ON MANIFOLDS WITH BOUNDARY AND CORNERS

SERAP GÜRER AND PATRICK IGLESIAS-ZEMMOUR

Abstract. We embed the category of manifolds with corners into the category {Diffeology} by the process of modeling, and then we prove an extension theorem for differential forms on corners and on manifolds with boundary.

Introduction

Half-spaces $\mathrm{H}^n = [0,\infty[\,\times \mathbf{R}^{n-1}$, corners $\mathrm{K}^n = [0,\infty[^n$, and all intermediate sectors $\mathrm{S}^n_k = [0,\infty[^k \times \mathbf{R}^{n-k}$, own a natural *subset diffeology* inherited from the standard smooth diffeology of $\mathbf{R}^n$. They are natural models for the unified procedure of modeling spaces [PIZ13, Chap. 4]. Half-spaces are the models for the category of *manifolds with boundary* [PIZ13, §4.7 & 4.16]. Corners are the models for the category of *manifold with corners*. Precisely,

Definition. — *A* n-manifold with corners *is a diffeological space diffeomorphic to the corner* K^n *at each point.*

A natural question is then to compare that definition with the traditional approach introduced originally in [Cer61, Dou62], and then used or developed by many authors, for example [ADLH73, GP74, Lee06, Joy10, etc.]. In this approach, smooth maps from corners into the real line are — by definition — the restrictions of smooth maps on some open neighborhood of the corner [Cer61] [Dou62] etc. In Diffeology, this heuristic becomes a theorem, where the corners are equipped with the subset diffeology. More precisely:

Theorem. — *Every map from* K^n *to* $\mathbf{R}$ *which is smooth when composed with a smooth parametrization*[1] $\mathrm{P}\colon \mathrm{U} \to \mathbf{R}^n$ *taking values in* K^n *is the restriction of a smooth map defined on some open neighborhood of* K^n.

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[1] We recall that a parametrization is just a map defined on an open subset of an Euclidean space.

Thanks to this theorem [PIZ13, §4.16], we show in (§4) that the two aproaches define the same objects, and then the same category. Hence, as a subcategory of {Diffeology}, manifolds with corners inherit automatically all the diffeological constructions: smooth maps, fiber bundles, homotopy, differential calculus, homology, cohomology, etc.

It is always progress when a convention, based on mathematicians' intuition, becomes a theorem in a well defined axiomatics. Here the axiomatics is the theory of Diffeology.

Now, noticing that $\mathrm{C}^\infty(\mathrm{K}^n, \mathbf{R})$ is the space of differential 0-forms $\Omega^0(\mathrm{K}^n)$, it is natural to ask about the behavior of differential k-forms on K^n, that is, $\Omega^k(\mathrm{K}^n)$ as it is defined in [PIZ13, §6.28]. Next, it has already been proved that differential forms on a half-space can be extended on a neighborhood of the half-space [GIZ16]. In (§7) we show that this property is also satisfied by manifolds with boundary. Precisely,

Theorem. — *Let* M *be a manifold with boundary imbedded in some manifold* N *as a* pièce à bord [Dou62]. *If* ∂M *is compact, then every differential k-form on* M *extends on a neighborhood of* M *in* N.

For the purpose of this question, we had to establish in (§5) a diffeological version of Taylor's series for real functions depending smoothly on parameters running in a diffeological space, and in (§6), a version of Whitney's theorem on extension of smooth real even functions.

Considering corners, we prove in (§9) the extension theorem:

Theorem. *Every differential form on the corner* K^n *is the restriction of a smooth form on an open neighborhood of* K^n *in* $\mathbf{R}^n$. *Precisely, the pullback* $j^*\colon \Omega^k(\mathbf{R}^n) \to \Omega^k(\mathrm{K}^n)$ *is surjective, where* j *denotes the inclusion from* K^n *into* $\mathbf{R}^n$.

In (§10) we give a variation of this theorem for other corners, made with powers of various half-lines [PIZ07]. In (§11) we show an application of these considerations to the construction of $\mathrm{SO}(2)^n$-invariant parasymplectic forms on $\mathbf{R}^{2n}$.

Smooth Structure on Corners

Let us recall the notions of *local smooth map* and *local diffeomorphism* in diffeology. Let X and X′ be two diffeological spaces and let f be a map defined from a subset A of X to X′, we denote $f : \mathrm{X} \supset \mathrm{A} \to \mathrm{X}'$. We say that f is a *local smooth* if A is a D-open subset of X, and f is smooth for the induced diffeology on A, see[2] [PIZ13, §2.1]. Then, we say that f is a local diffeomorphism if it is an injective and if $f^{-1}\colon \mathrm{X}' \supset f(\mathrm{A}) \to \mathrm{X}$ is also local smooth. That is, $f(\mathrm{A})$ is a D-open subset of X′ and $f^{-1} \restriction f(\mathrm{A})$ is smooth for the induced diffeology. The set of local smooth maps and diffeomorphisms from X to X′ are denoted by $\mathrm{C}^\infty_{\mathrm{loc}}(\mathrm{X}, \mathrm{X}')$ and $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{X}, \mathrm{X}')$.

[2]Actually f is local smooth if and only if, for all plot P in X, the composite $f \circ \mathrm{P}\colon \mathrm{P}^{-1}(\mathrm{A}) \to \mathrm{X}'$ is a plot. That implies in particular that A is D-open, by definition of the D-topology.

1. Corners as Diffeologies. We denote by K^n the *corner*

$$K^n = \{(x_i)_{i=1}^n \in \mathbf{R}^n \mid x_i \geq 0,\ i = 1, \dots, n\},$$

and we equip it with the subset diffeology. A plot in K^n is just a smooth parametrization to $\mathbf{R}^n$ but taking its values in K^n.

(A) The corner K^n is the diffeological n-power of the half-line $K = [0, \infty[\subset \mathbf{R}$, equipped with the subset diffeology.

(B) The corner K^n is *embedded* in $\mathbf{R}^n$, and closed. That is, the D-topology of the induction $K^n \subset \mathbf{R}^n$ coincides with the induced topology[3] of $\mathbf{R}^n$, see [PIZ13, §2.13].

(C) Let $X_0 = \{0\} \subset X_1 \subset \cdots \subset X_n = K^n$ be the natural filtration of K^n, where the *levels* X_j are defined by

$$X_j = \{(x_i)_{i=1}^n \in K^n \mid \text{there exist } i_1 < \cdots < i_{n-j} \text{ such that } x_{i_\ell} = 0\}.$$

Then, the *stratum*

$$S_j = X_j - X_{j-1}$$

is the subset of points in $\mathbf{R}^n$ that have j, and only j, coordinates strictly positive and the rest zero. The strata S_j are equipped with the subset diffeology[4],

$$S_j = \left\{ (x_i)_{i=1}^n \in \mathbf{R}^n \;\middle|\; \begin{array}{l} \text{There exist } i_1 < \cdots < i_j \text{ such that } x_{i_\ell} > 0, \\ \text{and } x_m = 0 \text{ for all } m \notin \{i_1, \dots, i_j\} \end{array} \right\}.$$

Then, S_j is D-open in X_j, $j \geq 1$. As a subset of X_j, S_j is the (diffeological) sum of $\binom{n}{j}$ connected components indexed by a string of j ones and $n-j$ zeros.

