we consider $y\mapsto[t\mapsto f_y(t)]$, from $\mathrm{T}^k\mathrm{M}$ to $\mathrm{C}^\infty(\mathbf{R},\mathbf{R})$, with $y=(x,v_1,\ldots,v_k)$ and $f_y(t)=\underline{\mathrm{A}}_t(x)(v_1,\ldots,v_k)$.

that is,

$$\mathrm{sq}^*(\alpha) = \mathrm{sq}^*(\underline{\alpha}) \quad \textit{i.e.} \quad \mathrm{sq}^*(\alpha - \underline{\alpha} \restriction \mathrm{M} \times [0,1[) = 0.$$

Consider now this version of the lemma (art. 7):

Lemma. — *Let* $\beta \in \Omega^k(\mathrm{M} \times [0,1[)$. *If* $\mathrm{sq}^*(\beta) = 0$, *then* $\beta = 0$.

◀ Proof of the Lemma. — Let $\mathrm{P} \colon \mathrm{U} \to \mathrm{M} \times [0,1[$ be a plot and let us show that $\beta(\mathrm{P}) = 0$. Let $\mathrm{U}' = \mathrm{P}^{-1}(\mathrm{M} \times]0,1[)$ and $\mathrm{P}' = \mathrm{P} \restriction \mathrm{U}'$. Since $\mathrm{sq} \restriction \mathrm{M} \times (]-1,0[\cup [0,+1[) \to \mathrm{M} \times]0,1[$ is a covering of manifolds, *i.e.* a local diffeomorphism everywhere, $\mathrm{sq}^*(\beta)(\mathrm{P}) \restriction \mathrm{U}' = 0$ implies $\beta(\mathrm{P}) \restriction \mathrm{U}' = 0$. By continuity $\beta(\mathrm{P}) \restriction \overline{\mathrm{U}'} = 0$. Then, let $\mathrm{U}'' = \mathrm{U} - \overline{\mathrm{U}'}$ and $\mathrm{P}'' = \mathrm{P} \restriction \mathrm{U}''$. But then $\mathrm{sq} \circ \mathrm{P}'' = \mathrm{P}''$, hence $\beta(\mathrm{P}) \restriction \mathrm{U}'' = \beta(\mathrm{P}'') = \beta(\mathrm{sq} \circ \mathrm{P}'') = \mathrm{sq}^*(\beta)(\mathrm{P}'') = 0$. Now $\beta(\mathrm{P}) \restriction \overline{\mathrm{U}'} = 0$ and $\beta(\mathrm{P}) \restriction \mathrm{U}'' = 0$, with $\mathrm{U} = \overline{\mathrm{U}'} \cup \mathrm{U}''$, implies $\beta(\mathrm{P}) = 0$. Therefore $\beta = 0$. ▶

Then, $\mathrm{sq}^*(\alpha - \underline{\alpha} \restriction \mathrm{M} \times [0,1[) = 0$ implies $\alpha = \underline{\alpha} \restriction \mathrm{M} \times [0,1[$.

The corollary is a particular application of Douady's theorem [ADLH73, Proposition 3.1] that embeds any manifold with corners into itself, as a *pièce à coins*, in our case as a *pièce à bord*. Since the two categories (usual and diffeological) of manifolds with corners (or boundary) coincide (art. 3), this embedding is also a embedding[10] as diffeological manifold with corners (boundary). Then, if the border is compact, then there is an open neighborhood of the boundary $\partial \mathrm{M}$ in N, diffeomorphic to $\partial \mathrm{M} \times]-1,+1[$ such that, restricted to $\partial \mathrm{M} \times [0,1[$, it is a diffeomorphism on a neighborhood of $\partial \mathrm{M}$ in M. And the propostion above applies. □

Extension of Differential Forms on Corners

In the previous section we extended the differential forms on manifolds with boundary, on open neighborhoods, after having pushed the manifolds as a *pièces à bords* [Dou62]. Next, we prove an extension theorem for corners (art. 8), which gives a local extension theorem for manifolds with corners[11].

