ON DIFFEOLOGY OF ORBIT SPACES

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ABSTRACT. We discuss the relationship between the orbit type stratification of a manifold M under the smooth action of a compact Lie group G and the Klein stratification (defined by local diffeomorphisms) of the resulting space of orbits M/G, when the latter is equipped with the quotient diffeology.

INTRODUCTION

The study of spaces arising from the smooth action of a compact Lie group G on a manifold M is a cornerstone of differential geometry and topology. This area has a rich history, built upon the foundational work of many distinguished mathematicians, including figures like Palais, Bredon, Mostov, and their contemporaries, who laid the groundwork for understanding compact transformation groups and their orbit spaces. The resulting orbit space M/G is in general not a manifold due to the presence of singularities at points corresponding to non-principal orbits. While classical approaches often describe these spaces topologically or with piecewise smooth structures, the framework of diffeology offers a powerful and unified method to endow M/G with a global smooth structure — the quotient diffeology — allowing for a consistent differential calculus throughout the space, even across singular regions.

This paper leverages the tools of diffeology to investigate the intrinsic geometric structure of these orbit spaces M/G. A key aspect of this investigation is the diffeological dimension map, a fundamental invariant. This map, defined on M/G, associates to each orbit $\mathcal{O} \in \mathrm{M}/\mathrm{G}$ the dimension of the space M/G at that point, denoted by $\dim_{\mathcal{O}}(\mathrm{M}/\mathrm{G})$. We establish the formula relating this diffeological dimension of M/G at a point $\mathcal{O} \in \mathrm{M}/\mathrm{G}$ to the geometry of $\mathcal{O} \subset \mathrm{M}$:

$$\dim_{\mathcal{O}}(\mathrm{M}/\mathrm{G}) = \dim(\mathrm{M}) - \dim(\mathcal{O}).$$

It is important to note that this formula is not a definition specific to orbit spaces, but an instance of the general definition of the dimension by minimax in diffeology [PIZ07,

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PIZ13], the first main invariant of the category. This result, while perhaps intuitive from the perspective of the local slice model $\mathrm{E}/\operatorname{Stab}_{\mathrm{G}}(x)$ (where $\mathcal{O} = \mathrm{G} \cdot x$), relies crucially on a key property of diffeological quotients of Euclidean spaces by subgroups of their orthogonal group: namely, that for $\mathrm{H} \subset \mathrm{O}(\mathrm{E})$, the diffeological dimension of E/H at its origin 0_{E} is $\dim(\mathrm{E})$. This implies that the "collapsing" effect of the group action on the dimension of the slice is fully captured by the diffeological dimension of the quotient at that point, leading directly to the formula. As a consequence, the diffeological dimension of M/G at singular points is greater than or equal to its dimension at principal points.

Beyond the dimension, the local structure of a diffeological space is characterized by its pseudogroup of local diffeomorphisms, whose orbits define the intrinsic Klein stratification [PIZ13]. For orbit spaces M/G, there is also the classical orbit-type stratification of the manifold M. A central theme of this paper is to understand the relationship between these two stratifications. We show that the canonical projection $\mathrm{orbit} : \mathrm{M} \to \mathrm{M}/\mathrm{G}$ induces a surjective map, not from the set of orbit-type strata in M to the set of Klein strata in M/G, but rather from the set of connected components of orbit-type strata in M to the set of Klein strata in M/G. This refined mapping establishes that the projection from M to its quotient M/G is a stratified subduction.

Crucially, we demonstrate through examples that this induced map is not necessarily injective. This non-injectivity manifests in various ways, such as different orbit types in M projecting to points in the same Klein stratum in M/G (e.g., $\mathrm{P}^2(\mathbf{C})/\mathrm{SO}(3)$), or even distinct connected components of the same orbit type mapping to the same Klein stratum (e.g., $\{\mathrm{P}_0, \mathrm{P}_2\} \subset \mathrm{P}^2(\mathbf{C})$ in $\mathrm{P}^2(\mathbf{C})/\mathrm{U}(1)$), while another component (e.g., $\{\mathrm{P}_1\} \subset \mathrm{P}^2(\mathbf{C})$) maps to a different Klein stratum. This occurs when the local slice models E/H corresponding to these points are diffeomorphic as diffeological spaces. This non-injectivity highlights a profound aspect of diffeological quotients: the resulting space M/G possesses an intrinsic geometric identity, defined by its diffeology (its smooth structure) and its local diffeomorphism pseudogroup. This identity, which is one of smooth equivalence, can be independent of the specific manifold M and group G used in its construction. This perspective motivates the potential definition and study of a class of singular spaces, tentatively termed *orthofolds*, characterized by being locally diffeomorphic to quotients of Euclidean spaces by orthogonal group actions.

The paper is organized as follows: Section I reviews the local structure of orbit spaces and its diffeological interpretation, introduces the diffeological dimension map and proves the dimension formula, with an application to toric geometry. Section II recalls the general theory of stratifications in diffeology. Section III defines and discusses the orbit-type and Klein stratifications. Section IV presents several illustrative examples. Finally, Section V establishes the relationship between the two stratifications and discusses the implications of the non-injectivity of the induced map.

I. Structure of Orbit Spaces

In this section we clarify the local structure of the diffeological quotient of a manifold M by the smooth action of a compact Lie group G.

1. Local Structure of Orbit Spaces.

Let G be a compact Lie group acting smoothly on a manifold M. Let $\mathcal{O} = \mathrm{G}(x)$ be the orbit of a point $x \in \mathrm{M}$, and let $\mathrm{H} = \mathrm{Stab}_{\mathrm{G}}(x)$ be its stabilizer.

The classical tube theorem (see, e.g., [Pal60, Bre72]) states that there exists a G-invariant neighbourhood $\mathrm{V} \subset \mathrm{M}$ of the orbit $\mathcal{O}$, and a vector space E (the normal slice at x) on which H acts orthogonally, such that V is G-equivariantly diffeomorphic to the associated bundle $\mathrm{G} \times_{\mathrm{H}} \mathrm{E} = (\mathrm{G} \times \mathrm{E})/\mathrm{H}$. Here, $h \in \mathrm{H}$ acts on $(g, \xi) \in \mathrm{G} \times \mathrm{E}$ by

$$h \cdot (g, \xi) = (gh^{-1}, h(\xi)),$$

and $g' \in \mathrm{G}$ acts on $[g, \xi]_{\mathrm{H}} \in \mathrm{G} \times_{\mathrm{H}} \mathrm{E}$ by left multiplication on the first factor:

$$g' \cdot [g, \xi]_{\mathrm{H}} = [g'g, \xi]_{\mathrm{H}}.$$

The diffeomorphism$\Phi : \mathrm{V} \to \mathrm{G} \times_{\mathrm{H}} \mathrm{E}$ is G-equivariant: $\Phi(g'(y)) = g' \cdot \Phi(y)$ for all $y \in \mathrm{V}, g' \in \mathrm{G}$. Then,

Proposition. *Considering the quotient* $(\mathrm{G} \times_{\mathrm{H}} \mathrm{E})/\mathrm{G}$*, the projection* $\mathrm{pr}_{\mathrm{G}} : \mathrm{G} \times \mathrm{E} \to \mathrm{E} \simeq \mathrm{G} \times_{\mathrm{G}} \mathrm{E}$ *descends into a diffeomorphism* $\phi : (\mathrm{G} \times_{\mathrm{H}} \mathrm{E})/\mathrm{G} \to (\mathrm{G} \times_{\mathrm{G}} \mathrm{E}/)/\mathrm{H} \simeq \mathrm{E}/\mathrm{H}$*, according to the diagram:*

$$\begin{tikzcd}
 & \mathrm{G} \times \mathrm{E} \arrow[dl, "\mathrm{pr}_{\mathrm{H}}"'] \arrow[dr, "\mathrm{pr}_{\mathrm{G}}"] & \\
\mathrm{G} \times_{\mathrm{H}} \mathrm{E} \arrow[d, "\mathrm{pr}_{\mathrm{H},\mathrm{G}}"'] & & \mathrm{G} \times_{\mathrm{G}} \mathrm{E} \arrow[d, "\mathrm{pr}_{\mathrm{G},\mathrm{H}}"] \\
(\mathrm{G} \times_{\mathrm{H}} \mathrm{E})/\mathrm{G} \arrow[rr, "\phi"'] & & (\mathrm{G} \times_{\mathrm{G}} \mathrm{E})/\mathrm{H}
\end{tikzcd}$$

Proof. This is a particular case of the general situation:

lemma. Let G and H be two diffeological groups acting smoothly on a diffeological space X. Assume their actions commute, i.e., $g(h(x)) = h(g(x))$ for all $g \in \mathrm{G}$, $h \in \mathrm{H}$ and $x \in \mathrm{X}$. Then, the iterated quotients are naturally diffeomorphic to the quotient by the product group:

$$(\mathrm{X}/\mathrm{H})/\mathrm{G} \simeq (\mathrm{X}/\mathrm{G})/\mathrm{H} \simeq \mathrm{X}/(\mathrm{G} \times \mathrm{H}).$$

Proof of Lemma. First of all $[x]_{\mathrm{H}} = \{h(x) \mid h \in \mathrm{H}\}$ and $\mathrm{X}/\mathrm{H} = \{\{h(x) \mid h \in \mathrm{H}\} \mid x \in \mathrm{X}\}$. Then, G acts on $[x]_{\mathrm{H}}$ by multiplication term by term. Indeed, $g[x]_{\mathrm{H}} = g\{h(x) \mid h \in \mathrm{H}\} = \{g(h(x)) \mid h \in \mathrm{H}\} = \{h(gx) \mid h \in \mathrm{H}\} = [gx]_{\mathrm{H}}$.

Now,

$$\begin{aligned}(\mathrm{X}/\mathrm{H})/\mathrm{G} &= \{\{g[x]_{\mathrm{H}} \mid g \in \mathrm{G}\} \mid x \in \mathrm{X}\} \\ &= \{\{g\{hx \mid h \in \mathrm{H}\} \mid g \in \mathrm{G}\} \mid x \in \mathrm{X}\} = \{\{\{ghx \mid h \in \mathrm{H}\} \mid g \in \mathrm{G}\} \mid x \in \mathrm{X}\} \\ &= \{ghx \mid h \in \mathrm{H}, g \in \mathrm{G}, x \in \mathrm{X}\} = \mathrm{X}/(\mathrm{G} \times \mathrm{H}),\end{aligned}$$

where $\mathrm{G}\times\mathrm{H}$ acts on X by $(g,h)x = ghx = hgx$. Hence, since $\mathrm{X}/(\mathrm{G}\times\mathrm{H}) = \mathrm{X}/(\mathrm{H}\times\mathrm{G})$ by commutation, one has $(\mathrm{X}/\mathrm{H})/\mathrm{G} = (\mathrm{X}/\mathrm{G})/\mathrm{H} = \mathrm{X}/(\mathrm{G}\times\mathrm{H})$. On the other hand, the projection from a diffeological space to its quotient is what is called a subduction. The composition of subductions is another subduction (they form a category [PIZ13, §1.47]). Therefore if we denote by $\mathrm{pr}_{\mathrm{H}} : \mathrm{X} \to \mathrm{X}/\mathrm{H}$, $\mathrm{pr}_{\mathrm{H,G}} : \mathrm{X}/\mathrm{H} \to (\mathrm{X}/\mathrm{H})/\mathrm{G}$, $\mathrm{pr}_{\mathrm{G}} : \mathrm{X} \to \mathrm{X}/\mathrm{G}$ and $\mathrm{pr}_{\mathrm{G,H}} : \mathrm{X}/\mathrm{G} \to (\mathrm{X}/\mathrm{G})/\mathrm{H}$, one has:

$$\mathrm{pr}_{\mathrm{G,H}} \circ \mathrm{pr}_{\mathrm{G}} = \mathrm{pr}_{\mathrm{H,G}} \circ \mathrm{pr}_{\mathrm{H}} = \mathrm{pr}_{\mathrm{G}\times\mathrm{H}}\,.$$

Therefore, the quotients $\mathrm{X}/(\mathrm{G}\times\mathrm{H})$, $(\mathrm{X}/\mathrm{H})/\mathrm{G}$ and $(\mathrm{X}/\mathrm{G})/\mathrm{H}$ are naturally diffeomorphic. □

Now, coming back to the original question, $(\mathrm{G}\times_{\mathrm{H}}\mathrm{E})/\mathrm{G} = ((\mathrm{G}\times\mathrm{E})/\mathrm{H})/\mathrm{G}$, but the action of G and $\mathrm{H} \subset \mathrm{G}$ commute, then $((\mathrm{G}\times\mathrm{E})/\mathrm{H})/\mathrm{G} = ((\mathrm{G}\times\mathrm{E})/\mathrm{G})/\mathrm{H}$. Since $(\mathrm{G}\times\mathrm{E})/\mathrm{G} \simeq \mathrm{E}$, then $(\mathrm{G}\times_{\mathrm{H}}\mathrm{E})/\mathrm{G} \simeq \mathrm{E}/\mathrm{H}$. □

And this is the diffeological interpretation of the slice theorem [Pal60]:

Corollary. *As a diffeological space, the quotient space* M/G *is locally diffeomorphic to some quotient* E/H *where* E *is an Euclidean space and* H *an orthogonal action. Precisely if* M *is locally diffeomorphic to* $\mathrm{G}\times_{\mathrm{H}}\mathrm{E}$*, then* M/G *is locally diffeomorphic to* E/H*.*

This is summarized by the following commutative diagram, where the maps Φ and ϕ are open embeddings.

$$\begin{array}{ccc} \mathrm{M} \supset \mathrm{V} & \xrightarrow{\ \Phi\ } & \mathrm{G}\times_{\mathrm{H}}\mathrm{E} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\mathrm{class}} \\ \mathrm{M}/\mathrm{G} \supset \mathrm{V}/\mathrm{G} & \xrightarrow[\ \phi\ ]{} & \mathrm{E}/\mathrm{H} \end{array}$$

where $\mathrm{class} : [g,\xi]_{\mathrm{G}} \mapsto [\xi]_{\mathrm{H}}$.

