HEAT, COLD AND GEOMETRY

P. Iglesias
(CNRS., Marseille)

J.M. Souriau
(Université de Provence, and CNRS. Marseille).

0. INTRODUCTION

Classical and relativistic mechanics can be formulated in terms of symplectic geometry; this formulation leads to a rigorous statement of the principles of statistical mechanics and of thermodynamics.

This analogy also brings to light however certain fundamental difficulties which remain hidden in the traditional approach through some ambiguities.

The "first principle" of thermodynamics can be formulated so as to avoid this ambiguity provided one accepts a detour through the principle of general relativity and the Einstein equations for gravitation.

The mathematical tools used are the theory of symplectic moments, certain cohomological formulae and the concept of distribution-tensor.

As the "second principle" we shall merely show how it is possible, by accepting a particular geometry status for temperature and entropy, to construct a relativistic model of a dissipative continuous medium. This model has the following properties :

a. it is predictive;

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b. all fits solutions satisfy both principles of thermodynamics and admit a detailed balance (energy-impulse, momentum).

c. it contains in particular all equilibrium situations of statistical mechanics, and also the relativistic theory of elasticity.

d. Finally its non-relativistic limit allows one to identify the usual thermodynamic variables and in particular it contains the theory of elasticity, the mechanics of perfect fluids, the theory of heat conduction (Fourier) and the theory of viscosity (Navier).

Nevertheless it is a schematic model which does not take into account phenomena such as capillarity, plasticity, electromagnetic effects, etc.

I. SYMPLECTIC FORMULATION OF DYNAMICS

Consider first of all an elementary dynamical system : a newtonian point mass of mass m, position $\vec{r}$, velocity $\vec{v}$, in a force field $(\vec{r},t) \to \vec{F}$(1); the triplet $y = (\vec{v},\vec{r},t)$ makes up an initial condition for a *motion* x; y travels through a manifold v, (évolution space); if one puts

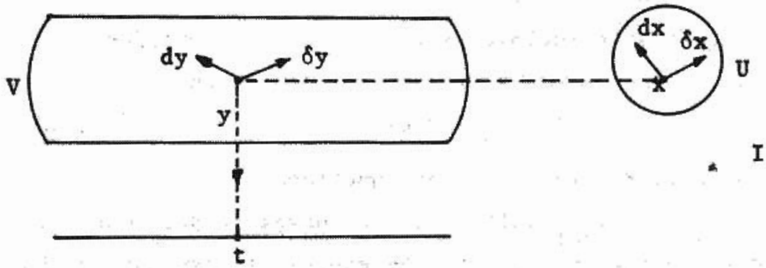
$$\sigma_V(dy)(\delta y) = \langle m\, \vec{d}v - \vec{F}\, dt, \delta\vec{r} - \vec{v}\, \delta t\rangle - \langle m\, \delta\vec{v} - \vec{F}\delta t, \vec{d}r - \vec{v}\, dt\rangle \tag{1.1.}$$

d and δ being two arbitrary variations, the brackets <,> representing scalar product in $\mathbb{R}^3$, one defines on V, a 2-form σ_V of rank 6; the equations of motion become $dy \in \ker(\sigma_V)$(2) ; if $\vec{F}$ is the gradient of some potential, σ_V is a *closed* form (its exterior derivative vanishes); σ_V is thus an absolute invariant integral of the equations of motion, discovered by E. Cartan, but in fact already described explicitly by Lagrange.

The set U of all possible motions has a structure of symplectic manifold (of dimension 6), provided with the closed and reversible 2-form σ_V, whose reciprocal image by the submersion $y \to x$ coincides with σ_V (fig. 1).

Such a scheme can be extended to general dynamical systems

(systems of many particles, spin particles, relativistic mechanics, etc.; see (20)) : in all cases, the space U of motions is a symplectic manifold (thus of even dimension) onto which the evolution space V is projected, each section t = constant of V is a "phase space",; however the identification of phase spaces corresponding to different times is an arbitrary operation, which depends on the system of reference chosen, and therefore is best avoided.



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II. SYMPLECTIC FORMULATION OF STATISTICAL MECHANICS

In this representation a *statistical state* μ is simply a probability law defined on U (i.e. a positive measure of weight 1); the set Prob(U) of these probability laws is a *convex* set, whose *extremal points* are the classical motions x (identified with the corresponding Dirac measures) (Fig. V).

The completely continuous states are characterized by a *density* of U, which is the product of the *Liouville density* [3] by a scalar which can be identified with the classical distribution function[4]. The *entropy* of a statistical state μ is defined to be the average value S of -Log ρ for this state; one can define a good class of states, the "Boltzmann states" (23), which make up a convex subset of Prob (U) and for which the integral of -log ρ is convergent; μ → S is a concave function on this convex.

III. THE PRINCIPLES OF THERMODYNAMICS

Statistical mechanics, as it has just been described, is capable of describing various real phenomena, but not *dissipative* phenomena (friction, heat conduction, viscosity, etc.) which make up the study of thermodynamics. The two "principles" of thermodynamics apply in fact only to idealized situations : dissipative transitions, in which a system is in a statistical state μ_{in} *before* the dissipative phenomena and reaches a new statistical state μ_{out} *after* the phenomena. The second principle (Carnot-Clausius) then reads

$$S(\mu_{out}) \geqslant S(\mu_{in}) \tag{3.1}$$

whereas the first principle expresses the *conservation* of *the mean value of the energy* E, which can be written

$$\mu_{out}(x \mapsto E) = \mu_{in}(x \mapsto E) \tag{3.2}$$

taking the measures to be linear functionals.

Both μ_{in} and μ_{out} belong to the convex of Boltzmann states giving a given mean value Q to the energy. It can happen that, on this convex, the concave function S be bounded; let S_Q be then its upper bound. Obviously

$$S(\mu_{in}) \leqslant S(\mu_{out}) \leqslant S_Q \ . \tag{3.3}$$

This gives a majorant of $S_Q - S(\mu_{in})$, known as a function of μ_{in} only, to the *entropy production* $S(\mu_{out}) - S(\mu_{in})$. It can also happen that the maximum of S on this convex be reached at a unique point μ_Q, known as a *Gibbs state* ; if $\mu_{in} = \mu_Q$, the entropy production vanishes and $\mu_{out} = \mu_{in}$: so Gibbs states cannot undergo dissipative phenomena; they constitute what is known as *thermodynamic equilibria*.