Proof. The point (A) is immediate, by definition. Consider (B), for any subset $U \subset K^n$ open for the induced topology, there exists (by definition) an open subset $\mathcal{O} \in \mathbf{R}^n$ such that $U = \mathcal{O} \cap K^n$. Then, for all plots P in K^n, $P^{-1}(U) = P^{-1}(\mathcal{O})$ is open, because plots are continuous. On the other hand, let $U \subset K^n$ be D-open. Then, $\mathrm{sq}^{-1}(U) \subset \mathbf{R}^n$ is open, where $\mathrm{sq} : \mathbf{R}^n \to K^n$ is the map $\mathrm{sq}(x_i)_{i=1}^n = (x_i^2)_{i=1}^n$. And $\mathrm{sq}^{-1}(U) \restriction K^n$ is open for the induced topology of $\mathbf{R}^n$. Now, the map sq restricted to K^n is a homeomorphism for the induced topology. Hence, since $U = \mathrm{sq}(\mathrm{sq}^{-1}(U) \restriction K^n)$, U is open for the induced topology of $\mathbf{R}^n$. Therefore the D-topology of the induction coincides with the induced topology, as we claimed.

For the point (C): let $x \in X_j$, then the number ν of coordinates of x that are 0 is at least $n-j$, *i.e.* $\nu \geq n-j$. Next, if $x \in X_j$ and $x \notin X_{j-1}$, then $\nu \geq n-j$ and $\nu < n-j+1$, thus, $\nu = n-j$. Therefore, $X_j - X_{j-1}$ is the subset of points in $\mathbf{R}^n$ that have exactly $n-j$ coordinates equal to 0 and the other j strictly positive:

[3]The standard topology of $\mathbf{R}^n$ is the D-topology of its standard smooth structure.

[4]Recall that, by transitivity of subset diffeology, to be a subspace of S_ℓ or K^n or of $\mathbf{R}^n$ is identical.

Consider now a point $x = (x_1, \ldots, x_n) \in \mathrm{S}_j - \mathrm{S}_{j-1}$. Since the j non-zero coordinates of x are strictly positive, there exists $\varepsilon > 0$ such that $x_i - \varepsilon > 0$, for all non-zero coordinate of x. The open n-parallelepiped $\mathrm{C}_x =]x_1 - \varepsilon, x_1 + \varepsilon[\times \cdots \times]x_n - \varepsilon, x_n + \varepsilon[\subset \mathbf{R}^n$ contains x, and $\mathrm{C}_x \cap \mathrm{S}_j \subset \mathrm{S}_j - \mathrm{S}_{j-1}$. Thus,

$$\mathrm{S}_j - \mathrm{S}_{j-1} = \bigcup_{x \in \mathrm{S}_j - \mathrm{S}_{j-1}} \mathrm{C}_x \cap \mathrm{S}_j.$$

Now, let $\mathrm{P} : \mathrm{U} \to \mathrm{S}_j$ be a plot for the subset diffeology. Then, $\mathrm{P}^{-1}(\mathrm{S}_j - \mathrm{S}_{j-1}) = \cup_{x \in \mathrm{S}_j - \mathrm{S}_{j-1}} \mathrm{P}^{-1}(\mathrm{C}_x \cap \mathrm{S}_j)$, but $\mathrm{P}^{-1}(\mathrm{C}_x \cap \mathrm{S}_j) = \mathrm{P}^{-1}(\mathrm{C}_x)$ since $\mathrm{val}(\mathrm{P}) \subset \mathrm{S}_j$. Next, since P is smooth as a map into $\mathbf{R}^n$ and C_x is open, $\mathrm{P}^{-1}(\mathrm{C}_x)$ is open and then $\mathrm{P}^{-1}(\mathrm{S}_j - \mathrm{S}_{j-1})$ is open. Therefore, $\mathrm{S}_j - \mathrm{S}_{j-1}$ is D-open in S_j. ☐

2. Smooth Maps on Corners. It has been proved that a map $f : \mathrm{K}^n \to \mathbf{R}^k$ is smooth in the sense of diffeology if and only if it is the restriction of a smooth map F defined on some open neighborhood $\mathcal{O}$ of K^n into $\mathbf{R}^k$ [PIZ13, §4.16]. That is, $f \in \mathrm{C}^\infty(\mathrm{K}^n, \mathbf{R}^k)$ if and only if, $f = \mathrm{F} \restriction \mathrm{K}^n$ and $\mathrm{F} \in \mathrm{C}^\infty(\mathcal{O}, \mathbf{R}^k)$.

Note. — It is worth noticing that this is not just a rephrasing of the usual approach which defines the real smooth maps from corners in $\mathbf{R}^k$, as restrictions of smooth maps defined on open neighborhoods, see for example [Cer61], [Dou62] and after [GP74] [Lee06] [Joy10] etc. In diffeology this property is a theorem which can be stated in a developed version:

Theorem. *Let $f : \mathrm{K}^n \to \mathbf{R}^k$ be some map. If for all smooth parametrizations $\mathrm{P} : \mathrm{U} \to \mathbf{R}^n$ such that $\mathrm{P}(\mathrm{U}) \subset \mathrm{K}^n$, $f \circ \mathrm{P} \in C^\infty(\mathrm{U}, \mathbf{R}^k)$, then there exists an open neighborhood $\mathcal{O}$ of K^n and $\mathrm{F} \in C^\infty(\mathcal{O}, \mathbf{R}^k)$ such that $f = \mathrm{F} \restriction \mathrm{K}^n$.*

3. Smooth Diffeomorphisms on Corners. We consider the corner $\mathrm{K}^n \subset \mathbf{R}^n$, equipped with the subset diffeology. For a smooth parametrization f of an Euclidean domain, we denote by $\mathrm{rk}(f)_x$ the rank of f at the point x, that is, the dimension of the image of the tangent linear map $\mathrm{D}(f)(x)$. Let us recall that an *étale map* is a smooth map which is a local diffeomorphism at each point. Then:

Lemma. Let $\mathrm{P} : \mathrm{U} \to \mathrm{K}^n$ be a plot. If $\mathrm{P}(r) \in \mathrm{S}_j$, then $\mathrm{rk}(\mathrm{P})_r \leq j$.

Theorem. Let $f \in \mathrm{Diff}(\mathrm{K}^n)$. Then, f respects the natural stratification, *i.e.* if $x \in \mathrm{S}_j$, then $f(x) \in \mathrm{S}_j$. Moreover, f and f^{-1} are the restrictions of two étale maps defined on some open neighborhoods of $\mathbf{K}^n$.

This theorem holds also if f is a local diffeomorphism.

Proof. Let us prove first the Lemma. Let $\mathrm{P} : \mathrm{U} \to \mathrm{K}^n$ be a plot and $\mathrm{P}(r') \in \mathrm{S}_j$. Then, $\mathrm{P}(r') = (\mathrm{P}_1(r'), \mathrm{P}_2(r'), \ldots, \mathrm{P}_n(r'))$ where there exist exactly $i_1 < \cdots < i_{n-j}$ indices such that $\mathrm{P}_{i_k}(r') = 0$. Since $\mathrm{P}_{i_k}(r) \geq 0$ for all $r \in \mathrm{U}$ and $\mathrm{P}_{i_k}(r') = 0$, then $\mathrm{D}(\mathrm{P}_{i_k})(r') = 0$. That is, $\mathrm{rk}(\mathrm{P})_r \leq j$.