Remark (The Orthofold Category). The local model E/H, where E is a Euclidean space and $\mathrm{H} \subset \mathrm{O}(\mathrm{E})$ is a subgroup acting orthogonally, appears fundamental. This could lead to a definition of a new subcategory of spaces within the category of Diffeological Spaces, which we might term the *orthofold category*. An *orthofold* would be a diffeological space that is locally diffeomorphic, at every point, to such a quotient E/H. This category would naturally include smooth manifolds (where H is trivial) and, as a significant subcategory, orbifolds (where H is finite, see [IKZ10]). Crucially, all orbit spaces M/G of manifolds by compact Lie group actions, as discussed in this paper, would be natural and founding examples of orthofolds. The study of orthofolds as a distinct class of diffeological spaces could be a fruitful area for future investigation.

As an illustration of this definition, and an application of the dimension formula 2.2:

Proposition (Orbifolds within Orthofolds). *A diffeological space* X *is an orthofold with a constant dimension map if and only if it is an orbifold. So, an orbifold is an orthofold of constant dimension.*

Proof. Let X be an orthofold. By definition, for any $x \in \mathrm{X}$, there is a local model diffeomorphic to E/H where E is a Euclidean space and $\mathrm{H} \subset \mathrm{O(E)}$ is a compact subgroup. The diffeological dimension of this local model at the point corresponding to x (the image of 0_{E}) is $\dim(\mathrm{E})$.

First of all: If X is an n-orbifold, it is, by definition (e.g., as a diffeological space [IKZ10]), locally diffeomorphic to E/H where $\dim(\mathrm{E}) = n$ and H is a *finite* subgroup of $\mathrm{O(E)}$. The diffeological dimension of such a local model E/H (with H finite) is $\dim(\mathrm{E}) = n$ at every point within that model. Since this holds for all local models, an n-orbifold has a constant dimension map equal to n.

Next: Assume X is an orthofold with a constant dimension map, say $\dim_y(\mathrm{X}) = d$ for all $y \in \mathrm{X}$. Consider any local model E/H for X. We must have $\dim(\mathrm{E}) = d$. For any point $[v]_{\mathrm{H}} \in \mathrm{E/H}$ (where $v \in \mathrm{E}$), the dimension of E/H at this point is given by the formula $\dim_{[v]_{\mathrm{H}}}(\mathrm{E/H}) = \dim(\mathrm{E}) - \dim(\mathcal{O}_v)$, where $\mathcal{O}_v = \mathrm{H} \cdot v$ is the orbit of v under H within E (this formula is analogous to Proposition 2.2 applied to the action of H on E). Since the dimension map is constant and equal to $d = \dim(\mathrm{E})$, we must have

$$\dim(\mathrm{E}) - \dim(\mathcal{O}_v) = \dim(\mathrm{E})$$

for all $v \in \mathrm{E}$. This implies $\dim(\mathcal{O}_v) = 0$ for all $v \in \mathrm{E}$. For a compact Lie group H acting orthogonally on E, if all its orbits $\mathrm{H} \cdot v$ are zero-dimensional, then H must be a discrete group [PIZ13, §1.81]. Since H is also compact, it must be a finite group. Therefore, every local model for X is of the form E/H with H finite. This means X is an orbifold. □

Note. It is worth noting that other classes of diffeological spaces also share the property of having a constant diffeological dimension, such as *quasifolds* [IZP21]. By definition, a quasifold is locally diffeomorphic to a quotient $\mathbf{R}^n/\Gamma$, where Γ is a countable subgroup of the affine group $\mathrm{Aff}(\mathbf{R}^n)$ such that the orbits are discrete. At a point x in such a local model, the isotropy group $\Gamma_x = \{\gamma \in \Gamma \mid \gamma(x) = x\}$ is a countable subgroup of $\mathrm{Aff}(\mathbf{R}^n)$ fixing x. The action of Γ_x on the tangent space at x (which serves as the slice model) is linear, given by the linear part of Γ_x, a countable subgroup of $\mathrm{GL}(n, \mathbf{R})$. For a quasifold to be an orthofold, its local models must be diffeomorphic to E/H where E is a Euclidean space and H is a compact subgroup of $\mathrm{O(E)}$. However, a countable subgroup of $\mathrm{GL}(n, \mathbf{R})$ can only be compact if it is finite. Therefore, quasifolds with infinite isotropy groups (i.e., strict quasifolds) cannot be locally diffeomorphic to quotients by compact groups, and thus strict quasifolds are not orthofolds.

2. The Dimension Map on the Quotient Space.

Let us recall some facts and constructions:

For the quotient M/G of a manifold by a compact Lie group action, the orbits of G are classified by their *orbit type*: two orbits have the same type if their respective stabilizers are conjugate in G. The *Principal Orbit Theorem* states that there exists an open dense G-invariant subset $\mathrm{M}_{\mathrm{pr}} \subset \mathrm{M}$ consisting of orbits of the same type (principal type), such that their stabilizer is conjugate to a subgroup of any other stabilizer. These are called *principal orbits*; orbits not of principal type are called *singular*.

The dimension map for diffeological spaces has been defined originally in [PIZ07], and then in [PIZ13, §2.22], as a minimax: the minimum dimension of a generating family, itself defined as the maximum dimension of its plots. The dimension map is a diffeological invariant. It applies to every kind of diffeological space and coincides with the standard dimension in the case of manifolds.

As a diffeological space, equipped with the quotient diffeology, the space of orbits M/G is accompanied by its dimension map $\mathcal{O} \mapsto \dim_{\mathcal{O}}(\mathrm{M}/\mathrm{G})$.[1] This map provides diffeologically invariant labels attached to points in $\mathcal{O} \in \mathrm{M}/\mathrm{G}$. These can be useful, for instance, when considering only the underlying D-topology of the quotient, as the dimension map can distinguish points that might otherwise be topologically similar.

One of the main properties of the dimension map, stemming from its nature as a diffeological invariant, is that:

Proposition 1. *The dimension map $\mathcal{O} \mapsto \dim_{\mathcal{O}}(\mathrm{M}/\mathrm{G})$ is constant on the Klein strata of M/G (i.e., on the orbits of the group of local diffeomorphisms of M/G). In some cases, this invariant is sufficient to distinguish between different Klein strata.*

The dimension in M/G can vary across the space. It captures geometric characteristics of the singularities, reflecting how orbits deviate from being principal. The simplest example is $\Delta_n = \mathbf{R}^n/\mathrm{O}(n)$. The space of orbits can be identified with the interval $[0, \infty)$, equipped with the pushforward of the standard diffeology of $\mathbf{R}^n$ by the map $\mathrm{sq} : x \mapsto \|x\|^2$. In this case, $\dim_t(\Delta_n)$ is equal to n for $t = 0$ (the singular point) and equal to 1 for $t > 0$ (principal points).

Proposition 2. *Let $\mathrm{orbit} : \mathrm{M} \to \mathrm{M}/\mathrm{G}$ be the projection from M onto the space of orbits M/G, equipped with the quotient diffeology. Pick a point $x \in \mathrm{M}$, let $\mathcal{O} = \mathrm{orbit}(x)$ be the orbit $\mathrm{G} \cdot x$ through x. The* diffeological dimension *of M/G at the point $\mathcal{O}$ obeys the formula:*

$$\dim_{\mathcal{O}}(\mathrm{M}/\mathrm{G}) = \dim(\mathrm{M}) - \dim(\mathcal{O}). \tag{♣}$$

From this central formula, we directly deduce the following properties concerning the diffeological dimension on principal and singular orbits in the quotient space. Note that if the quotient space is a manifold it coincides with the ordinary dimension. Actually, since the category of manifolds is a full subcategory of the category of diffeological spaces,

[1] An orbit $\mathcal{O} = \mathrm{G} \cdot x$ is indifferently a point $\mathcal{O} \in \mathrm{M}/\mathrm{G}$ or a subspace $\mathcal{O} \subset \mathrm{M}$.

all dimensions in this formula are diffeological dimensions [PIZ07]: the dimension of $\mathcal{O} \subset \mathrm{M}$ equipped with the subset diffeology is constant on $\mathcal{O}$ (which is a manifold) and equal to the dimension of the quotient $\mathcal{O} \simeq \mathrm{G}/\operatorname{Stab}_{\mathrm{G}}(x)$.

(a) The subspace $(\mathrm{M}/\mathrm{G})_{\mathrm{pr}} = \operatorname{orbit}(\mathrm{M}_{\mathrm{pr}})$ of principal orbits in M/G is an open dense subset (with respect to the D-Topology). Since all principal orbits $\mathcal{O}_{\mathrm{pr}}$ have the same dimension, the formula (♣) shows that the dimension map is constant on the open dense subset $(\mathrm{M}/\mathrm{G})_{\mathrm{pr}}$, with value $d_{\mathrm{pr}} = \dim(\mathrm{M}) - \dim(\mathcal{O}_{\mathrm{pr}})$.

(b) At singular orbits $\mathcal{O}_s$, $\dim_{\mathcal{O}_s}(\mathrm{Q}) \geq d_{\mathrm{pr}}$ since $\dim(\mathcal{O}_s) \leq \dim(\mathcal{O}_{\mathrm{pr}})$ (as $\operatorname{Stab}_{\mathrm{G}}(s)$ contains a conjugate of $\operatorname{Stab}_{\mathrm{G}}(\mathrm{pr})$). Thus, d_{pr} is the minimal dimension in M/G. Equality, $\dim_{\mathcal{O}_s}(\mathrm{M}/\mathrm{G}) = d_{\mathrm{pr}}$, can occur for certain types of singular orbits, often referred to as *exceptional orbits*.[2]

Proof. Let $\operatorname{orbit} : \mathrm{M} \to \mathrm{M}/\mathrm{G}$ denote the canonical projection, which, by definition of the quotient diffeology, is a subduction.

(i) *The set of principal orbits* $(\mathrm{M}/\mathrm{G})_{\mathrm{pr}}$ *is open and dense in* M/G. It is a standard result from the theory of compact Lie group actions that the set of principal points in M, denoted M_{pr}, forms an open and dense G-invariant subset of M (see, e.g., [Bre72, Ch. IV, Thm. 3.1]). Since orbit is an open map (as G is compact, or by properties of local subductions in diffeology [PIZ13, §2.18]) and M_{pr} is open and G-invariant, its image $(\mathrm{M}/\mathrm{G})_{\mathrm{pr}} = \operatorname{orbit}(\mathrm{M}_{\mathrm{pr}})$ is an open subset of M/G. Furthermore, since M_{pr} is dense in M and orbit is surjective and continuous (for the D-topologies), $(\mathrm{M}/\mathrm{G})_{\mathrm{pr}}$ is dense in M/G. The D-topology on the quotient M/G coincides with the standard quotient topology inherited from M [PIZ13, §1.20, §5.6].

(ii) *The dimension of* M/G *at the point* $\mathcal{O}$. The formula $\dim_{\mathcal{O}}(\mathrm{M}/\mathrm{G}) = \dim(\mathrm{M}) - \dim(\mathcal{O})$ relies on the following lemma.

Lemma. Let E be a finite-dimensional real vector space, and let $\mathrm{H} \subset \mathrm{O}(\mathrm{E})$ be a subgroup of its orthogonal group. Then $\dim_{[0_{\mathrm{E}}]_{\mathrm{H}}}(\mathrm{E}/\mathrm{H}) = \dim(\mathrm{E})$.

Proof of Lemma. Let $\Delta = \mathrm{E}/\mathrm{O}(\mathrm{E})$ and $\pi_{\mathrm{O}(\mathrm{E})} : \mathrm{E} \to \Delta$ be the projection. Let $\mathrm{X} = \mathrm{E}/\mathrm{H}$ and $\pi_{\mathrm{H}} : \mathrm{E} \to \mathrm{X}$ be the projection. Define $\mathrm{pr} : \mathrm{X} \to \Delta$ by $\mathrm{pr}([x]_{\mathrm{H}}) = [x]_{\mathrm{O}(\mathrm{E})}$. All three projections are subductions: $\pi_{\mathrm{O}(\mathrm{E})}$ and π_{H} by definition of quotient diffeology, and pr because it is induced by the identity on E and respects the equivalence relations [PIZ13, §1.51]. These maps send the origin of E to the respective origins of the quotient spaces: $\pi_{\mathrm{H}}(0_{\mathrm{E}}) = 0_{\mathrm{X}}$, $\pi_{\mathrm{O}(\mathrm{E})}(0_{\mathrm{E}}) = 0_{\Delta}$, and $\mathrm{pr}(0_{\mathrm{X}}) = 0_{\Delta}$.