IV. COVARIANT FORMULATION OF THE FIRST PRINCIPLE

The foregoing analysis applies to conservative systems; the function $x \mapsto E$, defined on a symplectic manifold U, permits by the Hamiltonian formalism to define a one-parameter group of symplectomorphisms of U[5]; calculations show that this group is lifted to the evolution space V by the group of time translations; in the case of a single particle

$$\vec{v} \to \vec{v}\ ,\ \vec{r} \to \vec{r}\ ,\ t \to t + \text{Cte} \qquad (4.1)$$

this is usually expressed somewhat incorrectly by saying that "time and energy are conjugate variables"[6].

Clearly translations (4.1) are linked to a particular frame: the first principle, as stated, does not respect relativistic covariance, even galilean[7] ;there must therefore exist a statement avoiding this drawback.

A radical solution is to replace the group (4.1) by the complete galilean group[8], or else in the relativistic case, by the Poincaré group[9].

The action of these groups on U by symplectomorphisms is defined in a natural way if the dynamical system is isolated; otherwise one considers a partial system, to which the "mechanism" made up by the given exterior system leaves only the symmetry corresponding to a subgroup of the galilean (or Poincaré) group. For example a fixed box containing a gas, which restricts the gas to the subgroup (4.1); but also a centrifugal machine, etc.

Let G then be the group of symmetries; we seek a quantity which plays the same role with respect to G as does the energy with respect to the group (4.1).

It is sufficient to achieve this to consider *all one-parameter subgroups of* G; each one will be characterized by an element Z of the Lie algebra $\mathcal{G}$ of G; to each one will correspond a hamiltonian which will be denoted by M(Z). Inspection shows that one can choose the additive constant which appears in each hamiltonian in such a way that the correspondence $Z \mapsto M(Z)$ be *linear;* M becomes thus a linear form on $\mathcal{G}$, thus an element of its dual $\mathcal{G}^\star$; there exists therefore an application $x \mapsto M$ of U into $\mathcal{G}^\star$; the variable M will be called the *moment* of the group; naturally therefore one replace the first principle (3.2) by the statement (19)

$$\mu_{out}(x \mapsto M) = \mu_{in}(x \mapsto M) \qquad (4.2)$$

without changing the second principle (3.1); the conclusions are similar. On the convex of Boltzmann states satisfying $\mu(x \leftrightarrow M)=Q^{(10)}$, there may exist a "Gibbs state" μ_Q having the largest entropy S_Q; as before, one obtains a majorant to the entropy production in a dissipative transition; and one arrives at the conclusion that Gibbs states are no longer susceptible to dissipative phenomena.

The distribution function of these Gibbs states is the exponential of an affine function of M, which can be written

$$\rho = e^{M\theta - z} \tag{4.3}$$

θ being an element of G ("geometric temperature"), z a number ("Planck's thermodynamic potential", see (VI)) which is obtained in terms of θ by writing that the weight of μ is 1,

$$z = \log \int_U e^{M\theta} \lambda(x)\, dx \,, \tag{4.4}$$

λ being the Liouville measure; z is a *convex* function of θ, which turns out to be the *Legendre transform* of $Q \leftrightarrow -S_Q$:

$$dQ\theta = - dS \,, \; Q\, d\theta = dz \,, \; Q\theta = z-S \,, \; \forall\, d. \tag{4.5}$$

All the classical formulae of thermodynamics are thus generalised but now the variables are provided with a geometrical status. For instance, the geometrical temperature θ, an element of the Lie algebra of the Galilea or Poincaré groups, can be interpreted as the *field of space-time vectors;* in the relativistic version, $\theta(x)$ is a *time like vector*, its orientation characterises the "arrow of time"; its direction is the 4-velocity of the equilibrium referential; its (Minkowski) length is $\beta = \frac{1}{kT}$ (k : Boltzmann's constant, T : absolute temperature). This temperature-vector had already been suggested by Planck in order to study relativistic thermodynamics, but its galilean counterpart is quite as relevant.

The formulae thus obtained can be applied correctly to a number of real situations : equilibrium of spin particles, centrifugal machines, rotation of celestial bodies, etc.

Furthermore new relations appear, linked to the non-commutativity of the group G, which give rise to some predictions; thus, under weak hypotheses, one can predict the existence of a *critical temperature* for an isolated system, beyond which no equilibrium state will exist; this fact is probably important in astrophysics (supernovae). For further details see (20) and (23).

V. GRAVITATIONAL SUSCEPTIBILITY

The covariant formulation (4.2) of the first principle removes thus a paradox, and at the same line increases the practical value of thermodynamics. However it leaves a conceptual problem.

During a dissipative transition, statistical mechanics is necessarily violated since $\mu_{out} \neq \mu_{in}$; the dynamical variable energy (or more generally the moment) *changes spectrum during the transition*[11]. As one can no longer appeal to conservation laws of classical or statistical mechanics, it is necessary to bring in *other laws of nature* in order to understand how the mean value of the energy is *conserved*, or more simply how it is *memorized*.

Somewhat unexpectly the answer is provided by general relativity; we shall see how in §7, after a study of preliminary concepts. Consider a dynamical system evolving in a *gravitational field*, field which is characterized in general relativity by its *potentials* $g_{\mu\nu}$; the space of all motions is always a symplectic manifold U, whose structure depends on the field.

Now we choose a compact K of space-time E_4 (see fig. II) wherein we perturb the $g_{\mu\nu}$. The new space of motions U' is still a symplectic manifold, which can be connected to U by the technique of *diffusion;* this technique will be described in the case of a spinless particle, whose motion is characterized by the world line; if this line does not meet K, it characterizes a motion equally in U as in U'.