Now, let us prove the first theorem. Let $x \in \mathrm{S}_j$. Let us assume that $x' = f(x) \in \mathrm{S}_k$ and $k \neq j$. We can choose $k > j$. There exists a smooth map F defined on an open neighborhood $\mathcal{O} \supset \mathrm{K}^n$, such that f and F coincide on K^n, $f = \mathrm{F} \restriction \mathrm{K}^n$. And also, there exists a smooth map G defined on an open neighborhood $\mathcal{O}' \supset \mathrm{K}^n$, such that f^{-1} and G coincide on K^n, $f^{-1} = \mathrm{G} \restriction \mathrm{K}^n$, [PIZ13, §4.16]. The restriction of G on S_k is a plot of K^n, and $\mathrm{G} \restriction \mathrm{S}_k \colon x' \mapsto x \in \mathrm{S}_j$. By the lemma, $\mathrm{rk}(\mathrm{G} \restriction \mathrm{S}_k)_{x'} \leq j$. But $\mathrm{G} \restriction \mathrm{S}_k = \mathrm{G} \circ j_k$, where $j_k : \mathrm{S}_k \hookrightarrow \mathrm{K}^n$ is identified with a plot. And we know that $(\mathrm{F} \circ \mathrm{G} \restriction \mathrm{S}_k)(t) = \mathrm{F} \circ \mathrm{G} \circ j_k(t) = \mathrm{F} \circ \mathrm{G}(j_k(t))$. But j_k takes values in $\partial \mathrm{K}^n$ (the border of K^n). Now, since f is a homeomorphism of K^n for the D-topology, it maps the border into the border, and G and f^{-1} coincide on the border. So we have $\mathrm{F} \circ \mathrm{G}(j_k(t)) = \mathrm{F} \circ f^{-1}(j_k(t))$. As well, F and f coincide on the border, and $\mathrm{F} \circ \mathrm{G}(j_k(t)) = f \circ f^{-1}(j_k(t)) = j_k(t)$. Thus, $\mathrm{rk}(\mathrm{F} \circ \mathrm{G} \restriction \mathrm{S}_k)_{x'} = \mathrm{rk}(j_k)_{x'} = k \leq \mathrm{rk}(\mathrm{G} \restriction \mathrm{S}_k)_{x'} \leq j$. But, we assumed that $k > j$ which is a contradiction, and $k = j$.

Now, consider the smooth parametrization $\mathrm{G} \circ \mathrm{F} \colon \mathcal{U} \to \mathbf{R}^n$, with $\mathcal{U} = \mathrm{F}^{-1}(\mathcal{O}')$. Then, $\mathcal{U} \supset \mathrm{K}^n$ and $\mathrm{G} \circ \mathrm{F} \restriction \mathrm{K}^n = \mathbf{1}_{\mathrm{K}^n}$. Hence, for all $x \in \mathring{\mathrm{K}}^n$, $\mathrm{D}(\mathrm{G} \circ \mathrm{F})(x) = \mathrm{D}(f^{-1} \circ f)(x) = \mathrm{D}(\mathbf{1}_{\mathring{\mathrm{K}}^n}) = \mathbf{1}_{\mathbf{R}^n}$, and by continuity, for all $x \in \mathrm{K}^n$, $\mathrm{D}(\mathrm{G} \circ \mathrm{F})(x) = \mathbf{1}_{\mathbf{R}^n}$. Therefore, for all $x \in \mathrm{K}^n$, $\mathrm{rk}(\mathrm{F})_x$ is maximum and equal to n. Hence, F is étale at each point of K^n. And obviously, the same for G. □

4. Manifolds with Corners, the diffeological way. Manifolds with corners have been introduced long ago in the usual framework of differential geometry, for example as *variétés à bord généralisées* by Cerf [Cer61, Chap. 1 §1.2], and then as *variétés à bords anguleux* by Douady [Dou62, §4]. Over time the various descriptions of manifolds with boundary or corners evolved to a commonly accepted definition, based on the heuristic that a real smooth map defined on a corner should be defined as the restriction of a smooth map defined on an open neighborhood of the corner. See for example Lee in [Lee06, pp. 251-252] or more recently Joyce in [Joy10, Chap. 2], from which we extract the following definition.

Usual Definition. — *Let* M *be a paracompact Hausdorff topological space. A n-chart with corners for* M *is a pair* (U, φ)*, where* U *is an open subset of* K^n*, and* φ *is a homeomorphism from* U *in* M*. Two charts with corners* (U, φ) *and* (V, ψ) *are said to be smoothly compatible if the composite map* $\psi^{-1} \circ \varphi \colon \varphi^{-1}(\psi(\mathrm{V})) \to \psi^{-1}(\varphi(\mathrm{U}))$ *is a diffeomorphism in the sense that it admits a smooth extension to an open set in* $\mathbf{R}^n$*. An n-atlas with corners for* M *is a pairwise compatible family of n-charts with corners covering* M*. A maximal atlas is an atlas which is not a proper subset of any other atlas. An n-manifold with corners is a paracompact Hausdorff topological space* M *equipped with a maximal n-atlas with corners.*

Now, following the general procedure of *modeling diffeologies* [PIZ13, Chap. 4], one can define what we understand as *manifolds with corners* in the category {Diffeology} [PIZ13, §4.16, Note]. We get:

Diffeological Definition. — *A n-manifold with corners is a diffeological space* X *which is locally diffeomorphic to the corner* K^n *at each point.*

In detail, that means that, for every point $x \in \mathrm{X}$ there exists a local diffeomorphism $\mathrm{F}\colon \mathrm{K}^n \supset \mathrm{U} \to \mathrm{X}$ such that $x \in \mathrm{F}(\mathrm{U})$. Such local diffeomorphisms will be called *charts* of X. Any covering family of charts will be called an *atlas* of X. The maximal atlas of X is $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{X})$ itself.

The manifolds with corners form a subcategory of {Diffeology} we denote {Manifolds with corners}, or simply {K-Manifolds}. The smooth maps between manifolds with corners are just the smooth maps between diffeological spaces. There exists an obvious hierarchy of manifolds categories:

$$\{\text{Manifolds}\} \subset \{\text{H-Manifolds}\} \subset \{\text{K-Manifolds}\} \subset \{\text{Diffeology}\},$$

where {H-Manifolds} is the category of manifolds with boundary [PIZ13, §4.16] and {Manifolds} the ordinary category of manifolds. Obviously, between {H-Manifolds} and {K-Manifolds} there exists also a hierarchy of subcategories of manifolds with corners, according to the maximum depth of their strata.

The first remark we can do, before entering in the detail of comparing the two definitions, is the difference of length of the two definitions: in one and a half line only diffeology defines a manifold with corners, instead of the many lines of the usual definition. The reason is that manifolds with corners are *a priori* diffeological spaces. That is, their smooth structure is already specified. To be a manifold with corners is just then a characteristic of that diffeology, to be generated by corners. In the usual approach, starting with a topological space needs to define at the same time its smooth structure and its special property of being a manifold with corners, which needs more words. The second remark we can do at this level is that corners in diffeology have already their own smooth structure, as diffeological subspaces of standard Euclidean spaces. And this is this diffeology which is in play in the definition. There is no need to add any extra consideration, everything needed is embedded in the theory from step one.

Let us compare now the two approaches.

Proposition. — *Let* $(\mathrm{M}, \mathscr{A})$ *be a n-manifold with corners according to the usual framework,* $\mathscr{A}$ *denoting the maximal atlas of* M. *The finest diffeology* $\mathscr{D}$ *on* M *such that the charts* $\mathrm{F} \in \mathscr{A}$ *are smooth is a diffeology of manifold with corners for which* $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M}) = \mathscr{A}$*, the D-topology of* $(\mathrm{M}, \mathscr{D})$ *coinciding with the given topology of* $(\mathrm{M}, \mathscr{A})$*. We shall denote* $\Phi\colon (\mathrm{M}, \mathscr{A}) \mapsto (\mathrm{M}, \mathscr{D})$ *this association. Conversely, let* $(\mathrm{M}, \mathscr{D})$ *be a diffeological n-manifold with corners. Equip* M *with its D-topology, then* $\mathscr{A} = \mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M})$ *is a maximal atlas equipping* M *with a usual structure of manifold with corners. Let* $\Psi\colon (\mathrm{M}, \mathscr{D}) \mapsto (\mathrm{M}, \mathscr{A})$ *be this association. Then* Φ *and* Ψ *are inverse of each other.*

Note 1. — As ordinary manifolds, the category {K-Manifolds} is closed for products and sums but is not closed for the other usual set theoretic constructions. On another note, as members of the category {Diffeology}, manifolds with corners inherits naturally smooth maps between them. There is no need of a specific definition. They inherit of course of all the diffeological constructions: fiber bundles, homotopy, differential calculus, homology, cohomology, etc.