[2]This happens when $\dim(\mathcal{O}_s) = \dim(\mathcal{O}_{\mathrm{pr}})$ even though $\mathcal{O}_s$ is not of principal type (e.g., when the stabilizer H_s of s has the same dimension as a principal stabilizer H_{pr} but is not conjugate to it, and $\mathrm{N}_{\mathrm{G}}(\mathrm{H}_s)/\mathrm{H}_s$ is finite).

$$\begin{tikzcd} \mathrm{E} \arrow[dr, "\pi_{\mathrm{H}}"] \arrow[ddr, bend right, "\pi_{\mathrm{O(E)}}"'] & \\ & \mathrm{X} \arrow[d, "\mathrm{pr}"] \\ & \Delta \end{tikzcd} \quad \text{and for origins:} \quad \begin{tikzcd} 0_{\mathrm{E}} \arrow[dr, mapsto, "\pi_{\mathrm{H}}"] \arrow[ddr, mapsto, bend right, "\pi_{\mathrm{O(E)}}"'] & \\ & 0_{\mathrm{X}} \arrow[d, mapsto, "\mathrm{pr}"] \\ & 0_{\Delta} \end{tikzcd}$$

Since π_{H} and pr are subductions preserving the origins (in the sense that the origin of the domain maps to the point we are considering as the "origin" in the codomain), by [PIZ13, §2.24, 2.26] (Dimension and Subductions/Pointed Subductions), we have:

$$\dim_{0_{\mathrm{X}}}(\mathrm{X}) \leq \dim_{0_{\mathrm{E}}}(\mathrm{E}), \text{ and } \dim_{0_{\Delta}}(\Delta) \leq \dim_{0_{\mathrm{X}}}(\mathrm{X}).$$

This yields the chain:

$$\dim_{0_{\Delta}}(\Delta) \leq \dim_{0_{\mathrm{X}}}(\mathrm{X}) \leq \dim_{0_{\mathrm{E}}}(\mathrm{E}).$$

From [PIZ13, Exercise 50] (with $\mathrm{E} = \mathbf{R}^n$), we know that $\dim_{0_{\Delta}}(\Delta) = \dim(\mathrm{E})$. Thus,

$$\dim(\mathrm{E}) \leq \dim_{0_{\mathrm{X}}}(\mathrm{X}) \leq \dim(\mathrm{E}).$$

This forces $\dim_{0_{\mathrm{X}}}(\mathrm{X}) = \dim(\mathrm{E})$, that is, $\dim_{[0_{\mathrm{E}}]_{\mathrm{H}}}(\mathrm{E}/\mathrm{H}) = \dim(\mathrm{E})$. ☐

Now, let $\mathcal{O} = \mathrm{orbit}(x) \in \mathrm{M}/\mathrm{G}$ be the orbit $\mathrm{G} \cdot x$, with stabilizer $\mathrm{H} = \mathrm{Stab}_{\mathrm{G}}(x)$. By the Corollary to Proposition 1, M/G is locally diffeomorphic around the point $\mathcal{O} \in \mathrm{M}/\mathrm{G}$ to E/H, where E is the slice at x. The point $\mathcal{O} \in \mathrm{M}/\mathrm{G}$ corresponds to $[0_{\mathrm{E}}]_{\mathrm{H}}$ in this local model, thus, by the Lemma, $\dim_{\mathcal{O}}(\mathrm{M}/\mathrm{G}) = \dim_{[0_{\mathrm{E}}]}(\mathrm{E}/\mathrm{H}) = \dim(\mathrm{E})$. The local model $\mathrm{V} = \mathrm{G} \times_{\mathrm{H}} \mathrm{E}$, where V is an open neighborhood of x, gives $\dim(\mathrm{M}) = \dim(\mathrm{G}) + \dim(\mathrm{E}) - \dim(\mathrm{H})$, that is, $\dim(\mathrm{M}) = \dim(\mathcal{O}) + \dim(\mathrm{E}) = \dim(\mathcal{O}) + \dim_{\mathcal{O}}(\mathrm{M}/\mathrm{G})$, and then: $\dim_{\mathcal{O}}(\mathrm{M}/\mathrm{G}) = \dim(\mathrm{M}) - \dim(\mathcal{O})$. ☐

3. Toric Actions and Depth.

The diffeological dimension map provides a natural way to quantify local complexity in quotient spaces arising from group actions, sometimes aligning with or refining existing topological or geometric invariants. Consider an effective action of the n-torus T^n on a $2n$-dimensional symplectic manifold $(\mathrm{M}^{2n}, \omega)$. Such actions are central to symplectic geometry, and their orbit spaces $\mathrm{Q}_n = \mathrm{M}^{2n}/\mathrm{T}^n$ have a well-studied combinatorial and topological structure, often described by Delzant polytopes or, more generally, as "manifolds with corners" locally.[3]

In many local models for these quotient spaces (e.g., near a fixed point or a point with a non-trivial stabilizer), the structure is identified with, or is locally homeomorphic to, a quotient of the form $(\mathbf{C}^k \times \mathbf{R}^{n-k})/\mathrm{T}^k_{\text{eff}}$ or, more simply for a standard model, $\mathbf{C}^n/\mathrm{T}^n$. The quotient $\mathbf{C}^n/\mathrm{T}^n$ (where T^n acts by $(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot (z_1, \ldots, z_n) =$

[3] The classification and local structure of such spaces are subjects of ongoing research, see for instance the work by Yael Karshon and Shintarô Kuroki [KS] on locally standard torus actions.

$(e^{i\theta_1}z_1,\ldots,e^{i\theta_n}z_n))$ is diffeomorphic to the positive orthant $[0,\infty[^n$ via the map with components $z \mapsto (|z_1|^2,\ldots,|z_n|^2)$.

In the topological analysis of such spaces, a label called the *depth* is often associated with points in the quotient. For a point $t = (t_1,\ldots,t_n) \in [0,\infty[^n$ representing an orbit, its depth is typically defined as the number of coordinates t_i that are equal to zero. This corresponds to the dimension of the subtorus in T^n that stabilizes points in $\mathbf{C}^n$ projecting to t.

Let us analyze this from a diffeological perspective. Consider the local model $\mathrm{Q}_n^{\mathrm{loc}} = \mathbf{C}^n/\mathrm{T}^n$. This space is endowed with the quotient diffeology, which is the pushforward of the standard diffeology of $\mathbf{C}^n$ by the map $\mathrm{sq}^n : (z_1,\ldots,z_n) \mapsto (|z_1|^2,\ldots,|z_n|^2)$. Let $t = \mathrm{sq}^n(z)$ be a point in $\mathrm{Q}_n^{\mathrm{loc}}$. According to the general formula derived in Proposition 2 above, the diffeological dimension of $\mathrm{Q}_n^{\mathrm{loc}}$ at t is given by:

$$\dim_t(\mathrm{Q}_n^{\mathrm{loc}}) = \dim(\mathbf{C}^n) - \dim(\mathcal{O}_z),$$

where $\mathcal{O}_z = \mathrm{T}^n \cdot z$ is the orbit of $z \in \mathbf{C}^n$ under the T^n action. The dimension of the orbit $\mathcal{O}_z$ is $\dim(\mathcal{O}_z) = \dim(\mathrm{T}^n) - \dim(\mathrm{Stab}_{\mathrm{T}^n}(z))$. The stabilizer $\mathrm{Stab}_{\mathrm{T}^n}(z)$ consists of $(e^{i\theta_1},\ldots,e^{i\theta_n}) \in \mathrm{T}^n$ such that $e^{i\theta_j}z_j = z_j$ for all $j = 1,\ldots,n$. This condition implies that $e^{i\theta_j}$ can be arbitrary if $z_j = 0$, and $e^{i\theta_j} = 1$, i.e., $\theta_j \equiv 0 \pmod{2\pi}$) if $z_j \neq 0$. If exactly k coordinates of z are zero (say $z_1 = \cdots = z_k = 0$ and $z_{k+1},\ldots,z_n \neq 0$), then $\mathrm{Stab}_{\mathrm{T}^n}(z)$ is isomorphic to T^k (the product of the first k circle factors of T^n). Thus, $\dim(\mathrm{Stab}_{\mathrm{T}^n}(z)) = k$. This integer k is precisely the depth of the point $t = \mathrm{sq}^n(z)$ in the quotient $[0,\infty[^n$, since $t_j = |z_j|^2$, so $t_j = 0 \iff z_j = 0$.

The dimension of the orbit is therefore $\dim(\mathcal{O}_z) = \dim(\mathrm{T}^n) - k = n-k$. Substituting this into the formula for the dimension of the quotient:

$$\dim_t(\mathrm{Q}_n^{\mathrm{loc}}) = \dim(\mathbf{C}^n) - (n-k) = 2n - (n-k) = n+k.$$

Thus, the depth, an important combinatorial invariant in the study of toric symplectic manifolds, is directly related to the diffeological dimension of the (local model of the) orbit space by the formula:

$$\dim_t(\mathrm{Q}_n^{\mathrm{loc}}) = n + \mathrm{depth}(t).$$

This shows how the diffeological dimension naturally incorporates and quantifies the "degree of singularity" captured by the depth in this context. For a point t in the interior of $[0,\infty[^n$ (depth $k=0$), $\dim_t(\mathrm{Q}_n^{\mathrm{loc}}) = n$. For a point at the origin of $[0,\infty[^n$ (depth $k=n$), $\dim_t(\mathrm{Q}_n^{\mathrm{loc}}) = 2n = \dim(\mathbf{C}^n)$.

A final remark: It is important to distinguish this quotient diffeology on $[0,\infty[^n$ (obtained from $\mathbf{C}^n/\mathrm{T}^n$) from its standard diffeology as a manifold with corners when viewed as a subset of $\mathbf{R}^n$. For the latter, as discussed in [PIZ13, §4.16, Exercise 51], the diffeological dimension is ∞ at any point on the boundary (e.g., $\dim_0[0,\infty[= \infty$), and n for interior points. In contrast, the quotient diffeology, which is the focus of this paper, yields $\dim_t(\mathrm{Q}_n^{\mathrm{loc}}) = n + \mathrm{depth}(t)$ for all t, reflecting the structure inherited from $\mathbf{C}^n$ and the T^n-action.

II. Stratifications

The space of orbits of a manifold by the action of a compact Lie group is naturally stratified by the types of orbits, called *orbit-type stratification*, as described below. We recall some aspects of the theory of stratifications in diffeology set out in [GIZ23].

4. Stratifications in Diffeology.

The classical theory of stratification is a mix of topology and differential geometry: given a topological space X, a stratification of X is a partition $\mathcal{S}$ of which basically satisfies the so-called *frontier condition*, that is:

[B] The closure of a stratum is a union of strata,

together with a manifold structure on the strata that is compatible with the topology. This is because interesting stratified spaces have in general not a global structure of manifold. They are generally subspaces of Cartesian spaces, often algebraic subspaces, even sometimes artificial decomposition of manifolds.

In diffeology, the approach is different, since a diffeology exists on almost any kind of space encountered in mathematics, one starts with a diffeological space X. Its smooth sructure is a given. Then, we define a stratification as a partition of X satisfying primarily the frontier condition [B], nothing more, regarding its D-Topology [PIZ13, §2.8]. Then, each stratum is endowed with the subset diffeology, and whether they are or not manifold doesn't change much the subject. There is no need to add a smooth structure by hand. We will see below that, according to the precise type of sub-structure we get, we classify the stratifications by different labels.

Let us denote by the letter $\wp$ the powerset functor that associates to a set the set of its subsets, and for a diffeological space, the set of its subspaces. Then, the frontier condition has a formal conceptual formulation:

[B] There is a map $\mathrm{clos} : \mathcal{S} \to \wp(\mathcal{S})$ such that for each stratum $\mathrm{S}_i \in \mathcal{S}$ there is a subset $\mathrm{clos}(\mathrm{S}_i) = \{\mathrm{S}_i^j\}_j$ such that $\mathrm{Clos}(\mathrm{S}_i) = \cup_j \mathrm{S}_i^j$,

where Clos denotes the topological closure

$$\mathrm{Clos} : \wp(\mathrm{X}) \to \wp(\mathrm{X}) \quad \text{such that} \quad \mathrm{Clos}(\mathrm{A}) = \overline{\mathrm{A}},$$

which is a projector. Let $\bigcup$ be the union operator:

$$\bigcup : \wp(\mathcal{S}) \to \wp(\mathrm{X}) \quad \text{such that} \quad \bigcup\left(\{\mathrm{S}_i\}_i\right) = \cup_i \mathrm{S}_i.$$

Then, we can rewrite the definition of a stratification of a diffeological space as follows:

[B] A stratification $\mathcal{S}$ is a partition of X such that the map $\mathrm{Clos} : \mathcal{S} \to \wp(\mathrm{X})$ lifts along $\bigcup : \wp(\mathcal{S}) \to \wp(\mathrm{X})$ by a map $\mathrm{clos} : \mathcal{S} \to \wp(\mathcal{S})$:

$$\begin{tikzcd} & \wp(\mathcal{S}) \arrow[d, "\cup"] \\ \mathcal{S} \arrow[ur, "\mathrm{clos}"] \arrow[r, "\mathrm{Clos}"'] & \wp(\mathrm{X}) \end{tikzcd}$$

5. Stratification Labels.

The differential theory of stratifications introduces certain constraints or particular behaviours, for example, that the strata be manifolds. This is a necessary condition in the specific framework of classical differential geometry, since there is no alternative to what is a smooth structure. This is no longer the case in diffeology, since any part of a diffeological space naturally inherits the subset diffeology. For this reason, in diffeology, we can separate the different properties of a stratification, independently of the frontier condition, with a series of characteristic labels, see [GIZ23] for a separate discussion on the subject.