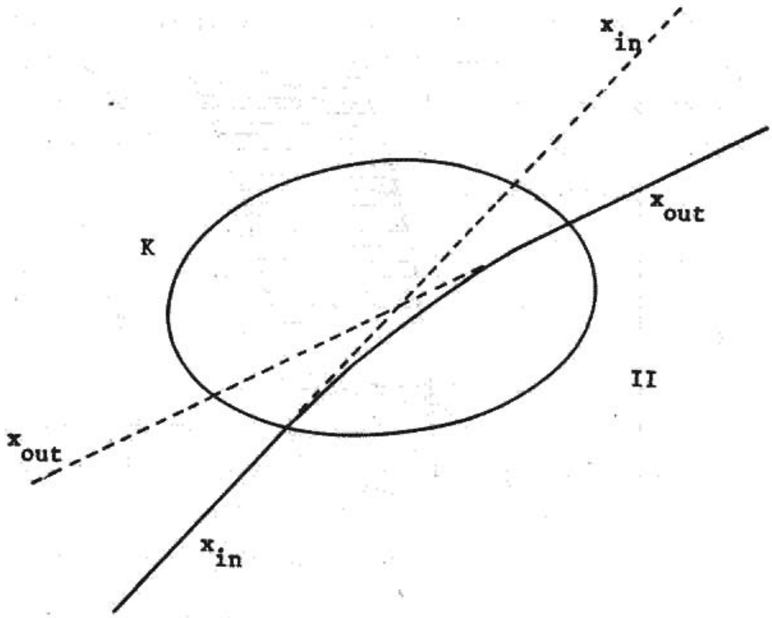
Consider now a motion in U, which we shall denote by X_{in} whose world line centers at some time into K; with the perturbed potentials, the line will deviate from the initial motion (dotted line in figure). When it leaves K, it takes a new path which can be identified to an element X_{out} of U; the correspondence $X_{in} \to X_{out}$, which characterizes globally *diffusion by scattering* is a local symplectomorphism of U (because U and U' are each symplectic, and their structures can be obtained by starting from the same evolution space). Such a quantity can be characterized by a certain dynamical variable called the diffusion *eikonal*.

We are interested here in *infinitesimal diffusion :* if one gives to the $g_{\mu\nu}$ a variation $\delta g_{\mu\nu}$ which vanishes outside K, the initial motion will undergo a displacement $\delta x = F(X)$ which derives from a certain hamiltonian ϕ (see §4); ϕ, so defined to within an additive constant, can be completely determined by putting it equal to zero on all paths which do not cross K[12].

For any motion $x \in U$, let T_x be the application which establishes a correspondence between ϕ and the tensor field $x \nrightarrow \delta g$ $(x \in E_4)$, T_x is a linear application, thus a priori a distribution[13]; knowing T_x allows one to predict how the particle will react to any "slight" modification of the gravitational field: this is why T_x will be called the *gravitational susceptibility* of the particle in the motion x.

We now use the *general relativity principle :* it states that a diffeomorphism A of time-space E_4, acting simultaneously on the potential (according to the standard formulae of differential geometry) and in the motion (here by direct image of the world line) is *unobservable* (see (21)). Let us choose A so that it leaves unchanged the points outside the compact K (fig. III); it modifies the potentials only in K, and its action in the particle leaves unchanged those parts of the world line outside K, the corresponding diffusion by scattering is thus zero.

Let us apply this result to the case $A = \exp(sF)$ (see 2.4), $s \in \mathbb{R}$, F = vector field vanishing outside K. One sees that $T_x(x \nrightarrow \delta g)$ vanishes if g is the derivative with respect to s, at s = 0, of the reciprocal image of A by $x \nrightarrow \delta g$; this variation by definition is the *Lie derivative* of g associated to the vector field $\delta X = F(X)$; we shall denote it as $\delta_L g$. Generally speaking a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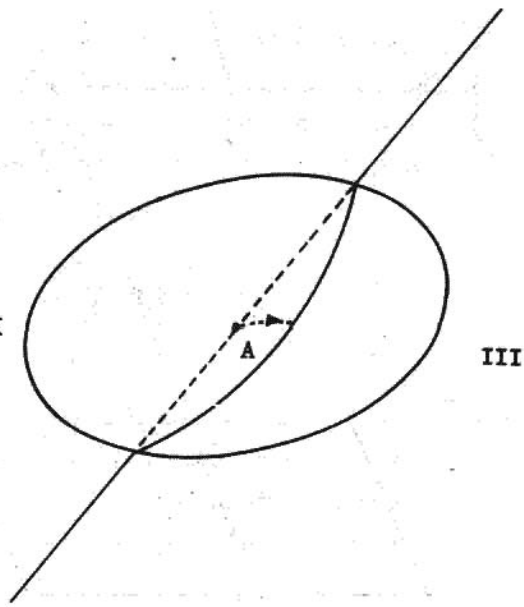
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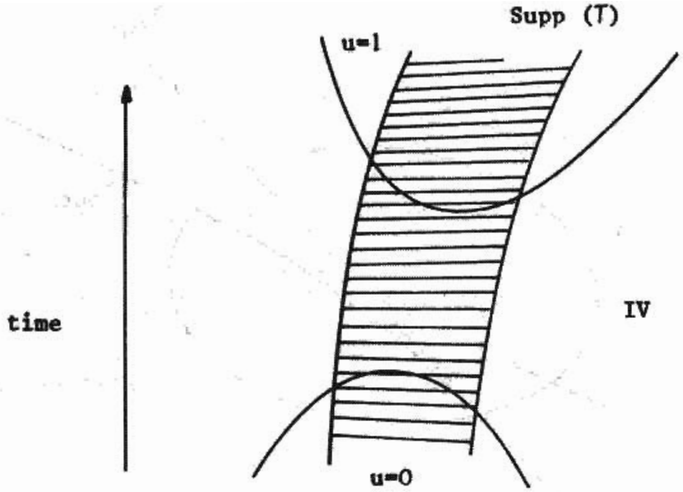
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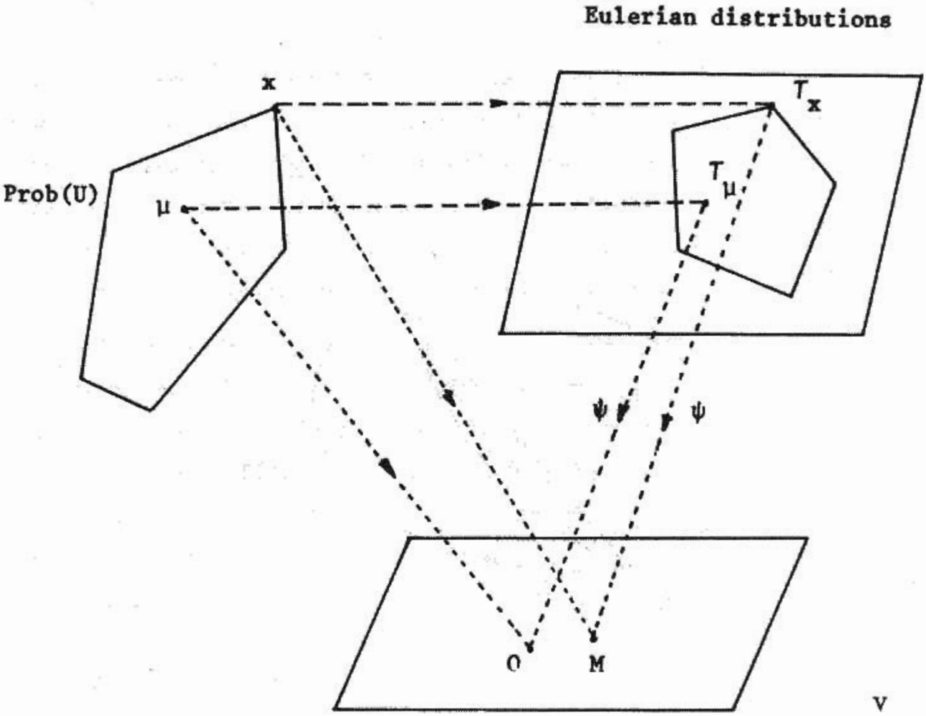
Eulerian distributions
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distribution T will be said to be *eulerian* if it satisfies the condition