NOTE 2. — The diffeology framework gives a new perspective on the definition of strata of a n-manifold with corners M, according to the *Klein structure and singularities of a diffeological space* [PIZ13, §1.42]. Indeed, thanks to (§3) one can define the different strata of M as the connected components[5] of the orbits of the pseudo group of local diffeomorphisms $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{M})$. That is,

$$\mathrm{Strat}(\mathrm{M}) = \{\mathscr{O}_i \in \pi_0(\mathscr{O}) \mid \mathscr{O} \in \mathrm{M}/\mathrm{Diff}_{\mathrm{loc}}(\mathrm{M})\}.$$

Moreover, Strat(M) does not capture only the decomposition of M is strata, but equipped with the quotient diffeology of M, it captures also its (transversal) smooth structure. Note also that the *regular part* $\mathrm{M}_{\mathrm{reg}} \subset \mathrm{M}$ — the principal orbit of $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{M})$ — is the union of strata of dimension n, it is a regular n-submanifold and an open dense subset of M.

Proof. Let us begin by a manifold with corners, according to the usual definition. The finest diffeology $\mathscr{D}$ making the charts $\mathrm{F} \in \mathscr{A}$ smooth is the set of parametrizations $\mathrm{P} \colon \mathrm{U} \to \mathrm{M}$ that satify the following: there is a covering of U by a family of open sets U_i, and for each index i a chart $\mathrm{F}_i \in \mathscr{A}$ and a smooth maps $\mathrm{Q}_i \colon \mathrm{U}_i \to \mathrm{K}^n$ such that $\mathrm{P} \restriction \mathrm{U}_i = \mathrm{F}_i \circ \mathrm{Q}_i$. We write $\mathrm{P} = \sup \mathrm{F}_i \circ \mathrm{Q}_i$.

Now, the charts $\mathrm{F} \in \mathscr{A}$ are smooth, by construction, and injective. Their domains are open for the induced topology of K^n, which is also the D-topology of K^n, according to above statement (B).

Let us show now that the topology of M and its D-topology coincide. Let first $\mathscr{O} \subset \mathrm{M}$ be an open subset of M. Let P be a plot of M, then $\mathrm{P} = \sup_i \mathrm{F}_i \circ \mathrm{Q}_i$ for some family of indices, with the F_i in $\mathscr{A}$ and the Q_i smooth parametrizations in K^n. Then, $\mathrm{P}^{-1}(\mathscr{O}) = (\sup \mathrm{F}_i \circ \mathrm{Q}_i)^{-1}(\mathscr{O}) = \cup_i \mathrm{Q}_i^{-1}(\mathrm{F}_i^{-1}(\mathscr{O}))$. And since the F_i and the Q_i are continuous, $\mathrm{P}^{-1}(\mathscr{O})$ is open. Thus, $\mathscr{O}$ is open for the D-topology. Conversely, let $\mathscr{O}$ be open for the D-topology. For all $x \in \mathscr{O}$, there exists $\mathrm{F}_x \in \mathscr{A}$ such that $x \in \mathrm{val}(\mathrm{F}_x)$. Since F_x is a plot for $\mathscr{D}$, $\mathrm{F}_x^{-1}(\mathscr{O})$ is open in K^n, and since F_x is a local homeomorphism from K^n to M, $\mathrm{F}_x \restriction \mathrm{F}_x^{-1}(\mathscr{O})$ is still a local homeomophism. Then, $\mathrm{val}(\mathrm{F}_x \restriction \mathrm{F}_x^{-1}(\mathscr{O}))$ is open in M. But $\mathrm{val}(\mathrm{F}_x \restriction \mathrm{F}_x^{-1}(\mathscr{O})) = \mathscr{O} \cap \mathrm{val}(\mathrm{F}_x)$, thus $\mathscr{O} = \cup_x \mathscr{O} \cap \mathrm{val}(\mathrm{F}_x)$ is a union of open subsets, then open in M. Therefore the topologies coincide.

[5]We choose this precise definition of strata by naturality and not by consideration of dimension. That is also coherent with the elementary exemples. Pick a square, its pseudo group of local diffeomorphism has three orbits, decomposed as follow: the principal orbit (the regular part) of dimension 2, the 1-dimensional orbit made of four edges (the 1-dimensional strata), and the 0- dimension orbit made of the four corners (the 0-dimensional strata).

Let us prove now that $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M}) = \mathscr{A}$. Let $\Phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M})$. Since the two topologies coincide, we know already that Φ is a local homeomorphism from K^n to M. Now let $\mathrm{F} \in \mathscr{A}$, thus $\mathrm{F}^{-1} \circ \Phi = \mathrm{F}^{-1} \circ (\sup \mathrm{F}_i \circ \mathrm{Q}_i)$, where $\Phi = \sup \mathrm{F}_i \circ \mathrm{Q}_i$, as previously. Hence, $\mathrm{F}^{-1} \circ \Phi = \sup(\mathrm{F}^{-1} \circ \mathrm{F}_i) \circ \mathrm{Q}_i$. But the $\mathrm{F}^{-1} \circ \mathrm{F}_i$ and the Q_i are smooth, and moreover local diffeomorphisms, thus $\mathrm{F}^{-1} \circ \Phi$ is a local diffeomorphism, and then also $\Phi^{-1} \circ \mathrm{F}$. Hence, since $\mathscr{A}$ is maximal, $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M}) \subset \mathscr{A}$. Next let $\mathrm{F} \in \mathscr{A}$. We now already that F is smooth, and a local homeomorphism for both topologies. Let us show that $\mathrm{F}^{-1} \colon \mathrm{val}(\mathrm{F}) \to \mathrm{K}^n$ is smooth. Let P be a plot in $\mathrm{val}(\mathrm{F}) \subset \mathrm{M}$, then $\mathrm{P} = \sup \mathrm{F}_i \circ \mathrm{Q}_i$. Hence, $\mathrm{F}^{-1} \circ \mathrm{P} = \sup(\mathrm{F}^{-1} \circ \mathrm{F}_i) \circ \mathrm{Q}_i$. Thus, F^{-1} is smooth and F is a local diffeomorphism. Therefore, $\mathscr{A} \subset \mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M})$, and then $(\mathrm{M}, \mathscr{D})$ is a diffeological manifold with corners such that $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M}) = \mathscr{A}$.

Conversely, let $(\mathrm{M}, \mathscr{D})$ be a diffeological manifold with corners. Equip M with its D-topology. Since local diffeomorphisms are local homeomorphisms for the D-topology, and since local diffeomorphisms of K^n admit smooth extensions on $\mathbf{R}^n$, then $\mathscr{A} = \mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M})$ gives M a structure of manifold with corners in the usual sense. The atlas $\mathscr{A}$ is obviously maximal.

Now, because in the two directions $\mathscr{A} = \mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M})$, the fact that Φ and Ψ are inverse of each other is pretty obvious. ☐

Extension of Differential Forms on Manifolds with Boundary

We already showed that any k-forms on half-spaces H^n can be extended to a neighborhood of $\mathrm{H}^n \subset \mathbf{R}^n$ [GIZ16]. Here we extend this result to manifolds with boundary.

Let us just recall that a differential k-form on a diffeological space X is a mapping α that associates with each plot P in X, a smooth k-form $\alpha(\mathrm{P})$ on dom(P), such that the smooth compatibility condition $\alpha(\mathrm{F} \circ \mathrm{P}) = \mathrm{F}^*(\alpha(\mathrm{P}))$ is satisfied, where F is any smooth parametrization in dom(P).