7. The Square Function Lemma. Let $\mathrm{sq} \colon \mathbf{R}^n \to \mathrm{K}^n$ be the smooth parametrization:

$$\mathrm{sq}(x_1, \ldots, x_n) = (x_1^2, \ldots, x_n^2).$$

Then $\mathrm{sq}^* \colon \Omega^k(\mathrm{K}^n) \to \Omega^k(\mathbf{R}^n)$ is injective. That is, for all $\alpha \in \Omega^k(\mathrm{K}^n)$, if $\mathrm{sq}^*(\alpha) = 0$, then $\alpha = 0$.

Proof. Note that each component of $\mathrm{S}_j - \mathrm{S}_{j-1}$ is diffeomorphic to $\mathbf{R}^j$. Hence, if $\mathrm{sq}^*(\alpha) = 0$, since $\mathrm{sq} \restriction \mathrm{sq}^{-1}(\mathrm{S}_j - \mathrm{S}_{j-1})$ is a 2-fold covering over $\mathrm{S}_j - \mathrm{S}_{j-1}$, $\alpha \restriction \mathrm{S}_j - \mathrm{S}_{j-1} = 0$. That is, for all plot Q in $\mathrm{S}_j - \mathrm{S}_{j-1}$, $\alpha(\mathrm{Q}) = 0$. Let then, for some $j \geq 1$, $\mathrm{P}_j \colon \mathrm{U}_j \to \mathrm{S}_j$ be a plot. In view of what precedes, the subset $\mathcal{O}_j = \mathrm{P}_j^{-1}(\mathrm{S}_j - \mathrm{S}_{j-1})$

[10] See embeddings in diffeology in [PIZ13, §2.13].

[11] The case of *pièces à coins* is a reserved for a future work.

is open, and $\alpha(\mathrm{P}_j \upharpoonright \mathcal{O}_j) = \alpha(\mathrm{P}_j) \upharpoonright \mathcal{O}_j = 0$. By continuity, $\alpha(\mathrm{P}_j) \upharpoonright \overline{\mathcal{O}}_j = 0$, where $\overline{\mathcal{O}}_j$ is the closure of $\mathcal{O}_j$. Let then $\mathrm{U}_{j-1} = \mathrm{U}_j - \overline{\mathcal{O}}_j$ and $\mathrm{P}_{j-1} = \mathrm{P}_j \upharpoonright \mathrm{U}_{j-1}$. Then, U_{j-1} is open and $\mathrm{P}_{j-1} \colon \mathrm{U}_{j-1} \to \mathrm{S}_{j-1}$ is a plot. This construction gives a descending recursion, starting with any plot $\mathrm{P} \colon \mathrm{U} \to \mathrm{K}^n$, by initializing $\mathrm{P}_n = \mathrm{P}$, $\mathrm{U}_n = \mathrm{U}$ and $\mathrm{S}_n = \mathrm{K}^n$. One has $\mathrm{P}_j = \mathrm{P} \upharpoonright \mathrm{U}_j$, $\mathrm{U}_{j-1} \subset \mathrm{U}_j$, the recursion ends with a plot P_0 with values in $\mathrm{S}_0 = \{0\}$, and $\alpha(\mathrm{P}_0) = 0$ since P_0 is constant. Therefore $\alpha = 0$. ☐

8. Differential Forms On Corners. The section (art. 2), which deals with smooth real functions on corners, is a particular case of the following theorem on differential forms of any degree:

THEOREM. *Any differential k-form on the corner* K^n*, equipped with the subset diffeology of* $\mathbf{R}^n$*, is the restriction of a smooth differential k-form defined on some open neighborhood of the corner.*

That means also that a differential form on a manifold with corners admits local smooth extensions, when it is represented in its charts.

NOTE. — Actually, the pullback : $j^* \colon \Omega^k(\mathbf{R}^n) \to \Omega^k(\mathrm{K}^n)$ is surjective, where j denotes the inclusion from K^n to $\mathbf{R}^n$.