- [B] For basic: the closure of a stratum is a union of strata.
- [LF] The stratification is locally finite (or not locally finite [~~LF~~]).
- [G] The strata are the orbits of a group, or a pseudo group of diffeomorphisms.
- [GK] The strata are the orbits of the group of diffeomorphisms, or the pseudo group of local diffeomorphisms.
- [M] All the strata are manifolds (or not [~~M~~]).
- [T_0] The strata are locally closed (in the D-topology): the space of strata is T_0 separated (or not [$\cancel{\mathrm{T}_0}$]).

The list is not exhaustive, for example we could add the label [C] for *locally conical stratification*, and so on.

Now, coming back to the *standard stratification* of diffeological spaces (which is not necessarily the most interesting of all), we could describe their family by the encoding [B]-[LF]-[M]-[T_0]. Another example, the stratification defined by a smooth action of a compact Lie group on a manifold will belong to the family [B]-[LF]-[G]-[M]-[T_0]. The manifolds with corners fall in the family [B]-[LF]-[GK]-[M]-[T_0] etc.

III. Stratifications Types

In this section we lay the basis for our discussion of the different types of stratification involved in quotients of manifolds by compact Lie groups.

6. Geometric Stratifications.

An important class of stratification is defined by the actions of groups or local groups on spaces,[4] Precisely:

Proposition. *Let* G *be a diffeological group acting smoothly on a diffeological space* X. *The partition of* X *into orbits is a basic stratification. We call such stratifications* geometric.

[4]We call local group acting on a space X any sub-pseudogroup of local diffeomorphisms of X.

Proof. Let $\mathcal{O}$ and $\mathcal{O}_y$ be two orbits of G. Assume that $x \in \overline{\mathcal{O}_y}$ and let $x' \in \mathcal{O}$. Let U' be an D-open neighbourhood of x' and $g \in \mathrm{G}$ such that $g(x) = x'$. Let $\mathrm{U} = g^{-1}(\mathrm{U}')$. Since $g \in \mathrm{Diff}(\mathrm{X})$, U is an open neighbourhood of x, and since $x \in \overline{\mathcal{O}_y}$ there exists $z \in \mathcal{O}_y \cap \mathrm{U}$, and $\mathcal{O}_y = \mathcal{O}_z$. Thus, $\mathcal{O}_y \cap \mathrm{U}' \neq \varnothing$, and $x' \in \overline{\mathcal{O}_y}$. Therefore, the closure of any orbit is a union of orbits. The partition of a diffeological space by a smooth action of a diffeological group is a basic stratification. ☐

7. Orbit Type Stratifications.

Our analysis involves comparing the classical orbit type stratification of M with the Klein stratification of M/G. We therefore begin by recalling the main theorem concerning the orbit type stratification of M.

Let G be a compact Lie group acting smoothly on a smooth manifold M. For each point $x \in \mathrm{M}$, its stabilizer (or isotropy group) is a closed subgroup of G. A key property is that for any $g \in \mathrm{G}$, $\mathrm{Stab}_{\mathrm{G}}(gx) = g\,\mathrm{Stab}_{\mathrm{G}}(x)g^{-1}$. This implies that all points on a single orbit $\mathrm{G} \cdot x$ have stabilizers conjugate to each other. Consequently, to each orbit $\mathcal{O} = \mathrm{G} \cdot x$, we can uniquely assign the conjugacy class $[\mathrm{Stab}_{\mathrm{G}}(x)]$ of its stabilizer subgroups. This conjugacy class is called the *orbit type* of $\mathcal{O}$ (and of any point $x \in \mathcal{O}$).

The set of points in M whose orbits have the same type (H) (i.e., whose stabilizers are conjugate to a closed subgroup $\mathrm{H} \subset \mathrm{G}$) forms a *stratum by orbit type*, denoted $\mathrm{M}_{(\mathrm{H})}$:

$$\mathrm{M}_{(\mathrm{H})} = \{x \in \mathrm{M} \mid [\mathrm{Stab}_{\mathrm{G}}(x)] = (\mathrm{H})\}.$$

Each $\mathrm{M}_{(\mathrm{H})}$ is a G-invariant submanifold of M (possibly empty). For the sake of self-consistency, we recall here the main statement (see for example [Bre72]) and its proof about the decomposition of the manifold M into orbit-type strata.

Proposition 1. *Let* G *be a compact Lie group acting smoothly on* M*. The collection of non-empty strata* $\{\mathrm{M}_{(\mathrm{H})}\}$*, where* (H) *ranges over all orbit types occurring in* M*, forms a partition of* M*. This partition is a basic stratification. We call this the* stratification by orbit types *of* M*, denoted by* $\mathrm{Str}_{\mathrm{OT}}(\mathrm{M}, \mathrm{G})$.

Proof. The fact that $\{\mathrm{M}_{(\mathrm{H})}\}$ forms a partition of M is clear. We need to show the frontier condition [B]. For compact Lie group actions, the set of orbit types (conjugacy classes of closed subgroups) is partially ordered. We define $(\mathrm{H}) \preceq (\mathrm{K})$ if H is conjugate to a subgroup of K. This means orbits of type (K) are "more singular" (have "larger" stabilizers in this sense) than or are of the same type as orbits of type (H). The fronftier condition states that for any orbit type (H):

$$\overline{\mathrm{M}_{(\mathrm{H})}} = \bigcup_{(\mathrm{K}) \text{ s.t. } (\mathrm{H}) \preceq (\mathrm{K})} \mathrm{M}_{(\mathrm{K})}.$$

This is a standard result in the theory of transformation groups (see, e.g., [Bre72, Chapter IV, Theorem 3.3] or similar results in [Pal60]). The proof relies on the Slice Theorem, which shows that if $y \in \overline{\mathrm{M}_{(\mathrm{H})}}$ and $\mathrm{Stab}_{\mathrm{G}}(y)$ is of type (K), then H must be conjugate to a subgroup of K, i.e., $(\mathrm{H}) \preceq (\mathrm{K})$. Thus, the closure of a stratum $\mathrm{M}_{(\mathrm{H})}$ is the union of $\mathrm{M}_{(\mathrm{H})}$ itself and all strata $\mathrm{M}_{(\mathrm{K})}$ corresponding to orbit types (K) that are strictly more singular than (H) (i.e., $(\mathrm{H}) \prec (\mathrm{K})$). This satisfies condition [B]. Furthermore, each stratum $\mathrm{M}_{(\mathrm{H})}$ is a locally closed submanifold of M [Bre72, Chapter IV, Thm. 3.3], inheriting a smooth structure. □

In terms of labels, introduced in Section II Article 5, the orbit-type stratification is standard and responds to the code [B][LF][M][T_0].

Let us now explicit the dependence of the stabilizer representations on the slice along the submanifold $\mathrm{M}_{(\mathrm{H})}$. For all point $x \in \mathrm{M}$, let us denote by N_x the tangent subspace $\mathrm{N}_x \subset \mathrm{T}_x(\mathrm{M})$, normal to the tangent space $\mathrm{T}_x(\mathrm{G}\cdot x) \subset \mathrm{T}_x(\mathrm{M})$, for a given G-invariant Riemannian metric on M. The subspace N_x representing the linear slice at x. While this statement is commonly found in literature, we include a proof for self-consistency.

Proposition 2. *Let* G *be a compact Lie group acting smoothly on a smooth manifold* M. *Let* H *be a closed subgroup of* G. *For any connected component* C *of the orbit type stratum* $\mathrm{M}_{(\mathrm{H})}$, *there exists a subgroup* K *conjugate to* H *such that* $\mathrm{Stab}_{\mathrm{G}}(x) = \mathrm{K}$ *for all* $x \in \mathrm{C}$. *Furthermore, for any two points* $x, x' \in \mathrm{C}$, *the slice representations* $\rho_x : \mathrm{K} \to \mathrm{O}(\mathrm{N}_x)$ *and* $\rho_{x'} : \mathrm{K} \to \mathrm{O}(\mathrm{N}_{x'})$ *are equivalent representations of* K.

Proof. All the elements of this proof are taken from Bredon [Bre72] and Palais [Pal60]. Let C be a connected component of the orbit type stratum $\mathrm{M}_{(\mathrm{H})}$. The set $\mathrm{M}_{(\mathrm{H})}$ is defined as $\{z \in \mathrm{M} \mid \mathrm{Stab}_{\mathrm{G}}(z) \text{ is conjugate to H}\}$. It is established that $\mathrm{M}_{(\mathrm{H})}$ is the disjoint union of the sets $\mathrm{M}_{\mathrm{K}} = \{z \in \mathrm{M} \mid \mathrm{Stab}_{\mathrm{G}}(z) = \mathrm{K}\}$ for all closed subgroups K conjugate to H. Furthermore, the set M_{K} is a smooth manifold (possibly empty). This follows from the Slice Theorem, which implies that M_{K} is locally diffeomorphic to the fixed-point set of the action of K on a slice, and the fact that the fixed-point set of a Lie group action on a manifold is a smooth submanifold. Since C is a connected subset of $\mathrm{M}_{(\mathrm{H})}$ and $\mathrm{M}_{(\mathrm{H})}$ is the disjoint union of the manifolds M_{K}, C must be entirely contained within a single M_{K} for some subgroup K conjugate to H. Thus, for all $z \in \mathrm{C}$, the stabilizer $\mathrm{Stab}_{\mathrm{G}}(z)$ is equal to K. In particular, for any two points $x, x' \in \mathrm{C}$, $\mathrm{Stab}_{\mathrm{G}}(x) = \mathrm{Stab}_{\mathrm{G}}(x') = \mathrm{K}$. This proves the first part of the proposition: the stabilizer is constant on any connected component of $\mathrm{M}_{(\mathrm{H})}$.

Now we show that the slice representations at any two points $x, x' \in \mathrm{C}$ are equivalent representations of K. The manifold M is equipped with a G-invariant Riemannian metric (which exists because G is compact). For any $z \in \mathrm{C} \subseteq \mathrm{M}_{\mathrm{K}}$, the normal space to the orbit $\mathrm{G}\cdot z$ at z is $\mathrm{N}_z = \mathrm{T}_z(\mathrm{G}\cdot z)^{\perp}$. The dimension of N_z is constant for all $z \in \mathrm{C}$. The collection of these normal spaces forms a smooth vector bundle $\mathrm{W}|_{\mathrm{C}} \to \mathrm{C}$ over the smooth manifold C, where the fiber over z is N_z. The slice representation at z is the action of the stabilizer K on the fiber N_z, $\rho_z : \mathrm{K} \to \mathrm{O}(\mathrm{N}_z)$.

Since C is a connected component of the smooth manifold M_{K}, for any two points $x, x' \in \mathrm{C}$, there exists a smooth path γ in C such that $\gamma(0) = x$ and $\gamma(1) = x'$.

Consider the pullback bundle $\gamma^*(\mathrm{W}|_{\mathrm{C}}) \to \mathbf{R}$.[5] This is a smooth vector bundle over the contractible base space, and is therefore trivial.

Let $\Phi : \mathbf{R} \times \mathrm{N}_x \to \gamma^*(\mathrm{W}|_{\mathrm{C}})$ be a smooth trivialization, where N_x is the fiber over $x = \gamma(0)$. For each $t \in [0,1]$, $\Phi_t = \Phi(t,\cdot) : \mathrm{N}_x \to \mathrm{N}_{\gamma(t)}$ is a smooth vector space isomorphism. The slice representation of K on $\mathrm{N}_{\gamma(t)}$ is $\rho_{\gamma(t)} : \mathrm{K} \to \mathrm{O}(\mathrm{N}_{\gamma(t)})$. We can use the isomorphism Φ_t to transport this representation to the fixed vector space N_x. Define a family of representations $\tilde\rho_t : \mathrm{K} \to \mathrm{GL}(\mathrm{N}_x)$ by:

$$\tilde\rho_t(k) = \Phi_t^{-1} \circ \rho_{\gamma(t)}(k) \circ \Phi_t \quad \text{for } k \in \mathrm{K}.$$

This family $\tilde\rho_t$ is a smooth path in the space of linear representations of K on N_x. Since $\rho_{\gamma(t)}$ is an orthogonal representation and Φ_t is an isomorphism, $\tilde\rho_t$ is equivalent to $\rho_{\gamma(t)}$ for each t.

The space of equivalence classes of smooth (or continuous) finite-dimensional representations of a compact group K is discrete (actually finite in our case). Since $\tilde\rho_t$ is a continuous path in the space of representations, and the equivalence classes form a discrete set, the representation $\tilde\rho_t$ must belong to the same equivalence class for all t.

In particular, $\tilde\rho_0$ is equivalent to $\tilde\rho_1$. $\tilde\rho_0 = \Phi_0^{-1} \circ \rho_{\gamma(0)}(k) \circ \Phi_0$. This representation is equivalent to ρ_x. $\tilde\rho_1 = \Phi_1^{-1} \circ \rho_{\gamma(1)}(k) \circ \Phi_1$. This representation is equivalent to $\rho_{x'}$.

Since $\tilde\rho_0$ is equivalent to $\tilde\rho_1$, it follows that ρ_x is equivalent to $\rho_{x'}$ as representations of the group $\mathrm{K} \sim \mathrm{H}$. □

8. Klein Stratifications.

In honor of Felix Klein's Erlangen Program, which characterized geometries by their transformation groups, the term *Klein strata* was introduced for diffeological spaces X [PIZ13, §1.42] to denote the orbits of the group of global diffeomorphisms, $\mathrm{Diff}(\mathrm{X})$. The fact that these orbits constitute a basic stratification, along with variants, was further discussed, for example in [PIZ25, p. 299].