$T(X \nrightarrow \delta_L g) = 0$ for any field $X \nrightarrow \delta X$ under compact support; (5.1)

we know therefore that $x \nrightarrow T_x$ is an application of the space of motions U into the vector space of eulerian distributions of E_4 (fig. V). Under certain hypotheses, a eulerian distribution allows one to associate a *conserved quantity* to any Killing vector Z of the metric g[13]; we shall give a brief description of this procedure in the case where E_4 is Minkowski space, and consequently Z is an element of the Lie algebra of the Poincaré group; the associated quantity is

$$I = T(X \nrightarrow \delta_L g) \text{ where } \delta X = uZ(x) \ , \qquad (5.2)$$

U being a function taken to be equal to zero in the past and equal to one in the future (fig. IV).

Contrary to what one may think by studying (5.1) and (5.2), I is not necessarily vanishing, because $X \nrightarrow u\, Z(x)$ is not a field with compact support; but the eulerian condition (5.1) allows one to show that I *does not depend on the choice of* u, by making some assumptions on the behaviour of T at infinity[14]. One can thus calculate I by making u jump from 0 to 1 in a small neighborhood of a space-like surface; the fact that the result is independent of the choice of this surface expresses the "preserved quantity" character of I.

Clearly the application $Z \nrightarrow I$ so defined is linear; it associates to T an element $\Psi(T)$ of the dual G^* of the Lie algebra G of the Poincaré group. It is obvious that the application

$$x \nrightarrow M = \Psi(T_x) \qquad (5.3)$$

is a moment of the Poincaré group (§4); we shall now find another property by *equivariance* considerations.

It is clear that the Poincaré group G acts on tensor fields with compact support, thus on distribution-tensors according to the formula

$$a(T)(a(X \nrightarrow \delta g)) = T(X \nrightarrow \delta g) \ , \quad \forall\, a \in G. \qquad (5.4)$$

It also acts on vector fields with compact support, and the Lie derivative of g is equivariant for this action, so that G acts on eulerian distributions. Finally G acts on Killing vectors, and this action coincides with the *adjoint representation* of G on $\mathcal{G}$.

Let us now suppose that a be an element of the *orthochrone* subgroup G↑(16) : the foregoing constructions show that

$$\Psi(a(\mathcal{T}))(a(Z)) = \Psi(\mathcal{T})(Z) \qquad (5.5)$$

and $\forall\, x \in U$, that

$$a(\mathcal{T}_x) = \mathcal{T}_{a(x)} \qquad (5.6)$$

whence

$$\Psi(\mathcal{T}_{a(x)})(a(Z) = \Psi(\mathcal{T}_x)(Z). \qquad (5.7)$$

This last formula has a *cohomological* interpretation : it expresses the vanishing of a certain symplectic cocycle and entails that (see(19)).

$$M[Z,Z'] = \sigma(Z(x))(Z'(x)) \ , \ \forall\, Z,Z' \in \mathcal{G}. \qquad (5.8)$$

This result allows one to fix the arbitrary constant which appeared in M (because the Lie algebra of the Poincaré group is equal to its derived algebra); it shows in particular that if Z is an infinitesimal time translation, then the integral I (5.2) is equal to the *relativistic energy* $E = mc^2$ of the system in the motion in question.

We have thus *factorized* the "moment" application $x \mapsto M$ by the composition of $x \mapsto T_x$ and $T_x \mapsto M$ (fig. V); this is the result which will be essential for thermodynamics. Let us indicate in detail what these results become for a single particle; the symplectic form (1.1), in general relativity, reads

$$\sigma_V(dy)(\delta y) = m\, g_{\mu\nu}[dX^\mu\ \hat{\delta U}^\nu - \delta X^\mu\ \hat{dU}^\nu] . \qquad (5.9)$$

In this case an initial condition y is a pair (X,U), X belonging to the world line, U being the unit tangent vector; the carets ^ represent covariant differentiation.

The calculation (somewhat technical) of the gravitational susceptibility yields

$$T_x(X \mapsto \delta g) = \int \frac{1}{2}\, m \quad \delta g_{\mu\nu}\, P^{\mu}\, \frac{dX^{\nu}}{ds}\, ds \qquad (5.10)$$

$S \mapsto X$ being an arbitrary parametrization of the world line of the particle (from the past towards future). We see that the distribution T_x is a *measure*, having as support this world line. It so happens that we know *all* the eulerian measures supported by a curve; they can be written as

$$T(X \mapsto \delta g) = \int \frac{1}{2}\, \delta g_{\mu\nu}\, P^{\mu}\, \frac{dX^{\nu}}{ds}\, ds \qquad (5.11)$$

with the supplementary conditions

$$\frac{dX}{ds} \text{ parallel to } P \ ; \ \frac{d\hat{P}}{ds} = 0 \qquad (5.12)$$

(the proof can be found in (2.1)); these conditions imply that the curve be a geodesic : a well-known fact for particles, which can also be found by using d'Alembert's principle $dy \in \ker(\sigma_V)$ in the form (5.9). The 4-momentum $P = mU$ appears thus as an element of the gravitational susceptibility; in the case of Minkowski space, the preserved quantity associated with an element Z of the Lie algebra G of the Poincaré group is

$$I = g_{\mu\nu}\, P^{\mu}\, Z(X)^{\nu} \qquad (5.13)$$

X being chosen arbitrarily on the world line; by varying Z in G , one can display energy, momentum, orbital momentum, etc.