Proof. Let $\omega \in \Omega^k(\mathrm{K}^n)$ and $\mathring{\mathrm{K}}^n = \{(x_i)_{i=1}^n \mid x_i > 0, i = 1, \ldots, n\}$. One has

$$\omega \upharpoonright \mathring{\mathrm{K}}^n = \sum_{i_1 < \cdots < i_k} a_{i_1 \ldots i_k}(x_1, \ldots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

with $i_j = 1, \ldots, n$ and $a_{i_1 \ldots i_k} \in \mathrm{C}^\infty(\mathring{\mathrm{K}}^n, \mathbf{R})$. Recall that $\mathrm{sq} \colon (x_i)_{i=1}^n \mapsto (x_i^2)_{i=1}^n$, then

$$\mathrm{sq}^*(\omega) = \sum_{i_1 < \cdots < i_k} \mathrm{A}_{i_1 \ldots i_k}(x_1, \ldots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $\mathrm{A}_{i_1 \ldots i_k} \in \mathrm{C}^\infty(\mathbf{R}^n, \mathbf{R})$. Let $\varepsilon_j : (x_1, \ldots, x_j, \ldots, x_n) \mapsto (x_1, \ldots, -x_j, \ldots, x_n)$, then $\mathrm{sq} \circ \varepsilon_j = \mathrm{sq}$ and $(\mathrm{sq} \circ \varepsilon_j)^*(\omega) = \varepsilon_j^*(\mathrm{sq}^*(\omega))$, that is, $\mathrm{sq}^*(\omega) = \varepsilon_j^*(\mathrm{sq}^*(\omega))$. Hence,

$$\begin{aligned}
\varepsilon_j^*(\mathrm{sq}^*(\omega)) \quad = \quad & \sum_{\substack{i_1 < \cdots < i_k \\ i_\ell \neq j}} \mathrm{A}_{i_1 \ldots i_k}(x_1, \ldots, -x_j, \ldots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
- \quad & \sum_{i_1 < \cdots \leq j \leq \cdots < i_k} \mathrm{A}_{i_1 \ldots j \ldots i_k}(x_1, \ldots, -x_j, \ldots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx_{i_k}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathrm{A}_{\substack{i_1 \ldots i_k \\ i_\ell \neq j}}(x_1, \ldots, -x_j, \ldots, x_n) &= \mathrm{A}_{i_1 \ldots i_k}(x_1, \ldots, x_j, \ldots, x_n), \\
\mathrm{A}_{i_1 \ldots j \ldots i_k}(x_1, \ldots, -x_j, \ldots, x_n) &= -\mathrm{A}_{i_1 \ldots j \ldots i_k}(x_1, \ldots, x_j, \ldots, x_n).
\end{aligned}$$

Hence,

$$\mathrm{A}_{i_1 \ldots j \ldots i_k}(x_1, \ldots, x_j = 0, \ldots, x_n) = 0.$$

Thus,

$$\mathrm{A}_{i_1\dots j\dots i_k}(x_1,\dots,x_j,\dots,x_n) = 2x_j\underline{\mathrm{A}}_{i_1\dots j\dots i_k}(x_1,\dots,x_j,\dots,x_n),$$

with $\underline{\mathrm{A}}_{i_1\dots j\dots i_k} \in \mathrm{C}^\infty(\mathbf{R}^n,\mathbf{R})$. Therefore, there are real smooth functions $\hat{\mathrm{A}}_{i_1\dots i_k}$ defined on $\mathbf{R}^n$ such that

$$\mathrm{A}_{i_1\dots i_k}(x_1,\dots,x_n) = 2^k x_{i_1}\dots x_{i_k}\hat{\mathrm{A}}_{i_1\dots i_k}(x_1,\dots,x_n).$$

Now,

$$\mathrm{sq}^*(\omega \upharpoonright \mathring{\mathrm{K}}^n) = \mathrm{sq}^*(\omega) \upharpoonright \{x_i \neq 0\}$$

implies

$$\begin{aligned} & \sum_{i_1<\dots<i_k} 2^k x_{i_1}\dots x_{i_k} a_{i_1\dots i_k}(x_1^2,\dots,x_n^2)\, dx_{i_1}\wedge\dots\wedge dx_{i_k} \\ = & \sum_{i_1<\dots<i_k} 2^k x_{i_1}\dots x_{i_k}\hat{\mathrm{A}}_{i_1\dots i_k}(x_1,\dots,x_n)\, dx_{i_1}\wedge\dots\wedge dx_{i_k}. \end{aligned}$$