5. Taylor's Series with Parameters. Let $[x \mapsto f_x] \in \mathrm{C}^\infty(\mathrm{X}, \mathrm{C}^\infty(\mathbf{R}, \mathbf{R}))$, with X any diffeological space and $\mathrm{C}^\infty(\mathbf{R}, \mathbf{R})$ equipped with the functional diffeology. Then, for all positive integer n there exists $a_n \in \mathrm{C}^\infty(\mathrm{X}, \mathbf{R})$ and there exists $[x \mapsto \varphi_{n,x}] \in \mathrm{C}^\infty(\mathrm{X}, \mathrm{C}^\infty(\mathbf{R}, \mathbf{R}))$, such that:

$$f_x(t) = a_0(x) + t a_1(x) + \cdots + t^{n-1} a_{n-1}(x) + t^n \varphi_{n,x}(t).$$

This is the Taylor's series of smooth real functions, with parameters in a diffeological space. It is based on a parameter version of Hadamard's Lemma, which is the order 1 of the series.

Proof. For all $x \in \mathrm{X}$ and all $t \in \mathbf{R}$, one has

$$f_x(t) = f_x(0) + t\, g_x(t) \quad \text{with} \quad g_x(t) = \int_0^1 f_x'(st)\, ds.$$

Since $(x,t) \mapsto \int_0^1 f_x'(st)\,ds$ is smooth, $[x \mapsto g_x]$ belongs to $\mathrm{C}^\infty(\mathrm{X}, \mathrm{C}^\infty(\mathbf{R},\mathbf{R}))$. The Taylor's series is then built by iteration. □

6. Whitney Theorem on Even Functions with Parameters. Let X be a diffeological space, let $[x \mapsto f_x] \in \mathrm{C}^\infty(\mathrm{X}, \mathrm{C}^\infty(\mathbf{R},\mathbf{R}))$. If, for all $t \in \mathbf{R}$ and all $x \in \mathrm{X}$, $f_x(t) = f_x(-t)$, then there exists $[x \mapsto g_x] \in \mathrm{C}^\infty(\mathrm{X}, \mathrm{C}^\infty(\mathbf{R},\mathbf{R}))$ such that $f_x(t) = g_x(t^2)$.

Proof. This is a direct adaptation of Whitney's original proof in [Whi43], a game of substitution. Thanks to Taylor's Series with Parameters (§5) we have $f_x(t) = a_0(x) + t a_1(x) + \cdots + t^{n-1} a_{n-1}(x) + t^n \varphi_{n,x}(t)$. But because f_x is even, the odd terms vanish, and we rewrite

$$f_x(t) = a_0(x) + t^2 a_1(x) + \cdots + t^{2n-2} a_{n-1}(x) + t^{2n} \varphi_{2n,x}(t).$$

Following Whitney, we put $\psi_{n,x}(u) = \psi_{n,x}(-u) = \varphi_{2n,x}(\sqrt{u})$ and

$$g_x(u) = a_0(x) + u a_1(x) + \cdots + u^{n-1} a_{n-1}(x) + u^n \psi_{n,x}(u).$$

According to Whitney, for every $x \in \mathrm{X}$, the function g_x is smooth. Let us check that $[x \mapsto g_x]$ is smooth. That is, for all plot $r \mapsto x_r$ in X, the parametrization $(r,t) \mapsto g_{x_r}(t)$ is smooth. Let then $(r,t) \mapsto \mathrm{F}(r,t) = f_{x_r}(t)$. The function F is even in t. According to Whitney [Whi43, Remark p.160], $\mathrm{F}(r,t) = \mathrm{G}(r,t^2)$, with $(r,u) \mapsto \mathrm{G}(r,u)$ smooth. Let us check then that $\mathrm{G}(r,u) = g_{x_r}(u)$. On the one hand, we have:

$$\mathrm{F}(r,t) = \alpha_0(r) + t^2 \alpha_1(r) + \cdots + t^{2n} \Phi_{2n}(r,t).$$

We put $\Psi_n(r,u) = \Psi_n(r,-u) = \Phi_{2n}(r,\sqrt{u})$ and, according to Whithney, we have:

$$\mathrm{G}(r,u) = \alpha_0(r) + u\alpha_1(r) + \cdots + u^n \Psi_n(r,u).$$

On the other hand, we have:

$$g_{x_r}(u) = a_0(x_r) + u a_1(x_r) + \cdots + u^n \psi_{n,x_r}(u).$$

Now, since $\mathrm{F}(r,t) = f_{x_r}(t)$ and $\partial^k[\mathrm{F}(r,t) - f_{x_r}(t)]/\partial t^k \,|_{t=0} = 0$ for all $k < 2n$, $a_i(x_r) = \alpha_i(r)$ for all i and r, and $\varphi_{2n,x_r}(t) = \Phi_{2n}(r,t)$. But $\varphi_{2n,x_r}(\sqrt{u}) = \psi_{n,x_r}(u)$ and $\Phi_{2n}(r,\sqrt{u}) = \Psi_n(r,u)$, hence $\psi_{n,x_r}(u) = \Psi_n(r,u)$. Thus, $g_{x_r}(u) = \mathrm{G}(r,u)$. Therefore $(r,u) \mapsto g_{x_r}(u)]$ is smooth, that is, $[x \mapsto g_x] \in \mathrm{C}^\infty(\mathrm{X}, \mathrm{C}^\infty(\mathbf{R},\mathbf{R}))$ and $f_x(t) = g_x(t^2)$. □

7. Differential Forms on Manifolds with Boundary. Let M be a n-manifold, and consider the $(n+1)$-manifold with boundary $\mathrm{M} \times [0,1[$. Let $\alpha \in \Omega^k(\mathrm{M} \times [0,1[)$, be a differential k-form [PIZ13, §6.28]. Then α extends to a k-form $\underline{\alpha}$ on an open neighborhood of $\mathrm{M} \times [0,1[$ in $\mathrm{M} \times]-1,+1[$.

Corollary. — Let M be a manifold with boundary. and $M \hookrightarrow N$ be an embedding as a *pièce à bord* [ADLH73]. If the boundary ∂M is compact[6], then every differential k-form on M extends to an open neighborhood of M in N.

Proof. First of all, consider $\alpha \restriction M \times]0,1[$. Then, there exist two smooth parametrizations $t \mapsto a_t \in \Omega^{k-1}(M)$ and $t \mapsto b_t \in \Omega^k(M)$, defined on $]0,1[$, with the space of forms equipped with the functional diffeology [PIZ13, §6.29], such that:

$$\alpha \restriction M \times]0,1[= dt \wedge a_t + b_t,$$

with

$$dt \wedge a_{t,x} \begin{pmatrix} \delta_1 x \\ \delta_1 t \end{pmatrix} \cdots \begin{pmatrix} \delta_k x \\ \delta_k t \end{pmatrix} = \sum_{i=1}^{k} (-1)^{i-1} \delta_i t \times a_{t,x}(\delta_1 x, \ldots, \widehat{\delta_i x}, \ldots, \delta_k x),$$

where $\delta_i x \in T_x(M)$, $\delta_i t \in \mathbf{R}$, and $\widehat{\delta_i x}$ means the vector $\delta_i x$ is omitted. And with an abuse of notation:

$$b_{t,x} \begin{pmatrix} \delta_1 x \\ \delta_1 t \end{pmatrix} \cdots \begin{pmatrix} \delta_k x \\ \delta_k t \end{pmatrix} = b_{t,x}(\delta_1 x) \cdots (\delta_k x).$$

Now, let

$$\mathrm{sq} : M \times]-1,+1[\to M \times [0,+1[\quad \text{defined by} \quad \mathrm{sq}(x,t) = (x,t^2).$$