The structures we have just displayed in the simplest case can be extended to a great variety of circumstances.

They can be transposed to the *classical mechanics* case; to each motion x we still associate a distribution T_x; the eulerian condition is expressed, no longer by a Lie derivative, but by a certain *connection* which takes into account the gravitational field in its newtonian form. One notes that the conserved quantities associated with the null field case bring out a Lie algebra of dimension 10 which is *not* that of the Galilean group, but somewhat paradoxically that of the *Caroll group,* which is a contraction of the Poincaré group obtained by letting the speed of light c go to zero. This phenomenon can be put together with the impossibility of choosing the galilean group moments in such a way as to satisfy formula (5.8) : an obstruction appears which is a class of symplectic cohomology and which is measured by the total mass of the system, thus non-vanishing.

Physically speaking, these facts indicate that in the formulation of classical mechanics by eulerian distribution, *mass conceals energy*.

The same method can be used to treat spin particles, both in classical and relativistic mechanics. The gravitational susceptibility involves, together with the 4-momentum P^{μ}, the antisymmetric *spin tensor* $s^{\mu\nu}$.

This method allows one to obtain in a simple way the collision and desintegration rules for particles: one has simply to write that the sum of the gravitational susceptibilities carried by the various world lines is still a eulerian distribution. This method can be extended to *electrodynamics* : one calculates the gravitational susceptibility for a simultaneous variation of the gravitation potentials $g_{\mu\nu}$ and the *electromagnetic potentials* A_{ρ}; for particles this susceptibility introduces, together with the 4-momentum and the spin tensor, the electric charge and the magnetic moment. The general relativity principle, which affected the group of diffeomorphisms in space-time, is generalized to an electromagnetic group; consequently the eulerian condition becomes

$$T(X \mapsto (\delta g, \delta A)) = 0 \text{ if } \delta g = \delta_L g \;,\; \delta A = \delta_L A + \frac{\delta\alpha}{\delta X} \quad (5.14)$$

$X \mapsto \delta X$ and $X \mapsto \alpha$ being a vector field and a scalar field with compact support respectively.

Note that these structures become particularly simple when written in the 5-dimensional space-time of Kaloza.

VI. LOCALISATION OF STATISTICAL STATES

Let μ be a statistical state of a dynamical system, i.e. a probability law of the manifold U of motions (fig. V).

If $X \nrightarrow \delta g$ is a variation with compact support of the gravitation potentials, we can establish a correspondence between every motion $X \in U$, and a diffusion Hamiltonian $\phi = T_x(X \nrightarrow \delta g)$; ϕ is a function of x, i.e. a dynamical variable; the state μ will be called *localisable* if this function is μ-integrable $X \nrightarrow \delta g$; we shall then put

$$T_\mu(X \nrightarrow \delta g) = \int_U T_x(X \nrightarrow \delta g)\mu(x)dx \quad ; \qquad (6.1)$$

this quantity is the *mean value* of ϕ is state μ.

One can check immediately that :

- the set of all localisable statistical states is a sub-convex of Prob(U), and contains its extremal points. (6.2)

-If μ is localisable, T_μ is a *eulerian distribution.* (6.3)

- In the case of Minkowski space, the element $\Psi(T_\mu)$ of g (see §5) is equal to the mean value Q of M in state μ(see fig. V). (6.4)

- It would appear that localisable states are the only ones met in nature; in particular Gibbs states are localisable. If the state μ has a distribution function of class $\mathbb{C}^\infty$ (on U), T_μ will be a *completely continuous* distribution (on E_4), which will be written

$$T(X \nrightarrow \delta g) = \int_{E_4} \frac{1}{2} \; \delta g_{\nu\rho} \; T^{\nu\rho} \; (X) \; dX \qquad (6.5)$$

The $T^{\nu\rho}$ being densities ($T^{\nu\rho} = T^{\rho\nu}$); these $T^{\nu\rho}$ are the components of a *tensorial density* in the Brillouin sense; they can also be written as $T^{\nu\rho}\lambda$, $T^{\nu\rho}$ now being the components of a symmetric tensor, and λ the riemannian density of space-line ; (6.5) becomes

$$T_\mu(X \leftrightarrow \delta g) = \int_{E_4} \frac{1}{2} T^{\nu\rho} \, \delta g_{\nu\rho} \, \lambda(X) \, dX \qquad (6.6)$$

the eulerian condition (5.1) is obtained from the Killing formula

$$[\delta_L g]_{\alpha\beta} = g_{\alpha\gamma} \; \hat{\partial}_\beta \delta X^\gamma + g_{\gamma\beta} \; \hat{\partial}_\alpha \, \delta X^\gamma \qquad (6.7)$$

one easily finds

$$\hat{\partial}_\nu T^{\nu\rho} = 0 \qquad (6.8)$$

where one recognizes the relativistic form of the Euler equations proposed by Einstein (4).

Consequently the localisation of a statistical state allows one to interpret it with the assistance of a continuous media, whose $T^{\nu\rho}$ defined by t6.6) make up the *energy tensor* and are *automatically* solutions of the Euler equations. This interpretation can be confirmed by detailed calculations; thus in the case of a particle, the T^{oo} component, which is interpreted as the *specific mass*, is the mean value of the relativistic mass $m/\sqrt{1-V^2/C^2}$ in a volume element in the neighborhood of the point X considered; the pressure, or more generally, the constraint tensor is interpreted as a measure of the random character of the speeds of the motions going through X; etc.