The notion of stratification is often closely tied to that of singularity. Since singularity is inherently a local concept, we consider the local geometry of a diffeological space. This is defined at each point by the germ of the diffeology there [PIZ13, §2.19]. The local geometry at a point is preserved under the action of local diffeomorphisms.[6] These local diffeomorphisms form a pseudogroup, which we denote by $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{X})$, rather than a group. Importantly, local diffeomorphisms can only map points to other points that share the same local geometry. This motivates the following definition, which specifies the variant of Klein strata primarily used in this work:

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[5] For the sake of diffeology, we consider stationary paths defined on the whole $\mathbf{R}$, that are constant on $]-\infty, 0]$ and also on $[1, +\infty[$.

[6] Actually, this would be a formal definition of the "local geometry" of the diffeological space X at a point x: its orbit by $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{X})$.

Definition. *Let* X *be a diffeological space. We call its (local)* Klein strata *the orbits of its pseudogroup of local diffeomorphisms,* $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{X})$. *They are the sets of points in* X *sharing the same local geometry.*

While both global $\mathrm{Diff}(\mathrm{X})$-orbits and local $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{X})$-orbits provide interesting partitions and are of interest, our focus on local structure and singularities leads us to primarily adopt the definition based on local diffeomorphisms for the remainder of this paper.

Proposition. *The set of (local) Klein strata of a diffeological space* X *forms a basic stratification [B], termed the* Klein stratification *of* X. *Hence, every diffeological space possesses a natural stratification intrinsic to its diffeology, revealed by the action of its local diffeomorphisms. This is the stratification of* X *by local geometry.*

Proof. The proof that the partition into orbits of local diffeomorphisms is a basic stratification [B] is analogous to the proof for geometric stratifications by group actions (see section 6). If a point p in a Klein stratum S_1 is in the closure of another Klein stratum S_2, then any point p' in S_1 (which is related to p by a finite chain of local diffeomorphisms covering a path from p to p') must also be in the closure of S_2. This holds because local diffeomorphisms map D-open sets to D-open sets [PIZ13, §2.1, §2.5], thereby preserving the property of neighborhoods intersecting S_2. □

IV. Examples

This section presents several examples of quotients of manifolds by compact group actions. These examples, though straightforward, are selected to illustrate the diverse situations encountered in the study of such orbit spaces.

9. Example $\mathbf{TS^2/SO(3)}$.

The group $\mathrm{SO}(3)$ acts diagonally on the tangent bundle $\mathrm{M} = \mathrm{TS}^2 = \{(x, v) \in \mathrm{S}^2 \times \mathbf{R}^3 \mid \langle x, v\rangle = 0\}$ by $\mathrm{A} \cdot (x, v) = (\mathrm{A}x, \mathrm{A}v)$. The stabilizers are:

$$\mathrm{Stab}_{\mathrm{SO}(3)}(x, v) = \begin{cases} \{\mathrm{id}\} & \text{if } v \neq 0 \text{ (principal type)},\\ \mathrm{SO}(2, x) & \text{if } v = 0 \text{ (singular type)}, \end{cases}$$

where $\mathrm{SO}(2, x)$ is the subgroup fixing x. The orbit type stratification of M consists of two strata: $\mathrm{M}_{(\{\mathrm{id}\})} = \{(x, v) \mid v \neq 0\}$ (the tangent bundle minus its zero section) and $\mathrm{M}_{(\mathrm{SO}(2))} = \{(x, v) \mid v = 0\}$ (the zero section, diffeomorphic to S^2).

The orbit space $\mathrm{Q} = \mathrm{M}/\mathrm{SO}(3)$ is diffeomorphic to the interval $[0, +\infty[$ equipped with the pushforward diffeology by the map $\mathrm{sq} : (x, v) \mapsto \|v\|^2$. We denote this space by $[0, +\infty[_{\mathrm{sq}}$. The invariant $\|v\|^2$ distinguishes the $\mathrm{SO}(3)$-orbits.

The Klein stratification of $[0, +\infty[_{\mathrm{sq}}$ consists of two strata:

* $\mathrm{K}_1 = \{0\}$, corresponding to the singular orbits from $\mathrm{M}_{(\mathrm{SO}(2))}$.
* $\mathrm{K}_2 =]0, \infty[$, corresponding to the principal orbits from $\mathrm{M}_{(\{\mathrm{id}\})}$.

The stratification by orbit types, $\mathrm{Q}_{(\{\mathrm{id}\})} =]0,\infty[$ and $\mathrm{Q}_{(\mathrm{SO}(2))} = \{0\}$, coincides with the Klein stratification.

The diffeological dimension map is (by Proposition 2):

* For $t \in \mathrm{K}_2 =]0,\infty[$ (principal points in Q): $\dim_t([0,+\infty[_{\mathrm{sq}}) = 1$.
* For $t = 0 \in \mathrm{K}_1$ (singular point in Q): The stabilizer of $(x,0)$ is $\mathrm{SO}(2,x)$ and $\dim_0([0,+\infty[_{\mathrm{sq}}) = 2$.

The dimension map separates the orbits. As a diffeological space, $\mathrm{TS}^2/\,\mathrm{SO}(3)$ is diffeomorphic to the half-line Δ_2.

10. Example $\mathbf{TP^2(R)/\,SO(3)}$.

Consider the tangent bundle of the real projective plane, $\mathrm{M} = \mathrm{TP}^2(\mathbf{R})$. A point in M is an equivalence class $[(x,v)] = \{(x,v),(-x,-v)\}$, where $(x,v) \in \mathrm{TS}^2$. The group $\mathrm{SO}(3)$ acts diagonally: $\mathrm{A}\cdot[(x,v)] = [(\mathrm{A}x,\mathrm{A}v)]$. The stabilizers are:

$$\mathrm{Stab}_{\mathrm{SO}(3)}[(x,v)] = \begin{cases} \mathbf{Z}_2 & \text{if } v \neq 0 \text{ (principal type)},\\ \mathrm{O}(2,x) & \text{if } v = 0 \text{ (singular type)}, \end{cases}$$

where the $\mathbf{Z}_2$ stabilizer for $v \neq 0$ is generated by the rotation of angle π around the axis defined by $x \wedge v$ (if x, v are identified with vectors in $\mathbf{R}^3$), and $\mathrm{O}(2,x)$ is the subgroup of $\mathrm{SO}(3)$ preserving the line through x. The orbit type stratification of M consists of two strata: $\mathrm{M}_{(\mathbf{Z}_2)} = \{[(x,v)] \mid v \neq 0\}$ and $\mathrm{M}_{(\mathrm{O}(2))} = \{[(x,v)] \mid v = 0\}$ (the zero section, diffeomorphic to $\mathrm{P}^2(\mathbf{R})$).

The orbit space $\mathrm{Q} = \mathrm{M}/\,\mathrm{SO}(3)$ is diffeomorphic to $[0,+\infty[$ equipped with the pushforward diffeology by the map $\mathrm{sq} : [(x,v)] \mapsto \|v\|^2$. We denote this space by $[0,+\infty[_{\mathrm{sq}}$. The Klein stratification of $[0,+\infty[_{\mathrm{sq}}$ consists of two strata: $\{0\}$ (from $\mathrm{M}_{(\mathrm{O}(2))}$) and $]0,\infty[$ (from $\mathrm{M}_{(\mathbf{Z}_2)}$).

The diffeological dimension map is (by Proposition 2):

* For $t \in]0,\infty[$ (principal points in Q): $\dim_t([0,+\infty[_{\mathrm{sq}}) = 1$.
* For $t = 0$ (singular point in Q): This point corresponds to orbits in the zero section $\mathrm{M}_{(\mathrm{O}(2))}$. The stabilizer of $[(x,0)]$ is $\mathrm{O}(2,x)$ and $\dim_0([0,+\infty[_{\mathrm{sq}}) = 2$.

In summary for $\mathrm{Q} \cong [0,+\infty[_{\mathrm{sq}}$:

* Orbit Type Strata (images in Q): $\mathrm{Q}_{(\mathrm{O}(2))} = \{0\}$, $\mathrm{Q}_{(\mathbf{Z}_2)} =]0,\infty[$.
* Klein Strata: $\{0\},]0,\infty[$.
* Dimensions: $\dim_t(\mathrm{Q}) = 2$ for $t = 0$, and $\dim_t(\mathrm{Q}) = 1$ for $t \in]0,\infty[$.

Here again the dimension map separates the points, and $\mathrm{TP}^2(\mathbf{R})/\,\mathrm{SO}(3)$ is diffeomorphic to the half-line Δ_2.

This example further illustrates that distinct G-manifolds, such as $(\mathrm{TS}^2,\mathrm{SO}(3))$ and $(\mathrm{TP}^2(\mathbf{R}),\mathrm{SO}(3))$ which have different orbit-type stratifications, can yield diffeomorphic orbit spaces (Δ_2) when equipped with the quotient diffeology.

11. Example $\mathrm{P}^2(\mathbf{R})/\mathrm{SO}(2)$.[7]

The projective space $\mathrm{P}^2(\mathbf{R})$ is the quotient of S^2 by the inversion $x \mapsto -x$. That is, the set of pairs $(x, -x)$ for all $x \in \mathrm{S}^2$. Let $k \in \mathrm{S}^2$ and $\mathrm{SO}(2, k)$ be the rotation around k. Since the inversion commutes with rotation, the action of $\mathrm{SO}(2, k)$ on S^2 descends to the quotient $\mathrm{P}^2(\mathbf{R})$, and

$$\mathrm{P}^2(\mathbf{R})/\mathrm{SO}(2,k) = [\mathrm{S}^2/\{\pm 1\}]/\mathrm{SO}(2,k) = [\mathrm{S}^2/\mathrm{SO}(2,k)]/\{\pm 1\}.$$

Now, $\mathrm{S}^2/\mathrm{SO}(2,k) \simeq [-1,+1]_\pi$ with $\pi : x \mapsto \langle x, k\rangle$. The inversion acts then on $[-1,+1]_\pi$ by $t \mapsto -t$. Thus, the quotient $\mathrm{P}^2(\mathbf{R})/\mathrm{SO}(2,k)$ is equivalent to the interval $[0,1]_{\pi^2}$ with

$$\mathrm{P}^2(\mathbf{R})/\mathrm{SO}(2,k) \simeq [0,1]_{\pi^2}, \ \text{ with } \left\{ \begin{array}{l} \pi^2 : \mathrm{P}^2(\mathbf{R}) \to [0,1] \\ \pi^2 : \{x, -x\} \mapsto \langle x, k\rangle^2. \end{array}\right.$$

The dimension of $[0,1]_{\pi^2}$ at the point 0 is 1, and at the point 1 is 2. In summary:

* Orbit Type Strata (images in Q): $\{0\}, \{1\}, \{]0,1[\}$.
* Klein Strata: $\{0\}, \{1\}, \{]0,1[\}$.
* $\dim_0([0,1]_{\pi^2}) = 1, \dim_1([0,1]_{\pi^2}) = 2, \dim_{t\in]0,1[}([0,1]_{\pi^2}) = 1.$

This example highlights how different singular points in the quotient can have different diffeological dimensions, reflecting the geometry of the orbits in the manifold that map to them. In this specific case, the orbit type strata in the quotient coincide with the Klein strata.

Its geometry, that is to say its diffeology, is interesting and can be described by the properties of its plots. A plot $\mathrm{P} : \mathrm{U} \to [0,1]$ is a plot if for any $r_0 \in \mathrm{U}$:

* If $\mathrm{P}(r_0) \in]0,1[$, there exists an open neighbourhood $\mathrm{V} \subseteq \mathrm{U}$ of r_0 such that $\mathrm{P} \restriction \mathrm{V}$ is an ordinary smooth map.
* If $\mathrm{P}(r_0) = 0$, there exists an open neighbourhood $\mathrm{V} \subseteq \mathrm{U}$ of r_0 and a smooth parametrization $\mathrm{Q} : \mathrm{V} \to \mathbf{R}$ such that $\mathrm{P} \restriction \mathrm{V}(r) = \mathrm{Q}(r)^2$.
* If $\mathrm{P}(r_0) = 1$, there exists an open neighbourhood $\mathrm{V} \subseteq \mathrm{U}$ of r_0 and a smooth parametrization $\mathrm{Q} : \mathrm{V} \to \mathbf{R}^2$ such that $\mathrm{P} \restriction \mathrm{V}(r) = 1 - \|\mathrm{Q}(r)\|^2$.

Another way to understand its structure would be to view it as a space obtained by gluing a piece of the half-line Δ_1 (near 0) and a piece of $1 - \Delta_2$ (near 1) over an open interval corresponding to the principal stratum. This shows that the structure of the quotient ultimately depends little on how it was formed, while retaining its smooth properties.

12. Example $(\mathrm{S}^2 \times \mathrm{S}^2)/\mathrm{SO}(3)$.

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[7] Thanks to Jordan Watts for suggesting this example.