Then, there exist two smooth parametrizations $t \mapsto A_t \in \Omega^{k-1}(M)$ and $t \mapsto B_t \in \Omega^k(M)$, defined on $]-1,+1[$, such that

$$\mathrm{sq}^*(\alpha) = dt \wedge A_t + B_t.$$

Consider $\varepsilon : (x,t) \mapsto (x,-t)$. Then, $\mathrm{sq} \circ \varepsilon = \mathrm{sq}$, thus $\mathrm{sq}^*(\alpha) = \varepsilon^*(\mathrm{sq}^*(\alpha))$, that is,

$$dt \wedge A_t + B_t = -dt \wedge A_{-t} + B_{-t}.$$

Hence, $t \mapsto A_t$ is odd, $A_{-t} = -A_t$. Thus $A_0 = 0$, and thanks to (§5)[7] there exists a smooth parametrization $t \mapsto \underline{A}_t$, defined on $]-1,+1[$ into $\Omega^{k-1}(M)$, such that:

$$A_t = 2t \times \underline{A}_t.$$

Hence,

$$\mathrm{sq}^*(\alpha) = 2t \times dt \wedge \underline{A}_t + B_t.$$

But,

$$\mathrm{sq}^*(\alpha \restriction M \times]0,1[) = \mathrm{sq}^*(\alpha) \restriction M \times]-1,0[\cup M \times]0,+1[.$$

That is,

$$2t \times dt \wedge a_{t^2} + b_{t^2} = 2t \times dt \wedge \underline{A}_t + B_t.$$

[6] The non-compact case seems to need more work, see for example [Bro62] [Con71].

[7] We apply the Hadamard's Lemma with parameters to the map $y \mapsto [t \mapsto f_y(t)]$, from $T^k M$ to $C^\infty(\mathbf{R},\mathbf{R})$, with $y = (x, v_1, \ldots, v_k)$ and $f_y(t) = A_t(x)(v_1, \ldots, v_k)$.

Hence,

$$\text{for all } t \neq 0 \quad a_{t^2} = \underline{\mathrm{A}}_t \quad \text{and} \quad b_{t^2} = \mathrm{B}_t.$$

Thus, $t \mapsto \underline{\mathrm{A}}_t$ and $t \mapsto \underline{\mathrm{B}}_t$, defined on $]-1,+1[$ are even. Then, according to (§6)[8], there exist two smooth parametrizations $t \mapsto \underline{a}_t$ and $t \mapsto \underline{b}_t$, defined on $]-1,+1[$, such that

$$\underline{\mathrm{A}}_t = \underline{a}_{t^2} \quad \text{and} \quad \mathrm{B}_t = \underline{b}_{t^2}.$$

Let us now define $\underline{\alpha} \in \Omega^k(\mathrm{M} \times]-1,+1[)$

$$\underline{\alpha} = dt \wedge \underline{a}_t + \underline{b}_t.$$

Then,

$$2t \times dt \wedge \underline{\mathrm{A}}_t + \underline{\mathrm{B}}_t = 2t \times dt \wedge \underline{a}_{t^2} + \underline{b}_{t^2},$$

that is,

$$\mathrm{sq}^*(\alpha) = \mathrm{sq}^*(\underline{\alpha}) \quad \textit{i.e.} \quad \mathrm{sq}^*(\alpha - \underline{\alpha} \restriction \mathrm{M} \times [0,1[) = 0.$$

Consider now this version of the lemma (§8):

Lemma. — *Let* $\beta \in \Omega^k(\mathrm{M} \times [0,1[)$. *If* $\mathrm{sq}^*(\beta) = 0$, *then* $\beta = 0$.

◀ Proof of the Lemma. — Let $\mathrm{P}\colon \mathrm{U} \to \mathrm{M} \times [0,1[$ be a plot and let us show that $\beta(\mathrm{P}) = 0$. Let $\mathrm{U}' = \mathrm{P}^{-1}(\mathrm{M} \times]0,1[)$ and $\mathrm{P}' = \mathrm{P} \restriction \mathrm{U}'$. Since $\mathrm{sq} \restriction \mathrm{M} \times (]-1,0[\cup [0,+1[) \to \mathrm{M} \times]0,1[$ is a covering of manifolds, *i.e.* a local diffeomorphims everywhere, $\mathrm{sq}^*(\beta)(\mathrm{P}) \restriction \mathrm{U}' = 0$ implies $\beta(\mathrm{P}) \restriction \mathrm{U}' = 0$. By continuity $\beta(\mathrm{P}) \restriction \overline{\mathrm{U}'} = 0$. Then, let $\mathrm{U}'' = \mathrm{U} - \overline{\mathrm{U}'}$ and $\mathrm{P}'' = \mathrm{P} \restriction \mathrm{U}''$. But then $\mathrm{sq} \circ \mathrm{P}'' = \mathrm{P}''$, hence $\beta(\mathrm{P}) \restriction \mathrm{U}'' = \beta(\mathrm{P}'') = \beta(\mathrm{sq} \circ \mathrm{P}'') = \mathrm{sq}^*(\beta)(\mathrm{P}'') = 0$. Now $\beta(\mathrm{P}) \restriction \overline{\mathrm{U}'} = 0$ and $\beta(\mathrm{P}) \restriction \mathrm{U}'' = 0$, with $\mathrm{U} = \overline{\mathrm{U}'} \cup \mathrm{U}''$, implies $\beta(\mathrm{P}) = 0$. Therefore $\beta = 0$. ▶

Then, $\mathrm{sq}^*(\alpha - \underline{\alpha} \restriction \mathrm{M} \times [0,1[) = 0$ implies $\alpha = \underline{\alpha} \restriction \mathrm{M} \times [0,1[$.

The corollary is a particular application of Douady's theorem [ADLH73, Proposition 3.1] that embeds any manifold with corners into itself, as a *pièce à coins*[9], in our case as a *pièce à bord*. Since the two categories (usual and diffeological) of manifolds with corners (or boundary) coincide (§4), this embedding is also a embedding[10] as diffeological manifold with corners (boundary). Then, if the border is compact, then there is an open neighborhood of the boundary $\partial\mathrm{M}$ in N, diffeomorphic to $\partial\mathrm{M} \times]-1,+1[$ such that, restricted to $\partial\mathrm{M} \times [0,1[$, it is a diffeomorphim on a neighborhood of $\partial\mathrm{M}$ in M. And the propostion above applies. □

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[8] As above, we consider $y \mapsto [t \mapsto f_y(t)]$, from $\mathrm{T}^k\mathrm{M}$ to $\mathrm{C}^\infty(\mathbf{R},\mathbf{R})$, with $y = (x, v_1, \ldots, v_k)$ and $f_y(t) = \underline{\mathrm{A}}_t(x)(v_1, \ldots, v_k)$.

[9] Piece with corners.

[10] See embeddings in diffeology in [PIZ13, §2.13].

Extension of Differential Forms on Corners

In the previous section we extended the differential forms on manifolds with boundary, on open neighborhoods, after having pushed the manifolds as a *pièces à bords* [Dou62]. Next, we prove an extension theorem for corners (§9). The general case of manifolds with corners as *pièces à coins* is a work in progress.

8. The Square Function Lemma. Let $\mathrm{sq} : \mathbf{R}^n \to \mathrm{K}^n$ be the smooth parametrization:

$$\mathrm{sq}(x_1, \ldots, x_n) = (x_1^2, \ldots, x_n^2).$$

Then $\mathrm{sq}^* : \Omega^k(\mathrm{K}^n) \to \Omega^k(\mathbf{R}^n)$ is injective. That is, for all $\alpha \in \Omega^k(\mathrm{K}^n)$, if $\mathrm{sq}^*(\alpha) = 0$, then $\alpha = 0$.