Let us consider the case of a system of N relativistic spinless particles of mass m, making up a Gibbs equilibrium in a box of volume V at a temperature T. In a frame linked to the box, the $T^{\nu\rho}$ tensor is diagonal, and can be expressed in terms of a *specific mass* ρ and a pressure P given by

$$\rho = \frac{Nm}{V} \frac{G''(x)}{-G'(x)} \quad p = \frac{Nm}{V} \frac{G''(x) - G(x)}{-3G'(x)} \qquad (6.9)$$

where $x = m/kT$, and G is defined by

$$G(x) = \int_1^\infty e^{-xs} \sqrt{s^2-1}\, ds. \qquad (6.10)$$

It so happens that $G(x) = \frac{K_1(x)}{x}$, K_1 being the modified Bessel function of order 1; G satisfies therefore the differential equation

$$G''(x) + \frac{3}{x} G'(x) - G(x) = 0 \qquad (6.11)$$

whence

$$pV = NkT; \qquad (6.12)$$

one recovers thus *exactly* the classical perfect gas law (Boyle-Mariotte-Charles-Gay-Lussac-Avogadro - Boltzmann); the first terms of the asymptotic expansion of K_1 (27) yield the formula

$$Q = \rho V = Nm + \frac{3}{2} NkT = \frac{Nm}{-2G'(m/kT)} \int_1^\infty e^{-ms/kT} (s-1)^{\frac{5}{2}} (s+1)^{\frac{1}{2}} ds. \qquad (6.13)$$

Here we have set c=1, the first term is the mass at absolute zero; the second term is the *mean classical value of the energy*, which allows one to calculate the specific heat of a monoatomic gas; the third (positive) term is the relativistic correction.

One also obtains Planck's thermodynamic potential (4.4)

$$z = N \log(-4\pi V m^3 G'(m/kT) F \qquad (6.14)$$

which indeed is a convex function of $\beta = 1/kT$; formula (4.5) then yields the entropy[17]

$$S = Kz + \frac{Q}{T} \qquad (6.15)$$

It is remarkable that the $T^{\nu\rho}$ tensor, which has been constructed as characteristic of the *gravitational susceptibility*, also characterizes the *gravitational action* of matter defined by the statistical state μ; for it is indeed this tensor which appears in the right- hand side of the Einstein gravitation equations :

$$R^{\nu\rho} - \frac{1}{2} Rg^{\nu\rho} + \Lambda\, g^{\nu\rho} = 8\pi G T^{\nu\rho} \qquad (6.16)$$

(Λ cosmological constant; G = Newton's constant; c = 1).

Using the definition (6.1), these equations can be written

$$\int_{E_4} \delta Z(X)dX = \int_U T_x(X \nrightarrow \delta g)\ \mu(x)dX\ ,\ \forall\ \delta g \qquad (6.17)$$

Z being the lagrangian density of the gravitational field

$$Z = \frac{2\Lambda - g^{\nu\rho}R_{\nu\rho}}{8\pi G}\ \lambda(\lambda = \text{riemannian density}) \qquad (6.18)$$

in this form, one notes that the distribution defined by the first member is *automatically* eulerian.

All this approach can be extended to the electromagnetic case; formulae (6.6) and (6.8) become

$$T_\mu(X \nrightarrow (\delta g, \delta A)) = \int_{E_4} [\tfrac{1}{2}\ T^{\nu\rho}\delta g_{\nu\rho} + J^\sigma \delta A_\sigma]\ \lambda(X)dX \qquad (6.19)$$

and

$$\hat{\partial}_\nu T^{\nu\rho} + F_\nu{}^\rho J^\nu = 0 \qquad \hat{\partial}_\sigma J^\sigma = 0 \qquad (6.20)$$

respectively.

Einstein's equations (6.16) are replaced by the coupled Einstein-Maxwell equations; the 4-vector J^σ is interpreted as the current-charge density.
Applying the preceding formulae to statistical states of particles with spin having a magnetic moment allows one to recover the principal characteristics of ferromagnetism (magnetic equivalence magnet-solenoid); gyromagnetic effects; magnetostriction ...) (see (21) and (23)).

VII. GRAVITATIONAL INTERPRETATION OF THE FIRST PRINCIPLE

Consider the case of a dissipative transition $\mu_{in} \to \mu_{out}$, and let us suppose there exists a *eulerian distribution* T that coincides with $T\mu_{in}$ before the *dissipative phenomena* and with $T\mu_{out}$ afterwards; in other words a distribution T that interpolates between $T\mu_{in}$ and $T\mu_{out}$.

We can then associate to Q a conserved quantity Q which will acquire the same value for $T\mu_{in}$ as for $T\mu_{out}$, as it can be calculated at an arbitrary time; we also know that for $T\mu_{in}$, Q is equal to the mean value of the Poincaré moment M; similarly for $T\mu_{out}$; consequently the *first principle,* in its covariant form (4.2) will be assured. We only need to set (notation 6.17)

$$T(X \nrightarrow \delta g) = \int_{E_4} \delta\mathcal{L}(X)dX \; ; \qquad (7.1)$$

indeed it is known that T is a eulerian distribution that interpolates between $T\mu_{in}$ and $T\mu_{out}$, as the Einstein equations (6.16) are valid before and after the dissipative phenomena[18]. It is thus the gravitational potentials g that remember the mean value of the moment M and which guarantee the validity of the first principle (in its covariant form (4.2)).

VIII. THE DISSIPATIVE MEDIUM MODEL

Relativity, thermodynamics, matter and geometry

Consider a Gibbs state of a particle in a symmetrical gravitational field; there exists in space-time a conservative current[19] S whose integral on a space-like hypersurface is equal to the statistical entropy

$$S^\mu = T^{\mu\nu}\theta_\nu - F\theta^\mu \qquad (8.1)$$

T being given here by the construction explained in §6, F is the specific free energy of the system [20] and θ is the *temperature vector*. The conservation of the *entropy current* amounts to the equation

$$\partial_\mu S^\mu = 0. \qquad (8.2)$$

In the case of a dissipitative continuous medium, we shall assume as so many authors (2), (3), (8) ... (12) that the geometrization of the second principle is obtained by the permanence, out of equilibrium, of the entropy current, whose flux S satisfies then

$$\hat{\text{div}}\; S = \hat{\partial}_\mu S^\mu \geqslant 0. \qquad (8.3)$$

At any point div S is interpreted as *the specific entropy production.*

Still in the case of dissipative processes, we shall also assume the permanence of the temperature vector, a less strong hypothesis than the often adopted one of local thermodynamic equilibrium.