Hence,

$$\hat{\mathrm{A}}_{i_1\dots i_k}(x_1,\dots,x_n) = a_{i_1\dots i_k}(x_1^2,\dots,x_n^2) \quad \text{for } x_i \neq 0, i=1,\dots,n.$$

Thus $(x_1,\dots,x_n) \mapsto \hat{\mathrm{A}}_{i_1\dots i_k}(x_1,\dots,x_n)$, which belongs to $\mathrm{C}^\infty(\mathbf{R}^n,\mathbf{R})$, is even in each variable. Therefore, according to Whitney (*op. cit.*), there exist

$$\underline{a}_{i_1\dots i_k} \in \mathrm{C}^\infty(\mathbf{R}^n,\mathbf{R}),$$

such that

$$\hat{\mathrm{A}}_{i_1\dots i_k}(x_1,\dots,x_n) = \underline{a}_{i_1\dots i_k}(x_1^2,\dots,x_n^2).$$

One deduces:

$$\underline{a}_{i_1\dots i_k}(x_1,\dots,x_n) = a_{i_1\dots i_k}(x_1,\dots,x_n), \text{ for all } (x_1,\dots,x_n) \in \mathring{\mathrm{K}}^n.$$

Then, defining the k-form $\underline{\omega}$ on $\mathbf{R}^n$ by

$$\underline{\omega} = \sum_{i_1<\dots<i_k} \underline{a}_{i_1\dots i_k}(x_1,\dots,x_n)\, dx_{i_1}\wedge\dots\wedge dx_{i_k},$$

one has already

$$\underline{\omega} \upharpoonright \mathring{\mathrm{K}}^n = \omega \upharpoonright \mathring{\mathrm{K}}^n.$$

Let us show that $\underline{\omega} \upharpoonright \mathrm{K}^n = \omega$. That is, let us check that for all plot $\mathrm{P}\colon \mathrm{U} \to \mathbf{R}^n$, $\mathrm{P}^*(\underline{\omega}) = \omega(\mathrm{P})$. Actually, one has

$$\mathrm{sq}^*(\omega) = \mathrm{sq}^*(\underline{\omega} \upharpoonright \mathrm{K}^n).$$

Indeed:

$$\begin{aligned}
\mathrm{sq}^*(\omega) &= \sum_{i_1\dots i_k} \mathrm{A}_{i_1\dots i_k}(x_1,\dots,x_n)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\
&= \sum_{i_1\dots i_k} 2^k x_{i_1}\dots x_{i_k}\hat{\mathrm{A}}_{i_1\dots i_k}(x_1,\dots,x_n)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\
&= \sum_{i_1\dots i_k} 2^k x_{i_1}\dots x_{i_k}\underline{a}_{i_1\dots i_k}(x_1^2,\dots,x_n^2)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}.
\end{aligned}$$

And, on the other hand:

$$\mathrm{sq}^*(\underline{\omega}\upharpoonright \mathrm{K}^n) = \sum_{i_1\dots i_k} 2^k x_{i_1}\dots x_{i_k}\underline{a}_{i_1\dots i_k}(x_1^2,\dots,x_n^2)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}.$$

Thus, $\mathrm{sq}^*(\omega - \underline{\omega}\upharpoonright \mathrm{K}^n) = 0$. Therefore, according to the lemma (art. 7), $\omega - \underline{\omega}\upharpoonright \mathrm{K}^n = 0$. And then, ω is the restriction on K^n of the smooth k-form $\underline{\omega}$ on $\mathbf{R}^n$. □