We consider the group $\mathrm{SO}(3)$ acting diagonally on $\mathrm{S}^2 \times \mathrm{S}^2$ by $\mathrm{A} \cdot (x, y) = (\mathrm{A}x, \mathrm{A}y)$. The stabilizers are:

$$\mathrm{Stab}_{\mathrm{SO}(3)}(x, y) = \begin{cases} \{\mathrm{id}\} & \text{if } x \neq y \text{ and } x \neq -y \text{ (principal type)}, \\ \mathrm{SO}(2, x) & \text{if } x = y \text{ or } x = -y \text{ (singular type)}, \end{cases}$$

where $\mathrm{SO}(2, x)$ is the subgroup fixing x (isomorphic to $\mathrm{SO}(2)$). The orbit type stratification of $\mathrm{M} = \mathrm{S}^2 \times \mathrm{S}^2$ thus consists of two strata: $\mathrm{M}_{(\{\mathrm{id}\})}$ and $\mathrm{M}_{(\mathrm{SO}(2))}$.

The orbit space $\mathrm{Q} = \mathrm{M}/\,\mathrm{SO}(3)$ is diffeomorphic to the interval $[-1, +1]$ equipped with the pushforward diffeology from M by the map $\pi : (x, y) \mapsto \langle x, y\rangle$ (the standard inner product in $\mathbf{R}^3$). We denote this diffeological space by $[-1, +1]_\pi$. The map π induces a diffeomorphism $\bar\pi : \mathrm{Q} \to [-1, +1]_\pi$. The invariant $\langle x, y\rangle$ distinguishes the $\mathrm{SO}(3)$-orbits.

The Klein stratification of $[-1, +1]_\pi$ (by orbits of local diffeomorphisms) consists of two strata:

* $\mathrm{K}_1 = \{\pm 1\}$, corresponding to the images of singular orbits from $\mathrm{M}_{(\mathrm{SO}(2))}$.
* $\mathrm{K}_2 = \,]-1, 1[$, corresponding to the images of principal orbits from $\mathrm{M}_{(\{\mathrm{id}\})}$.

In this case, the stratification by images of orbit types, $\mathrm{Q}_{(\{\mathrm{id}\})} = \,]-1, 1[$ and $\mathrm{Q}_{(\mathrm{SO}(2))} = \{\pm 1\}$, coincides with the Klein stratification.

The diffeological dimension map is (see Proposition 2):

* For $t \in \mathrm{K}_2 = \,]-1, 1[$ (principal points in Q): $\dim_t([-1, +1]_\pi) = 1$.
* For $t \in \mathrm{K}_1 = \{\pm 1\}$ (singular points in Q): These points correspond to orbits like $\mathrm{G} \cdot (x, x)$ in M, with stabilizer $\mathrm{SO}(2, x)$ and $\dim_t([-1, +1]_\pi) = 2$.

The dimension map separates the orbits. As a diffeological space, $(\mathrm{S}^2 \times \mathrm{S}^2)/\,\mathrm{SO}(3)$ is diffeomorphic to I_2, a space that we discuss further.

13. Example $\mathbf{P^2(C)/\,SO(3)}$.

Consider the projective space $\mathrm{P}^2(\mathbf{C})$ as the quotient of $\mathrm{S}^5 \subset \mathbf{R}^3 \times \mathbf{R}^3$ by $\mathrm{SO}(2)$. Let $(\mathrm{X}, \mathrm{Y}) \in \mathrm{S}^5$:

$$\mathrm{SO}(2) \ni \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} : \begin{pmatrix} \mathrm{X} \\ \mathrm{Y} \end{pmatrix} \mapsto \begin{pmatrix} \cos(\theta)\mathrm{X} - \sin(\theta)\mathrm{Y} \\ \sin(\theta)\mathrm{X} + \cos(\theta)\mathrm{Y} \end{pmatrix}.$$

And $\mathrm{A} \in \mathrm{SO}(3)$ acts diagonally:

$$\mathrm{A} : (\mathrm{X}, \mathrm{Y}) \mapsto (\mathrm{AX}, \mathrm{AY}).$$

This action descends to the quotient and defines an action of $\mathrm{SO}(3)$ on $\mathrm{P}^2(\mathbf{C})$. The space of orbits $\mathrm{P}^2(\mathbf{C})/\,\mathrm{SO}(3)$ is realized as the interval $[0, 1]$ by the projection:

$$\pi : (\mathrm{X}, \mathrm{Y}) \mapsto 1 - 4\|\mathrm{X} \wedge \mathrm{Y}\|^2. \tag{1}$$

That is,

$$\mathrm{P}^2(\mathbf{C})/\,\mathrm{SO}(3) \simeq [0, 1]_\pi.$$

There are two singular orbits: $\mathcal{O}_\wedge$ and $\mathcal{O}_\perp$ images of the subspaces,

$$\mathrm{X} \wedge \mathrm{Y} = 0 \text{ for } \mathcal{O}_\wedge, \text{ and } \|\mathrm{X} \wedge \mathrm{Y}\|^2 = \frac{1}{4} \text{ for } \mathcal{O}_\perp.$$

The orbit $\mathcal{O}_\wedge$ is made of collinear pairs X // Y and the orbit $\mathcal{O}_\perp$ is made of orthogonal pairs $\mathrm{X} \perp \mathrm{Y}$. The description of this action was primarily described in [PI91]. We summarize the result here:

The tubular neighbourhood of the orbit $\mathcal{O}_\wedge$ has type $\mathrm{SO}(3) \times_{\mathrm{O}(2)} \mathbf{R}^2$, where $\mathrm{O}(2) \subset \mathrm{SO}(3)$ acts by $\mathrm{A}(\mathrm{R}, \xi) = (\mathrm{RA}^{-1}, \mathrm{A}\xi)$. The singular stabilizer $\mathrm{O}(2) \subset \mathrm{SO}(3)$ is the subgroup generated by the rotations $\mathrm{SO}(2)$ around some axis and the reflection with respect to the orthogonal plane to this axis. This is a semi direct product of $\mathrm{SO}(2)$ by $\{\pm 1\}$. The orbit $\mathcal{O}_\wedge$ is of type $(\mathrm{SO}(3)/\mathrm{SO}(2))/\{\pm 1\} \simeq \mathrm{P}^2(\mathbf{R})$. The principal stabilizer has type $\mathbf{Z}/2\mathbf{Z}$ corresponding to the reflections. The local quotient is $\mathbf{R}^2/\mathrm{O}(2) = \Delta_2$ and the dimension of $\mathrm{P}^2(\mathbf{C})/\mathrm{SO}(3)$ at the point $\mathcal{O}_\wedge$ is 2.

The tubular neighbourhood of the orbit $\mathcal{O}_\perp$ is of type F_2 (in [PI91]), that is $\mathrm{SO}(3) \times_{\mathrm{SO}(2)} \mathbf{R}^2$, where $\mathrm{A} \in \mathrm{SO}(2)$ acts by $\mathrm{A} \cdot (\mathrm{R}, \xi) = (\mathrm{RA}^{-1}, \mathrm{A}^2\xi)$. The principal stabilizer has (of course) type $\mathbf{Z}/2\mathbf{Z}$, identifying $\mathrm{SO}(2)$ with $\mathrm{U}(1) \subset \mathbf{C}$ it would be $\{e^{i\pi}, 1\}$. The singular stabilizer for the orbit $\mathcal{O}_\perp$ is $\mathrm{SO}(2)$, so $\mathcal{O}_\perp$ has type $\mathrm{SO}(3)/\mathrm{SO}(2) \simeq \mathrm{S}^2$. The quotient space is then diffeomorphic to $\mathbf{R}^2/\mathrm{SO}(2)$ with $\mathrm{SO}(2)$ acting by $\mathrm{A} \cdot \xi = \mathrm{A}^2\xi$. But the orbit space is still the space of concentric circles which coincides with the standard Δ_2. The dimension of $\mathrm{P}^2(\mathbf{C})/\mathrm{SO}(3)$ at the point $\mathcal{O}_\wedge$ is also 2.

In summary:

(a) The quotient of $\mathrm{P}^2(\mathbf{C})/\mathrm{SO}(3)$ is diffeomorphic to $[0,1]_\pi$.

(b) The point 0 corresponding to the orbit $\mathcal{O}_\perp$, with singular stabilizer $\mathrm{SO}(2)$, has a local structure of $\Delta_2 = \mathbf{R}^2/\mathrm{SO}(2)$ and $\dim_{\mathcal{O}_\perp}(\mathrm{P}^2(\mathbf{C})/\mathrm{SO}(3)) = 2$.

(c) The point 1 corresponding to the orbit $\mathcal{O}_\wedge$, with singular stabilizer $\mathrm{O}(2)$, has a local structure of $1 - \Delta_2$ at 1, equivalent to $\mathbf{R}^2/\mathrm{O}(2)$, and $\dim_{\mathcal{O}_\wedge}(\mathrm{P}^2(\mathbf{C})/\mathrm{SO}(3)) = 2$.

(d) There are three orbit-type strata: $\{0\}$, $\{1\}$, $\{]0,1[$.

(e) There are two Klein strata: $\{0,1\}$, $\{]0,1[\}$, since the quotient $\mathrm{P}^2(\mathbf{C})/\mathrm{SO}(3) \simeq [0,1]_\pi$ has the same local structure Δ_2 on the neighbourhood of 0 and 1. This shows clearly that even if the projection $\pi : \mathrm{M} \to \mathrm{M}/\mathrm{G}$ is a stratified subduction (as will be made explicit further), this map is not necessarily injective.

Note. Diffeologically speaking, the quotients of $\mathrm{S}^2 \times \mathrm{S}^2$ and $\mathrm{P}^2(\mathbf{C})$ by $\mathrm{SO}(3)$ are diffeomorphic. It is noteworthy that $\mathrm{S}^2 \times \mathrm{S}^2$ can admit a family of non-equivariantly isomorphic (or non-equivariantly symplectomorphic) $\mathrm{SO}(3)$-actions, indexed by $m \in \mathbf{Z} - \{0\}$ (but with same singular stabilizers), as detailed in [PI91]. They nevertheless all yield this same diffeological quotient space $[-1,+1]_\pi$, while the actions differ (e.g., the effective action of the $\mathrm{SO}(2)$ stabilizer on the 2-dimensional slice at singular points is by $\mathrm{A} \mapsto \mathrm{A}^m$), the resulting local quotient model $\mathbf{R}^2/\mathrm{SO}(2)_{\text{action } \mathrm{A}^m}$ is always diffeomorphic to Δ_2.

This implies that the diffeological dimension map and the overall structure of the quotient space are identical for all these actions, highlighting that the diffeological quotient captures certain fundamental geometric features determined by the invariant theory and local singularity types, which can be common across distinct G-manifold structures.

14. Example: $\mathrm{P}^2(\mathbf{C})/\mathrm{U}(1)$.

This example is extracted from [Aud91]. Consider the action of the compact Lie group $\mathrm{G} = \mathrm{U}(1)$ on the smooth manifold $\mathrm{M} = \mathrm{P}^2(\mathbf{C})$ given, in projective notation, by:

$$\tau \cdot [z_1 : z_2 : z_3] = [z_1 : \tau z_2 : \tau^2 z_3] \quad \text{for } \tau \in \mathrm{U}(1)$$

We analyze the stabilizers of points in $\mathrm{P}^2(\mathbf{C})$ under this action to determine the stratification of M and the resulting structure of the orbit space $\mathrm{P}^2(\mathbf{C})/\mathrm{U}(1)$.

A point $[z_1 : z_2 : z_3] \in \mathrm{P}^2(\mathbf{C})$ is stabilized by $\tau \in \mathrm{U}(1)$ if it exists $\lambda \in \mathbf{C}^*$ such that

$$z_1 = \lambda z_1,\ \tau z_2 = \lambda z_2 \ \text{ and } \ \tau^2 z_3 = \lambda z_3.$$

Let $\mathrm{P}_0, \mathrm{P}_1, \mathrm{P}2$ be the three points:

$$\mathrm{P}_0 = [1 : 0 : 0],\ \mathrm{P}_1 = [0 : 1 : 0],\ \mathrm{P}_2 = [0 : 0 : 1],$$

we get the following stabilizers:

(1) $\mathrm{Stab}_{\mathrm{U}(1)}(\mathrm{P}_0, \mathrm{P}_1, \mathrm{P}_2) = \mathrm{U}(1)$.
(2) $\mathrm{Stab}_{\mathrm{U}(1)}([z_1 : 0 : z_3]) = \{\pm 1\}$, with $z_1 \neq 0$ and $z_3 \neq 0$.
(3) $\mathrm{Stab}_{\mathrm{U}(1)}([z_1 : z_2 : z_3]) = \{1\}$, with $z_2 \neq 0$ and $[z_1 : z_2 : z_3] \neq \mathrm{P}_1$.

Stratification by Exact Stabilizer: The sets of points with a specific exact stabilizer are:

(1) $\mathrm{M}_{\mathrm{U}(1)} = \{\mathrm{P}_0, \mathrm{P}_1, \mathrm{P}_2\}$ are the singular fixed points.
(2) $\mathrm{M}_{\{\pm 1\}} = \{[z_1 : 0 : z_3] \mid z_1 \neq 0, z_3 \neq 0\}$ are the singular exceptional points.
(3) $\mathrm{M}_{\{1\}} = \{\mathrm{P} = [z_1 : z_2 : z_3] \mid z_2 \neq 0 \text{ and } \mathrm{P} \neq \mathrm{P}_1\}$ are the principal points.

These sets form a partition and a stratification of M. Since $\mathrm{U}(1)$ is abelian, conjugacy classes of subgroups are just the subgroups themselves, and the orbit types strata are just these ones.

Connected Components of Strata:

* $\mathrm{M}_{\{1\}}$ the principal stratum is connected.
* $\mathrm{M}_{\{\pm 1\}}$ is a singular exceptional stratum diffeomorphic to $\mathbf{C}^*$, which is connected.
* $\mathrm{M}_{\mathrm{U}(1)} = \{\mathrm{P}_0, \mathrm{P}_1, \mathrm{P}_2\}$ is the stratum of fixed points, it is discrete and has 3 connected components.