Proof. Note that each component of $\mathrm{S}_j - \mathrm{S}_{j-1}$ is diffeomorphic to $\mathbf{R}^j$. Hence, if $\mathrm{sq}^*(\alpha) = 0$, since $\mathrm{sq} \restriction \mathrm{sq}^{-1}(\mathrm{S}_j - \mathrm{S}_{j-1})$ is a 2-fold covering over $\mathrm{S}_j - \mathrm{S}_{j-1}$, $\alpha \restriction \mathrm{S}_j - \mathrm{S}_{j-1} = 0$. That is, for all plot Q in $\mathrm{S}_j - \mathrm{S}_{j-1}$, $\alpha(\mathrm{Q}) = 0$. Let then, for some $j \geq 1$, $\mathrm{P}_j : \mathrm{U}_j \to \mathrm{S}_j$ be a plot. In view of what precedes, the subset $\mathcal{O}_j = \mathrm{P}_j^{-1}(\mathrm{S}_j - \mathrm{S}_{j-1})$ is open, and $\alpha(\mathrm{P}_j \restriction \mathcal{O}_j) = \alpha(\mathrm{P}_j) \restriction \mathcal{O}_j = 0$. By continuity, $\alpha(\mathrm{P}_j) \restriction \overline{\mathcal{O}}_j = 0$, where $\overline{\mathcal{O}}_j$ is the closure of $\mathcal{O}_j$. Let then $\mathrm{U}_{j-1} = \mathrm{U}_j - \overline{\mathcal{O}}_j$ and $\mathrm{P}_{j-1} = \mathrm{P}_j \restriction \mathrm{U}_{j-1}$. Then, U_{j-1} is open and $\mathrm{P}_{j-1} : \mathrm{U}_{j-1} \to \mathrm{S}_{j-1}$ is a plot. This construction gives a descending recursion, starting with any plot $\mathrm{P} : \mathrm{U} \to \mathrm{K}^n$, by initializing $\mathrm{P}_n = \mathrm{P}$, $\mathrm{U}_n = \mathrm{U}$ and $\mathrm{S}_n = \mathrm{K}^n$. One has $\mathrm{P}_j = \mathrm{P} \restriction \mathrm{U}_j$, $\mathrm{U}_{j-1} \subset \mathrm{U}_j$, the recursion ends with a plot P_0 with values in $\mathrm{S}_0 = \{0\}$, and $\alpha(\mathrm{P}_0) = 0$ since P_0 is constant. Therefore $\alpha = 0$. ☐

9. Differential Forms On Corners. The section (§2) above deals with smooth real functions on corners, that is, $\Omega^0(\mathrm{K}^n)$. It is a particular case of the following theorem on differential forms of any degree:

Theorem. *Any differential k-form on the corner* K^n*, equipped with the subset diffeology of* $\mathbf{R}^n$*, is the restriction of a smooth differential k-form defined on some open neighborhood of the corner.* Precisely, the pullback : $j^* : \Omega^k(\mathbf{R}^n) \to \Omega^k(\mathrm{K}^n)$ is surjective, where j denotes the inclusion from K^n to $\mathbf{R}^n$.

Proof. Let $\omega \in \Omega^k(\mathrm{K}^n)$ and $\mathring{\mathrm{K}}^n = \{(x_i)_{i=1}^n \mid x_i > 0, i = 1, \ldots, n\}$. One has

$$\omega \restriction \mathring{\mathrm{K}}^n = \sum_{i_1 < \cdots < i_k} a_{i_1 \ldots i_k}(x_1, \ldots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

with $i_j = 1, \ldots, n$ and $a_{i_1 \ldots i_k} \in \mathrm{C}^\infty(\mathring{\mathrm{K}}^n, \mathbf{R})$. Recall that $\mathrm{sq} : (x_i)_{i=1}^n \mapsto (x_i^2)_{i=1}^n$, then

$$\mathrm{sq}^*(\omega) = \sum_{i_1 < \cdots < i_k} \mathrm{A}_{i_1 \ldots i_k}(x_1, \ldots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $\mathrm{A}_{i_1 \ldots i_k} \in \mathrm{C}^\infty(\mathbf{R}^n, \mathbf{R})$. Let $\varepsilon_j : (x_1, \ldots, x_j, \ldots, x_n) \mapsto (x_1, \ldots, -x_j, \ldots, x_n)$, then $\mathrm{sq} \circ \varepsilon_j = \mathrm{sq}$ and $(\mathrm{sq} \circ \varepsilon_j)^*(\omega) = \varepsilon_j^*(\mathrm{sq}^*(\omega))$, that is, $\mathrm{sq}^*(\omega) = \varepsilon_j^*(\mathrm{sq}^*(\omega))$.

Hence,

$$\begin{aligned}\varepsilon_j^*(\mathrm{sq}^*(\omega)) &= \sum_{\substack{i_1<\cdots<i_k\\ i_\ell\neq j}} \mathrm{A}_{i_1\ldots i_k}(x_1,\ldots,-x_j,\ldots,x_n)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}\\ &\quad - \sum_{i_1<\cdots\leq j\leq\cdots<i_k} \mathrm{A}_{i_1\ldots j\ldots i_k}(x_1,\ldots,-x_j,\ldots,x_n)\, dx_{i_1}\wedge\cdots\wedge dx_j\wedge\cdots\wedge dx_{i_k}.\end{aligned}$$

Then,

$$\begin{aligned}\mathrm{A}_{\substack{i_1\ldots i_k\\ i_\ell\neq j}}(x_1,\ldots,-x_j,\ldots,x_n) &= \mathrm{A}_{i_1\ldots i_k}(x_1,\ldots,x_j,\ldots,x_n),\\ \mathrm{A}_{i_1\ldots j\ldots i_k}(x_1,\ldots,-x_j,\ldots,x_n) &= -\mathrm{A}_{i_1\ldots j\ldots i_k}(x_1,\ldots,x_j,\ldots,x_n).\end{aligned}$$

Hence,

$$\mathrm{A}_{i_1\ldots j\ldots i_k}(x_1,\ldots,x_j=0,\ldots,x_n)=0.$$

Thus,

$$\mathrm{A}_{i_1\ldots j\ldots i_k}(x_1,\ldots,x_j,\ldots,x_n)=2x_j\underline{\mathrm{A}}_{i_1\ldots j\ldots i_k}(x_1,\ldots,x_j,\ldots,x_n),$$

with $\underline{\mathrm{A}}_{i_1\ldots j\ldots i_k}\in \mathrm{C}^\infty(\mathbf{R}^n,\mathbf{R})$. Therefore, there are real smooth functions $\hat{\mathrm{A}}_{i_1\ldots i_k}$ defined on $\mathbf{R}^n$ such that

$$\mathrm{A}_{i_1\ldots i_k}(x_1,\ldots,x_n)=2^k x_{i_1}\ldots x_{i_k}\hat{\mathrm{A}}_{i_1\ldots i_k}(x_1,\ldots,x_n).$$

Now,

$$\mathrm{sq}^*(\omega\restriction \mathring{\mathrm{K}}^n)=\mathrm{sq}^*(\omega)\restriction\{x_i\neq 0\}$$

implies

$$\begin{aligned}&\sum_{i_1<\cdots<i_k} 2^k x_{i_1}\ldots x_{i_k} a_{i_1\ldots i_k}(x_1^2,\ldots,x_n^2)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}\\ =\;&\sum_{i_1<\cdots<i_k} 2^k x_{i_1}\ldots x_{i_k}\hat{\mathrm{A}}_{i_1\ldots i_k}(x_1,\ldots,x_n)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}.\end{aligned}$$