9. Other Corners. The *half-line* $\Delta_k = \mathbf{R}^k/\mathrm{O}(k)$ is identified to the interval $[0,\infty[$, equipped with the pushforward of the smooth diffeology of $\mathbf{R}^k$ by the projection $\nu_k\colon \mathrm{X}\mapsto \|\mathrm{X}\|^2$, see [PIZ07, IZW14]. Then, with each half-line we can associate a n-corner Δ_k^n. Note that, according to definition (art. 3), the only one of these corners being a manifold with corners is $\mathrm{K}^n = \Delta^n$. Note also that the identity j_k^n from Δ_k^n to $\mathrm{K}^n \subset \mathbf{R}^n$ is still a smooth map. Now,

Proposition. — *The pullback* $j_k^{n*}\colon \Omega^*(\mathrm{K}^n)\to\Omega^*(\Delta_k^n)$ *is surjective.*

As well as for the standard corner K^n, every differential form on Δ_k^n is the pullback of some smooth form on $\mathbf{R}^n$.

Proof. The proof is a copy from the proof of (art. 8) because 1) the map sq is smooth, 2) the D-topology of $\Delta_k^n \subset \mathbf{R}^n$ coincides with the induced topology. Then, 3) the interior of each stratum is some power of the open interval, and the rest follows. □

10. An Application. We can apply the theorems above to describe the closed 2-forms on manifolds, invariant with respect to some action of a Lie group. Any closed 2-form ω on a manifold M, invariant by the Hamiltonian action of a compact group[12] G, is characterized by its *moment map* $\mu\colon \mathrm{M}\to\mathscr{G}^*$, and for each moment map, a closed 2-form $\varepsilon\in \mathrm{Z}^2(\mathrm{M}/\mathrm{G})$. Let us be more precise: the space of G-invariant closed 2-forms $\mathrm{Z}^2_{\mathrm{G}}(\mathrm{M})$ is a vector space, the space of G-equivariant maps from M to $\mathscr{G}^*$ is also a vector space, and the map associating its moment map[13] μ with each invariant closed 2-form ω is linear. What we claim is that the kernel of this map is exactly $\mathrm{Z}^2(\mathrm{M}/\mathrm{G})$, where M/G is equipped with the quotient diffeology. Denoting by $\mathscr{E}q_\bullet(\mathrm{M},\mathscr{G}^*)\subset$

[12]There could a diffeological generalisation possible here to non compact group.

[13]The manifold M is supposed to be connected. To have a unicity of the moment maps we decide to fix their value to 0 at some base point $m_0\in\mathrm{M}$, for example.

$\mathscr{E}q(\mathrm{M}, \mathscr{G}^*)$ the space of moment maps of G-invariant closed 2-forms on M, as a subset of smooth equivariant maps, one has this exact sequence of smooth linear maps:

$$0 \longrightarrow \mathrm{Z}^2(\mathrm{M}/\mathrm{G}) \longrightarrow \mathrm{Z}^2_{\mathrm{G}}(\mathrm{M}) \longrightarrow \mathscr{E}q_\bullet(\mathrm{M}, \mathscr{G}^*) \longrightarrow 0.$$

Now, if an equivariant map is easy to conceive, it is more problematic for a differential form on the space of orbits, which is generally not a manifold. This is where the above theorem can help, because it happens that M/G is not far to be a manifold with boundary or corners, as show the following example.

Consider the simple case $\mathrm{M} = \mathbf{R}^{2n}$, equipped with the standard symplectic form $\omega = \sum_{i=1}^n dq_i \wedge dp_i$. It is invariant by the group $\mathrm{SO}(2)^n$ acting naturally, each factor on its respective copy of $\mathbf{R}^2$. The quotient space $\mathscr{Q}^n = \mathbf{R}^{2n}/\mathrm{SO}(2)^n$ is equivalent to the *other corner* Δ_2^n, with $\Delta_2 = \mathbf{R}^2/\mathrm{O}(2)$. Thus, thanks to (art. 8)-(art. 9), for each 2-form ε on the quotient $\mathscr{Q}^n$ there exists a 2-form $\underline{\varepsilon}$ on $\mathbf{R}^n$, such that $\varepsilon = j_k^{n*}(\underline{\varepsilon})$. Then, the 2-form ω is characterized by the moment map μ and $\underline{\varepsilon} \restriction \mathrm{K}^n$, with $\underline{\varepsilon} \in \Omega^k(\mathbf{R}^n)$.

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