As discussed in the proof of Article 7 Proposition 2: the connected components of the orbit-type strata $\mathrm{M}_{(\mathrm{H})}$ are precisely the connected components of the sets M_K for K conjugate to H. In this example, since $\mathrm{U}(1)$ is abelian, the connected components of the orbit-type strata are the connected components of the M_H strata listed above.

Slice Representations on Connected Components:

(a) For points in $x \in \mathrm{M}_{\{1\}}$, the slice representation is trivial and constant on the whole principal stratum.

(b) For points in $\mathrm{M}_{\{\pm 1\}}$, the slice representation is an action of $\mathbf{Z}_2$ on N_x. This representation is constant up to equivalence on $\mathrm{M}_{\{\pm 1\}}$ since it is connected. Thus, we can choose the point $x = (0, 1) \in \mathrm{M}_{\{\pm 1\}}$. The local model is the orthogonal subspace of $\mathbf{C} \times \mathbf{C}$ tangent to the orbit $\mathrm{U}(1) \cdot (0, 1)$. That is, $\mathbf{C} \times i\mathbf{R}$, and the linear action of the stabilizer is $\varepsilon : (w, it) \to (w, \varepsilon it)$, with $\varepsilon \in \{\pm 1\}$.

(c) For points in $\mathrm{M}_{\{\pm 1\}}$, the normal spaces to the orbits are just $\mathbf{C} \times \mathbf{C}$, and the slice representations on the connected components of $\mathrm{M}_{\mathrm{U}(1)}$ are:

* At P_0: $\rho_0(\tau) \cdot (w_2, w_3) = (\tau w_2, \tau^2 w_3)$.
* At P_1: $\rho_1(\tau) \cdot (w_1, w_3) = (\bar{\tau} w_1, \tau w_3)$.
* At P_2: $\rho_2(\tau) \cdot (w_1, w_2) = (\bar{\tau}^2 w_1, \bar{\tau} w_2)$.

Comparing the slice representations ρ_0, ρ_1, ρ_2, we see that ρ_0 is equivalent to ρ_2 by mapping $\tau \mapsto \bar{\tau}$ and exchanging $w_1 \leftrightarrow w_3$.

$(*)$ The decomposition into orbits of $\mathrm{U}(1)$ in the local model $\mathrm{E} = \mathbf{C} \times \mathbf{C}$ corresponding to $\rho_0 \simeq \rho_2$ is:

$$\mathrm{E}_{\mathrm{U}(1)} = \{0\},\ \mathrm{E}_{\{\pm 1\}} = \{0\} \times \mathbf{C}^*,\ \mathrm{E}_{\{1\}} = \mathbf{C}^* \times \mathbf{C}.$$

We notice that the singular orbit $\mathrm{E}_{\mathrm{U}(1)}$ is in the closure of the exceptional line $\mathrm{E}_{\{\pm 1\}}$.

$(*)$ The decomposition into orbits of $\mathrm{U}(1)$ in the local model $\mathrm{E} = \mathbf{C} \times \mathbf{C}$ corresponding to ρ_1 is:

$$\mathrm{E}_{\mathrm{U}(1)} = \{(0, 0)\},\ \mathrm{E}_{\{1\}} = \mathbf{C} \times \mathbf{C} - \{(0, 0)\}.$$

We notice the absence of the exceptional line $\mathrm{E}_{\{\pm 1\}}$, the only singular strata is $\{\mathrm{P}_1\}$.

Local Model and Klein Strata: Based on the slice representations,

(a) The neighbourhood of $\{\mathrm{P}_0\}$ and $\{\mathrm{P}_2\}$ in $\mathrm{P}^2(\mathbf{C})/\mathrm{U}(1)$ represented by the local model $\mathrm{E}/\rho_0 \simeq \mathrm{E}/\rho_2$ is diffeomorphic to $\mathbf{C} \times \Delta_1$, where $\Delta_1 = \mathbf{R}/\{\pm 1\}$ is an orbifold. Thus, any open neighbourhood of $\{\mathrm{P}_0\}$ or $\{\mathrm{P}_2\}$ in $\mathrm{P}^2(\mathbf{C})/\mathrm{U}(1)$ contains orbifold points.

(b) The neighbourhood of $\{\mathrm{P}_1\}$ in $\mathrm{P}^2(\mathbf{C})/\mathrm{U}(1)$ represented by the local model E/ρ_1 contains only principal points at the exclusion of $\{\mathrm{P}_1\}$. The singular point $\{\mathrm{P}_1\}$ is isolated among the principal points in the neighbourhood which does not contains orbifold points. Now, the isotropy group (or structure group) of an orbifold is a diffeological invariant [IKZ10, Lemma 21], thus, $\{\mathrm{P}_1\} \in \mathrm{P}^2(\mathbf{C})/\mathrm{U}(1)$ cannot be mapped to either $\{\mathrm{P}_0\}$ or $\{\mathrm{P}_2\}$ by a local diffeomorphism because of these orbifold points in their neighbourhoods. Thus, the point $\{\mathrm{P}_1\} \in \mathrm{P}^2(\mathbf{C})/\mathrm{U}(1)$ is fixed by local diffeomorphisms.

(c) Therefore, based on the distinct local geometries captured by these local models, the Klein stratification of $\mathrm{P}^2(\mathbf{C})/\mathrm{U}(1)$ consists of 4 strata:

$$\mathrm{K}_{\mathrm{princ}},\ \mathrm{K}_{\{\pm 1\}},\ \mathrm{K}_{\mathrm{fixed},1} = \{\{\mathrm{P}_0)\}, \{\mathrm{P}_2\}\} \text{ and } \mathrm{K}_{\mathrm{fixed},2} = \{\mathrm{P}_1\}.$$

Moreover, the dimension map on $\mathrm{P}^2(\mathbf{C})/\mathrm{U}(1)$ takes the value 3 on $\mathrm{K}_{\mathrm{princ}} \cup \mathrm{K}_{\{\pm 1\}}$ and 4 on $\mathrm{K}_{\mathrm{fixed},1} \cup \mathrm{K}_{\mathrm{fixed},2}$. It does not distinguish between $\mathrm{K}_{\mathrm{princ}}$ and $\mathrm{K}_{\{\pm 1\}}$, nor between $\mathrm{K}_{\mathrm{fixed},1}$ and $\mathrm{K}_{\mathrm{fixed},2}$.

This example particularly also shows that the projection $\mathrm{orbit} : \mathrm{M} \to \mathrm{M}/\mathrm{G}$ does not necessarily map orbit-type strata onto Klein strata, but specifically illustrates how distinct connected components of the same orbit-type stratum ($\mathrm{M}_{\mathrm{U}(1)}$ fixed points) can map to different Klein strata (e.g., $\{\mathrm{P}_1\}$), while other components (e.g., $\{\{\mathrm{P}_0)\}, \{\mathrm{P}_2\}\}$) map to the same Klein stratum ($\mathrm{K}_{\mathrm{fixed},1}$).

15. Example: Geodesics on the 2-Torus.

To illustrate the broader applicability of Klein stratifications beyond quotients by compact Lie group actions, we now consider the diffeological space of geodesics on the 2-torus. The space of oriented unparametrized geodesics of the flat 2-torus $\mathrm{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ is not a manifold but is a natural diffeological space [PIZ25, p. 268, 279].[8] It is diffeomorphic to the quotient space

$$\mathrm{Geod}(\mathrm{T}^2) \simeq (\mathrm{S}^1 \times \mathbf{R})/\mathbf{Z}^2,$$

where $\mathrm{S}^1 \times \mathbf{R}$ represents $\mathrm{Geod}(\mathbf{R}^2)$, the space of affine lines in the Euclidean plane $\mathbf{R}^2$ (parameterized by direction $u \in \mathrm{S}^1$ and signed distance $\rho \in \mathbf{R}$ from the origin). In this model, the $\mathbf{Z}^2$-action is defined for $k \in \mathbf{Z}^2$ by

$$k \cdot (u, \rho) = (u, \rho + \langle u, k \rangle),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbf{R}^2$. Let $\mathrm{class}(u, \rho)$ denote the orbit of (u, ρ) under this $\mathbf{Z}^2$-action. Then

$$\mathrm{class}(u, \rho) = (u, [\rho]_u), \quad \text{where} \quad [\rho]_u = \{\rho + \langle k, u \rangle \mid k \in \mathbf{Z}^2\}.$$

If $u = (\cos\theta, \sin\theta)$, then $[\rho]_u = \{\rho + n\cos\theta + m\sin\theta \mid (n, m) \in \mathbf{Z}^2\}$.

The space $\mathrm{Geod}(\mathrm{T}^2)$ admits a natural projection $\mathrm{pr}_1 : \mathrm{Geod}(\mathrm{T}^2) \to \mathrm{S}^1$ mapping $(u, [\rho]_u) \mapsto u$. The fiber over $u \in \mathrm{S}^1$ is $\mathrm{pr}_1^{-1}(u) = \mathrm{T}_u = \{(u, [\rho]_u) \mid \rho \in \mathbf{R}\} \cong \mathbf{R}/\Gamma_u$, where $\Gamma_u = \{\langle k, u \rangle \mid k \in \mathbf{Z}^2\}$ is the (diffeologically) discrete subgroup of $\mathbf{R}$ generated by projections of $\mathbf{Z}^2$ onto the line in direction u.

* If the slope of u is rational (i.e., $\tan\theta \in \mathbf{Q} \cup \{\infty\}$), Γ_u is isomorphic to $a\mathbf{Z}$ for some $a \neq 0$. Then $\mathrm{T}_u \cong \mathbf{R}/a\mathbf{Z} \simeq \mathrm{S}^1$ (a circle).
* If the slope of u is irrational, Γ_u is a dense subgroup of $\mathbf{R}$. Then $\mathrm{T}_u \cong \mathbf{R}/\Gamma_u$ is an irrational torus.

Thus, $\mathrm{Geod}(\mathrm{T}^2)$ is a diffeological space whose fibers T_u can be circles or these irrational tori, depending on the direction u.

The Klein strata of $\mathrm{Geod}(\mathrm{T}^2)$ (orbits under $\mathrm{Diff}(\mathrm{Geod}(\mathrm{T}^2))$) are described by the following results, adapted from [PIZ25, §72].

[8] Note that in contrast, the space of oriented unparametrized geodesics of the 2-sphere S^2 is a manifold diffeomorphic to S^2 itself.

Proposition 1. *Any diffeomorphism* $f \in \mathrm{Diff}(\mathrm{Geod}(\mathrm{T}^2))$ *induces a diffeomorphism* $\varphi : \mathrm{S}^1 \to \mathrm{S}^1$ *on the base space of directions, such that* $\mathrm{pr}_1 \circ f = \varphi \circ \mathrm{pr}_1$. *This induced map* φ *is realized by an element* $\mathrm{M} \in \mathrm{GL}(2, \mathbf{Z})$ *acting on* S^1 *by* $u \mapsto \mathrm{M}u/\|\mathrm{M}u\|$. *This yields a group homomorphism* $\Psi : \mathrm{Diff}(\mathrm{Geod}(\mathrm{T}^2)) \to \mathrm{GL}(2, \mathbf{Z})$ *given by* $\Psi(f) = \mathrm{M}$. *This homomorphism fits into a short exact sequence:*

$$1 \to \mathrm{J} = \ker(\Psi) \to \mathrm{Diff}(\mathrm{Geod}(\mathrm{T}^2)) \xrightarrow{\Psi} \mathrm{GL}(2, \mathbf{Z}) \to 1,$$

where J *is the group of gauge transformations mapping the fibers to themselves.*

Proposition 2. *The Klein strata of* $\mathrm{Geod}(\mathrm{T}^2)$ *under the action of* $\mathrm{Diff}(\mathrm{Geod}(\mathrm{T}^2))$ *are precisely the preimages under* pr_1 *of the orbits of* $\mathrm{GL}(2, \mathbf{Z})$ *on* S^1. *A Klein stratum consists of all fibers* T_u *where* u *belongs to a single* $\mathrm{GL}(2, \mathbf{Z})$*-orbit in* S^1. *Consequently, a Klein stratum will either consist entirely of fibers that are circles (if it corresponds to the* $\mathrm{GL}(2, \mathbf{Z})$*-orbit of rational directions) or entirely of fibers that are diffeomorphic irrational tori (if it corresponds to a* $\mathrm{GL}(2, \mathbf{Z})$*-orbit of irrational directions). Then, the space* $\mathcal{K}$ *of the Klein strata of* $\mathrm{Geod}(\mathrm{T}^2)$ *is*

$$\mathcal{K} := \mathrm{Geod}(\mathrm{T}^2) / \mathrm{Diff}(\mathrm{Geod}(\mathrm{T}^2)) \simeq \mathrm{S}^1 / \mathrm{GL}(2, \mathbf{Z}).$$

In terms of labels, introduced in Section II Article 5, the Klein stratification of $\mathrm{Geod}(\mathrm{T}^2)$ responds to the code [B][~~LF~~][GK][~~M~~][$\cancel{\mathrm{T}_0}$].