Hence,

$$\hat{\mathrm{A}}_{i_1\ldots i_k}(x_1,\ldots,x_n)=a_{i_1\ldots i_k}(x_1^2,\ldots,x_n^2)\quad \text{for } x_i\neq 0, i=1,\ldots,n.$$

Thus $(x_1,\ldots,x_n)\mapsto \hat{\mathrm{A}}_{i_1\ldots i_k}(x_1,\ldots,x_n)$, which belongs to $\mathrm{C}^\infty(\mathbf{R}^n,\mathbf{R})$, is even in each variable. Therefore, according to Schwartz's Theorem [Sch75][11], there exist

$$\underline{a}_{i_1\ldots i_k}\in \mathrm{C}^\infty(\mathbf{R}^n,\mathbf{R}),$$

such that

$$\hat{\mathrm{A}}_{i_1\ldots i_k}(x_1,\ldots,x_n)=\underline{a}_{i_1\ldots i_k}(x_1^2,\ldots,x_n^2).$$

One deduces:

$$\underline{a}_{i_1\ldots i_k}(x_1,\ldots,x_n)=a_{i_1\ldots i_k}(x_1,\ldots,x_n),\ \text{for all } (x_1,\ldots,x_n)\in\mathring{\mathrm{K}}^n.$$

[11] Which is a generalisation of a famous theorem due to Whitney [Whi43]. It could be also deduced from it easily.

Then, defining the k-form $\underline{\omega}$ on $\mathbf{R}^n$ by

$$\underline{\omega} = \sum_{i_1 < \cdots < i_k} \underline{a}_{i_1 \ldots i_k}(x_1, \ldots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

one has already

$$\underline{\omega} \restriction \mathring{K}^n = \omega \restriction \mathring{K}^n.$$

Let us show that $\underline{\omega} \restriction K^n = \omega$. That is, let us check that for all plot $P \colon U \to \mathbf{R}^n$, $P^*(\underline{\omega}) = \omega(P)$. Actually, one has

$$\mathrm{sq}^*(\omega) = \mathrm{sq}^*(\underline{\omega} \restriction K^n).$$

Indeed:

$$\begin{aligned} \mathrm{sq}^*(\omega) &= \sum_{i_1 \ldots i_k} A_{i_1 \ldots i_k}(x_1, \ldots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum_{i_1 \ldots i_k} 2^k x_{i_1} \ldots x_{i_k} \hat{A}_{i_1 \ldots i_k}(x_1, \ldots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum_{i_1 \ldots i_k} 2^k x_{i_1} \ldots x_{i_k} \underline{a}_{i_1 \ldots i_k}(x_1^2, \ldots, x_n^2)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \end{aligned}$$

And, on the other hand:

$$\mathrm{sq}^*(\underline{\omega} \restriction K^n) = \sum_{i_1 \ldots i_k} 2^k x_{i_1} \ldots x_{i_k} \underline{a}_{i_1 \ldots i_k}(x_1^2, \ldots, x_n^2)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Thus, $\mathrm{sq}^*(\omega - \underline{\omega} \restriction K^n) = 0$. Therefore, according to the lemma (§8), $\omega - \underline{\omega} \restriction K^n = 0$. And then, ω is the restriction on K^n of the smooth k-form $\underline{\omega}$ on $\mathbf{R}^n$. □

10. Other Corners. The *half-line* $\Delta_k = \mathbf{R}^k/O(k)$ is identified to the interval $[0, \infty[$, equipped with the pushforward of the smooth diffeology of $\mathbf{R}^k$ by the projection $\nu_k \colon X \mapsto \|X\|^2$, see [PIZ07, IZW14]. Then, with each half-line we can associate a n-corner Δ_k^n. Note that, according to definition (§4), the only one of these corners being a manifold with corners is $K^n = \Delta^n$. Note also that the identity j_k^n from Δ_k^n to $K^n \subset \mathbf{R}^n$ is still a smooth map. Now,

Proposition. — *The pullback $j_k^{n*} \colon \Omega^*(K^n) \to \Omega^*(\Delta_k^n)$ is surjective.*

As well as for the standard corner K^n, every differential form on Δ_k^n is the pullback of some smooth form on $\mathbf{R}^n$.

Proof. The proof is a copy from the proof of (§9) because 1) the map sq is smooth, 2) the D-topology of $\Delta_k^n \subset \mathbf{R}^n$ coincides with the induced topology, same proof as in (§1). Then, 3) the interior of each stratum is some power of the open interval, and the rest follows. □

11. An Application. Among the possible applications of the theorems above there is already one worthy of mention. It is about the description of closed 2-forms, invariant with respect to the action of a Lie group. As it has been shown in particular in the classification of SO(3)-symplectic manifolds [Igl84, Igl91], any closed 2-form ω on a manifold M, invariant by a compact group[12] G, is characterized by its *moment map* $\mu\colon \mathrm{M} \to \mathscr{G}^*$ (we assume the action Hamiltonian), and for each moment map, a closed 2-form $\varepsilon \in \mathrm{Z}^2(\mathrm{M}/\mathrm{G})$. Let us be more precise: the space of G-invariant closed 2-forms $\mathrm{Z}^2_\mathrm{G}(\mathrm{M})$ is a vector space, the space of G-equivariant maps from M to $\mathscr{G}^*$ is also a vector space, and the map associating its moment map[13] μ with each invariant closed 2-form ω is linear. What we claim is that the kernel of this map is exactly $\mathrm{Z}^2(\mathrm{M}/\mathrm{G})$, where M/G is equipped with the quotient diffeology. Denoting by $\mathscr{E}q_\bullet(\mathrm{M},\mathscr{G}^*) \subset \mathscr{E}q(\mathrm{M},\mathscr{G}^*)$ the space of moment maps of G-invariant closed 2-forms on M, as a subset of smooth equivariant maps, one has this exact sequence of smooth linear maps:

$$0 \longrightarrow \mathrm{Z}^2(\mathrm{M}/\mathrm{G}) \longrightarrow \mathrm{Z}^2_\mathrm{G}(\mathrm{M}) \longrightarrow \mathscr{E}q_\bullet(\mathrm{M},\mathscr{G}^*) \longrightarrow 0.$$

Now, if an equivariant map is easy to conceive, it is more problematic for a differential form on the space of orbits, which is generally not a manifold. This is where the above theorem can help, because it happens that M/G is not far to be a manifold with boundary or corners, as show the following example.

Consider the simple case $\mathrm{M} = \mathbf{R}^{2n}$, equipped with the standard symplectic form $\omega = \sum_{i=1}^n dq_i \wedge dp_i$. It is invariant by the group $\mathrm{SO}(2)^n$ acting naturally, each factor on its respective copy of $\mathbf{R}^2$. The quotient space $\mathscr{Q}^n = \mathbf{R}^{2n}/\mathrm{SO}(2)^n$ is equivalent to the *other corner* Δ_2^n, with $\Delta_2 = \mathbf{R}^2/\mathrm{O}(2) = \mathbf{R}^2/\mathrm{SO}(2)$. Thus, thanks to (§9)-(§10), for each 2-form ε on the quotient $\mathscr{Q}^n$ there exists a 2-form $\underline{\varepsilon}$ on $\mathbf{R}^n$, such that $\varepsilon = j_k^{n*}(\underline{\varepsilon})$. Then, the 2-form ω is characterized by the moment map μ and $\underline{\varepsilon} \upharpoonright \mathrm{K}^n$, with $\underline{\varepsilon} \in \Omega^k(\mathbf{R}^n)$.

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Serap Gürer — Galatasaray University, Ortaköy, Çırağan Cd. No:36, 34349 Beşiktaş/İstanbul, Turkey.

Patrick Iglesias-Zemmour — I2M CNRS, Marseille France & The Hebrew University of Jerusalem, Israel.