The duality between the momentum-energy current and the universe metric on the one hand, and the geometrical nature of the entropy current and the temperature vector on the other suggests there exists a duality entropy-temperature that a complete geometrization of the second principle should clarify.

relativity	thermodynamics	
T	S	matter
$\updownarrow$	$\updownarrow$	
g	θ	geometry

The vectorial character of θ makes it the infinitesimal generator of a one parameter group and thus gives it a strictly *kinematic* role. Its lines of current are the molecules, their abstract set makes up a manifold V_3 called the reference body which corresponds to our three dimensional intuition of "space".

Programm for a model

To construct a model we have to choose a system of fundamental variables from which will be written the equations of motion. What we shall ask of a *thermodynamic* model of a continuous medium is that it takes into account the *kinematic variables*

$$(g,\theta)$$

and the dual variables

$$(T,S)$$

in such a way that all the solutions of the equations of motion satisfy the two principles of thermodynamics

$$\hat{\partial}_\mu T^{\mu\nu} = 0 \text{ and } \partial_\mu S^\mu \geqslant 0 .$$

Furthermore we shall require that the motions contain as special case the Gibbs equilibria for which θ is a Killing vector and S is given by (8.1) and $\partial_\mu S^\mu = 0$.

The characterization of equilibria through infinitesimal isometries emphasizes the role of the tensor γ defined as

$$\gamma = \frac{1}{2}\,\delta_L g \ ; \ \delta X = \theta$$

i.e.

$$\gamma_{\mu\nu} = \frac{1}{2}[\hat{\partial}_\mu\theta_\nu + \hat{\partial}_\nu\theta_\mu]$$

γ will be called the *friction tensor*.

It seems reasonable to interpret γ as the source of *dissipative phenomena* as thermodynamic equilibria are characterized by $\gamma = 0$.

A simple phenomenological model satisfying this programm

Classical thermodynamic media are characterized by a function of state, the dissipation function (ex. cf (18)) relating the constraints to the amount of deformation. By noticing that the amount of deformation and the constraint are the spatial parts of the friction tensor γ and the momentum-energy tensor T respectively, it seems reasonable to generalize this hypothesis to relativity by assuming the existence of a *generating function* ϕ[21] relating γ and T.

More precisely we suppose that

ϕ depends on the variables $(x,g,\theta,\gamma,q,\frac{\partial q}{\partial x})$ where q represents the molecule of V_3 going through x

$$T^{\mu\nu} = \frac{\partial\phi}{\partial\gamma_{\mu\nu}}\ ,\ \text{i.e. } \delta\phi = T^{\mu\nu}\delta\gamma_{\mu\nu},\ \text{for any variation} \qquad (8.6)$$

of γ, the other variables kept fixed.

In order to satisfy the principle of general relativity, we shall suppose that ϕ is invariant[22] under diffeomorphisms of V_4. We take this statement to be the rigorous expression of the *principle of objectivity* or of *material indifference*, proposed by many authors in the framework of classical mechanics (16),(17). Calculations show that ϕ depends only on the following variables:

molecule	q	
reciprocal of the temperature	$\beta = [g_{\mu\nu}\theta^{\mu}\theta^{\nu}]^{1/2}$	
conformation	$h^{ij} = g^{\mu\nu}\partial_{\mu}q^{i}\partial_{\nu}q^{j}$	
deformation rate	$k^{ij} = -2\gamma^{\mu\nu}\partial_{\mu}q^{i}\partial_{\nu}q^{j}$	(8.7)
thermal field	$\chi^{i} = 2\partial_{\mu}q^{i}\gamma^{\mu}{}_{\rho}\theta^{\rho}$	
thermal velocity	$a = \frac{1}{\beta}\gamma_{\mu\nu}\theta^{\mu}\theta^{\nu}$	

We have proved the following theorem (22), (7) :

if ϕ is convex on γ ;

if there exists a state function F which depends only on q,β,Q, such that

$$\frac{\partial\phi}{\partial\gamma_{\mu\nu}} = \frac{\partial F}{\partial g_{\mu\nu}} \quad (23) \qquad (8.8)$$

and if one sets (24)

$$S^{\mu} = T^{\mu\nu}\,\Theta_{\nu} - F\theta^{\mu} \qquad (8.9)$$

then the equations of motion defined by the Einstein equations

$$R^{\mu\nu} - \frac{1}{2}\,Rg^{\mu\nu} = 8\,\pi\,GT^{\mu\nu} \qquad (8.10)$$

generate the realization of the two principles of thermodynamics

$$\hat{\partial}_{\mu}T^{\mu\nu} = 0 \quad \text{and} \quad \hat{\partial}_{\mu}s^{\mu} \geqslant 0.$$

IX. INTERPRETATION OF THE MODEL

The friction tensor γ

The thermodynamic variables k,χ and a of table 8.7 are built up from the friction tensor γ; one can show that, along the motion, they take as values :

$k^{ij} = \frac{dh^{ij}}{ds}$, with $\frac{dx}{ds} = \theta$; k measures thus the variation, along the lines of current, of the deformation which justifies it being called the rate of deformation.

$a = \frac{d\beta}{ds}$; $\frac{dx}{ds} = \theta$; "a" measures the variation of temperature along the lines of current.

χ^i is the relativistic equivalent of the classical temperature gradient.

Momentum-energy tensor and entropy

The calculation of T and S yields

$$T_{\mu\nu} = \alpha U_\mu U_\nu - 2\partial_\mu q^i \partial_\nu q^j \Lambda_{ij} + (U_\mu \partial_\nu q^i c_i + U_\nu \partial_\mu q^i c_i)$$
$$S^\mu = (\alpha - F)\theta^\mu + \beta g^{\mu\nu}\partial_\nu q^i c_i \qquad (9.1)$$

where

$\alpha = \frac{\partial\phi}{\partial a}$ is the internal energy in the normal sense of thermodynamics.

$c_i = \frac{\partial\phi}{\partial\chi_i}$ is the heat convexion.

$\Lambda_{ij} = \frac{\partial\phi}{\partial k^{ij}}$ is the constraint.

The expression for the entropy shows that the non convective heat remains, as in usual thermodynamics, the difference between internal and free energy; furthermore the model distinguishes quite naturally that part of the entropy flux vector which is proportional to θ and its orthogonal part, the convective heat flux.