Remark on the Nature of Strata in $\mathbf{Geod(T^2)}$. This example illustrates the robustness of the Klein stratification in diffeology, extending beyond spaces typically considered in classical stratification theory. The Klein strata here are generally not manifolds. For instance, a stratum corresponding to the dense $\mathrm{GL}(2, \mathbf{Z})$-orbit of rational directions in S^1 is itself a dense subset of $\mathrm{Geod}(\mathrm{T}^2)$, and its closure is the entire space. Despite this complex topological behavior, where strata may not be submanifolds and frontier conditions are met in a broad sense, the Klein stratification reveals crucial geometric information. It precisely distinguishes sets of fibers based on their diffeomorphic type: fibers T_u and $\mathrm{T}_{u'}$ belong to the same Klein stratum if and only if u and u' are in the same $\mathrm{GL}(2, \mathbf{Z})$-orbit. This demonstrates that the Klein stratification provides a meaningful structural decomposition even for intricate diffeological spaces.

16. Example: Orbifolds.

An important class of diffeological spaces whose Klein stratification has a clear classical counterpart is that of orbifolds. As defined in [IKZ10],[9] an n-orbifold is a diffeological space everywhere locally diffeomorphic to $\mathbf{R}^n/\Gamma$ for some finite subgroup $\Gamma \subset \mathrm{GL}(n, \mathbf{R})$, which may change from point to point. The group Γ, unique up to conjugation in $\mathrm{GL}(n, \mathbf{R})$ for a given local model at a point, is called the *structure group* or *isotropy group* at that point [IKZ10, Definition 27].

[9] As was shown in this paper, the diffeological definition of an orbifold is equivalent to the orginal definition of a "V-manifold" by Ichiro Satake in [Sat56].

The Klein stratification of an orbifold[10] is described by the following proposition:

Proposition. *The Klein strata of orbifolds are precisely the sets of points whose structure groups are conjugate.*

Therefore, the Klein stratification of an orbifold coincides precisely with the classical stratification defined by the conjugacy classes of its structure groups. This coincidence was also noted in [GIZ23]. The diffeology of an orbifold thus inherently encodes and distinguishes these classical singularity types.

This proposition is a direct consequence of the fundamental Lemma 21 in [IKZ10] :

Lemma 21. *Let* $\Gamma \subset \mathrm{GL}(n, \mathbf{R})$ *and* $\Gamma' \subset \mathrm{GL}(n', \mathbf{R})$ *be finite subgroups. A local diffeomorphism between* $\mathbf{R}^n/\Gamma$ *and* $\mathbf{R}^{n'}/\Gamma'$ *mapping the origin to the origin exists if and only if* $n = n'$ *and* Γ *is conjugate to* Γ' *in* $\mathrm{GL}(n, \mathbf{R})$.

Furthermore, as noted in Proposition "Orbifolds within Orthofolds" in Section 1, the diffeological dimension of an n-orbifold is constant and equal to n at every point. This constancy of dimension is a property that distinguishes orbifolds from more general orthofolds.

In terms of labels, introduced in Section II Article 5, it has been proved in [GIZ23] that the Klein stratification of an orbifold responds to the code [B][LF][GK][M][T_0].

V. Orbit Type versus Klein

This section clarifies the relationship between the orbit-type stratification on the manifold and the Klein stratification on the space of orbits, equipped with the quotient diffeology.

17. Stratified Orbit Map.

Let M be a manifold with a smooth action of a compact group G. We have seen that M is stratified by orbit type on the one hand, and on the other hand, the space of orbits M/G is stratified by the action of the local diffeomorphisms. It is thus natural to compare these two different stratifications, since one is a stratification of the manifold M and the other one is a stratification of the quotient space M/G.

We have seen that the space of orbits M/G is locally modeled by the quotients E/H where E and H are the slices and the stabilizers of orbits. Precisely, for any point $\mathcal{O} \in \mathrm{M}/\mathrm{G}$, there is a local diffeomorphism from E/H to M/G mapping $0 \in \mathrm{E}/\mathrm{H}$ to $\mathcal{O}$, where E is a slice at $x \in \mathcal{O}$ and $\mathrm{H} = \mathrm{Stab}_{\mathrm{G}}(x)$. Thus, we have the natural property:

Proposition. *Let* $x, x' \in \mathrm{M}$ *be points belonging to the same connected component of an orbit-type stratum* $\mathrm{M}_{(\mathrm{H})}$. *Then their orbits* $\mathcal{O} = \mathrm{orbit}(x)$ *and* $\mathcal{O}' = \mathrm{orbit}(x')$ *belong to the same Klein stratum in* M/G. *In other words, the subduction* $\mathrm{orbit} : \mathrm{M} \to \mathrm{M}/\mathrm{G}$ *is stratified with respect to the stratification by connected components of the orbit-type strata on* M *and the Klein stratification on* M/G.

[10] Which is not necessarily the orbit space of the action of a finite group on a manifold.

Let us then denote by $\mathrm{Str}^*_{\mathrm{OT}}(\mathrm{M}, \mathrm{G})$ the set of connected components of orbit-type strata of M under G, and $\mathrm{Str}_{\mathrm{K}}(\mathrm{M}/\mathrm{G})$ the set of Klein strata of M/G. The projection $\mathrm{orbit} : \mathrm{M} \to \mathrm{M}/\mathrm{G}$ descends into a surjective map $\mathrm{orbit}^* : \mathrm{Str}^*_{\mathrm{OT}}(\mathrm{M}, \mathrm{G}) \to \mathrm{Str}_{\mathrm{K}}(\mathrm{M}/\mathrm{G})$, according to the diagram:

$$\begin{tikzcd}
\mathrm{M} \arrow[r, "\mathrm{orbit}"] \arrow[d, "\mathrm{Str}^*_{\mathrm{OT}}"'] & \mathrm{M}/\mathrm{G} \arrow[d, "\mathrm{Str}_{\mathrm{K}}"] \\
\mathrm{Str}^*_{\mathrm{OT}}(\mathrm{M}, \mathrm{G}) \arrow[r, "\mathrm{orbit}^*"'] & \mathrm{Str}_{\mathrm{K}}(\mathrm{M}/\mathrm{G})
\end{tikzcd}$$

Note. This reveals a more nuanced situation than one might initially expect regarding the map $\mathrm{orbit} : \mathrm{M} \to \mathrm{M}/\mathrm{G}$ being stratified with respect to the orbit-type stratification on M and the Klein stratification on M/G. This is perhaps a more interesting situation. One might have conjectured an isomorphism from $\mathrm{Str}_{\mathrm{OT}}(\mathrm{M}, \mathrm{G})$ to $\mathrm{Str}_{\mathrm{K}}(\mathrm{M}/\mathrm{G})$, but this conjecture is refuted by Example 13 ($\mathrm{SO}(3)$ acting on $\mathrm{P}^2(\mathbf{C})$), where two non-equivalent singular orbits ($\mathcal{O}_\wedge$ and $\mathcal{O}_\perp$) are mapped to the same fixed-point orbit in $\mathrm{P}^2(\mathbf{C})/\mathrm{SO}(3)$. One might then weaken this conjecture to assume that the map $\mathrm{orbit} : \mathrm{M} \to \mathrm{M}/\mathrm{G}$ is merely stratified with respect to the orbit-type stratification on M and the Klein stratification on M/G, without requiring it to induce an isomorphism. Even this weaker conjecture is refuted by Example 14 ($\mathrm{U}(1)$ acting on $\mathrm{P}^2(\mathbf{C})$), where distinct connected components of the same orbit-type stratum (the fixed points stratum) are mapped to different Klein strata. Therefore, the result we have proved is the strongest possible on a general basis: that the map $\mathrm{orbit} : \mathrm{M} \to \mathrm{M}/\mathrm{G}$ is stratified with respect to the stratification by connected components of the orbit-type strata on M and the Klein stratification on M/G. And, as shown by Example 14, the induced map on the level of strata, $\mathrm{orbit}^* : \mathrm{Str}^*_{\mathrm{OT}}(\mathrm{M}, \mathrm{G}) \to \mathrm{Str}_{\mathrm{K}}(\mathrm{M}/\mathrm{G})$, is not necessarily injective.

It also reveals a fundamental aspect of diffeological quotients: once an orbit space $\mathrm{Q} = \mathrm{M}/\mathrm{G}$ is formed and its diffeology identified, it possesses an intrinsic geometric nature that can be independent of its specific "constructor" (M, G). For instance, non-equivalent $\mathrm{SO}(3)$-actions on $\mathrm{S}^2 \times \mathrm{S}^2$ [PI91] yield the same orbit space as $\mathrm{P}^2(\mathbf{C})/\mathrm{SO}(3)$. Despite originating from different manifolds or distinct (non-equivalent) group actions, they can all result in diffeological orbit spaces that are diffeomorphic to a canonical space, say $\mathrm{I}_2 = [0, 1]_2$. This space I_2 can be defined intrinsically by its plots, without prior reference to any M/G construction: A parametrization $\mathrm{P} : \mathrm{U} \to [0, 1]$ (where U is an open domain in some $\mathbf{R}^k$) is a plot of I_2 if for every $r_0 \in \mathrm{U}$:

(1) If $\mathrm{P}(r_0) \in\ ]0, 1[$, then there exists an open neighbourhood $\mathrm{V} \subseteq \mathrm{U}$ of r_0 such that $\mathrm{P} \restriction \mathrm{V}$ is an ordinary smooth map (into $\mathbf{R}$, with values in $]0, 1[$).
(2) If $\mathrm{P}(r_0) \in \{0, 1\}$ (i.e., an endpoint), then there exists an open neighbourhood $\mathrm{V} \subseteq \mathrm{U}$ of r_0 and a smooth parametrization $\mathrm{Q} : \mathrm{V} \to \mathbf{R}^2$ such that $\mathrm{P} \restriction \mathrm{V}(r) = \|\mathrm{Q}(r)\|^2$ or $\mathrm{P} \restriction \mathrm{V}(r) = 1 - \|\mathrm{Q}(r)\|^2$, depending if $\mathrm{P}(r_0) = 0$ or 1.

This intrinsic definition characterizes the diffeology of spaces like Δ_2 at its origin. This construction suggests other "canonical singular spaces" independent of any specific quotient origin.[11] For example:

One could generalize I_2 to $\mathrm{I}_{n,m} = [0,1]_{n,m}$, a space whose diffeology is defined by plots that, near the endpoint 0, locally factor through the squared norm map from $\mathbf{R}^n$, and near the endpoint 1, locally factor through the map $v \mapsto 1 - \|v\|^2$ from $\mathbf{R}^m$. Locally, near its endpoints 0 and 1, such an $\mathrm{I}_{n,m}$ is locally diffeomorphic to $\mathbf{R}^n/\mathrm{O}(n)$ or $\mathbf{R}^m/\mathrm{O}(m)$ (specifically, to Δ_n and $1-\Delta_m$ near its endpoints). Yet, these spaces, such as I_2 and the more general $\mathrm{I}_{n,m}$, are defined directly by their plots, without needing to first find a specific manifold M and a compact group G such that $\mathrm{I}_{n,m} \simeq \mathrm{M}/\mathrm{G}$.[12]

This intrinsic definability is what it means for these diffeological geometric constructs to be "independent of a quotient construction" and to "have a life of their own." They become fundamental building blocks or model singularities within the category of diffeological spaces.

VI. Concluding Remarks

In this paper, we have demonstrated how the framework of diffeology, particularly through the notion of Klein stratification and its intrinsic dimension map, provides valuable tools for analyzing the structure of orbit spaces M/G arising from the smooth action of a compact Lie group G on a manifold M.

We established the fundamental formula $\dim_{\mathcal{O}}(\mathrm{M}/\mathrm{G}) = \dim(\mathrm{M}) - \dim(\mathcal{O})$, which offers a clear understanding of how the diffeological dimension of the quotient space at a point $\mathcal{O}$ reflects the dimension of the corresponding orbit $\mathcal{O} \subset \mathrm{M}$. This formula highlights that singular points in the orbit space (corresponding to lower dimensional orbits in M) tend to have higher diffeological dimensions.

Furthermore, we investigated the relationship between the classical orbit-type stratification of M and the Klein stratification of M/G (defined by local diffeomorphisms). We showed that the canonical projection $\mathrm{orbit} : \mathrm{M} \to \mathrm{M}/\mathrm{G}$ induces a surjective stratified map from the space of connected components of orbit-type strata in M to the space of Klein strata in M/G.

Crucially, our examples demonstrate that this induced map is not necessarily injective. Distinct orbit types strata in M can map to points belonging to the same Klein stratum in M/G, and even distinct connected components of the same orbit-type stratum can map to points belonging to different Klein strata. This finding reveals a profound aspect of diffeological quotients: the resulting space M/G possesses an intrinsic smooth identity, defined by its diffeology and its local diffeomorphism pseudogroup, which can unify

[11]This is why we introduced this new category of *orthofolds* to clarify this point that we may start from group action on manifolds but we land in the category of orthofolds.

[12]Actually, we met the *orthofold* $\mathrm{I}_{1,2}$ as the quotient $\mathrm{P}^2(\mathbf{R})/\,\mathrm{SO}(2)$.

points originating from different orbit structures in the manifold M. This intrinsic identity is fundamentally tied to the local structure of the quotient, which is locally diffeomorphic to quotients of Euclidean spaces by orthogonal group actions, motivating the potential study of such spaces as a distinct class (orthofolds).

These results underscore the utility of diffeology in providing a consistent and nuanced language and tools for the study of singular spaces arising from group actions. By capturing the complete smooth structure that remains after reduction, the diffeological approach offers a richer understanding of the quotient space's geometry than its topology alone, providing advantages even for quotients with a well-behaved topology.

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