X . NON DISSIPATIVE LIMIT OF THE MODEL

It follows from the strict convexity of the generating function ϕ that the only non dissipative motions of this model are the thermodynamic equilibria for which the friction vanishes. Nevertheless the hypothesis ϕ affine in γ gives the non dissipative approximation for which every motion satisfies

$$\hat{\partial}_\mu \delta^\mu = 0. \qquad (10.1)$$

The energy-momentum tensor becomes

$$T_{\mu\nu} = \beta \frac{\partial F}{\partial\beta} U_\mu U_\nu + F g_{\mu\nu} - 2\partial_\mu q^i \partial_\nu q^j \frac{\partial F}{\partial h^{ij}} \qquad (10.2)$$

and the entropy

$$S^\mu = \frac{\partial F}{\partial\beta}\theta^\mu \qquad (10.3)$$

and one recovers usual thermodynamics.

Introducing a matter current characterized by its flux N

$$N^{\mu} = nU^{\mu}$$
$$\partial_{\mu}N^{\mu} = 0 \tag{10.4}$$

allows one, by putting

$$W = F - n\,c^{2} \tag{10.5}$$

to write

$$T_{\mu\nu} = (nc^{2}+Q)U_{\mu}U_{\nu} + wg_{\mu\nu} - 2\partial_{\mu}q^{i}\partial_{\nu}q^{j}\,\frac{\partial}{\partial h^{ij}} \tag{10.6}$$

$$S^{\mu} = Q\theta^{\mu} \;,\; Q = \beta\frac{\partial W}{\partial \beta}$$

The internal energy α becomes then

$$\alpha = nc^{2} + Q + W \tag{10.7}$$

and Q can be written, by setting $S^{\mu} = SU^{\mu}$ and $\beta = \frac{1}{kT}$,

$$Q = \frac{S}{kT} \;.$$

The expression (10.6) for the energy-momentum tensor allows one to interpret W as the elastic energy of the relativistic theory of elasticity. Formula (10.8) leads to the interpretation of Q as the usual heat of thermodynamics, and one recovers in (10,7) the usual expression for the internal energy.

Thus the non dissipative limit of the model encorporates completely the usual definitions and equations of thermodynamics. Furthermore the equations of motion $\partial_{\mu}T^{\mu\nu} = 0$ and $\partial_{\mu}S^{\mu} = 0$ express the relativistic balance of thermodynamics of reversible process of energy, entropy ... and *one recovers the variational theory of elastic media.*

The equations $\hat{\partial}_{\mu}S^{\mu} = 0$ and $S^{\mu} = \beta\frac{\partial F}{\partial \beta}\theta^{\mu}$ allow one to eliminate the temperature β as independent variable (by using a Legendre transformation). It is then possible to construct a Lagrangian density depending only on q and h from which the equations of motions can be derived. This lagrangian coincides with that of the relativistic theory of elasticity (24). Going from this lagrangian to the classical limit gives Hamilton's principle as applied to continuous media.

Perfect fluids and statistical mechanics

The perfect fluid is obtained by letting F depend on h only through its determinant, i.e. the matter density n (10.4) (see (24), (15)). In this case, we have

$$T = \frac{\partial[\beta F]}{\partial\beta} U_\mu U_\nu - (\frac{\partial F}{\partial n} n - F)(g_{\mu\nu} - U_\mu U_\nu)$$
$$S^\mu = \sigma N^\mu \ , \ \sigma = \zeta - \frac{\partial\zeta}{\partial\beta}\beta \ , \zeta = -\frac{\beta F}{n} \tag{10.9}$$

and the pressure p

$$p = \frac{\partial F}{\partial n} n - F \tag{10.10}$$

appears as the Legendre transform of F and is expressed in terms of the chemical potential $\mu = \frac{\partial F}{\partial n}$.

Furthermore, by interpreting ζ as the specific Planck potential per molecule, one recovers for the entropy per molecule σ the expression given by statistical mechanics (8.6). This model is thus, in the non-dissipative limit, in agreement with the predictions of statistical mechanics. Introducing the 1-form

$$H_\mu = hU_\mu \ , \text{ with } h = \frac{p+\rho}{\eta} \quad (25) \tag{10.11}$$

and its exterior derivative

$$\Omega_{\lambda\mu} = \partial_\lambda H_\mu - \partial_\mu H_\lambda \tag{10.12}$$

we can replace the equations of motion (10.4) and (8.4) by

$$\hat{\partial}_\mu N^\mu = 0 \ , \ \Omega_{\mu\nu}\theta^\mu + \partial_\nu s = 0.$$

This last equation shows that

$$\delta S = 0 \text{ and } \delta_L \Omega = 0 \quad \text{for } \delta X = \theta. \tag{10.14}$$

It follows that s is constant along each line of current and that Ω is an integral invariant of the field $X \nrightarrow \theta$; its rank (4,2 or 0) is thus constant along each line of current, as well as the pseudo-scalar

$$\pi = pf(\Omega)\ \frac{\beta}{n} \tag{10.15}$$

where $pf(\Omega)$ is the pfaffian of the form [26]. In general $\pi \neq 0$, Ω is of rank 4 and the sign of π defines an orientation of space. There exist also important classes of motions for which $\pi = 0$.

- The isentropic motions (those in which s keeps the same value along all lines of current); then (10.13) shows that $\theta \in \ker(\Omega)$; in general the rank of Ω is 2; the kernel of Ω defines a foliation whose leaves of dimension 2, can be interpreted as vortex lines carried away by the fluid. These notions are barotopic: there exists an equation of state, indexed by the value of s, obtained by eliminating β and n between p and ρ : the particular enthalpy h (10.11) coincides with the index defined by Lichnerowicz (15).

- Non isentropic motions in which rank $(\Omega) = 2$ (it is sufficient that this be true at some arbitrary time); they make up the relativistic equivalent of the *oligotropic* motions of Casal (1); the "leaves" of Ω are described on the hypersurfaces $S = C^{st}$.

- The motions where $\Omega = 0$ (here again it is sufficient to verify this at some arbitrary time); (10.13) shows that they are isentropic; they constitute the *irrotational* notions, in the sense of Lichnerowi.z (15).

There exist solutions with discontinuities on a hypersurface Σ' of V_4 (*shock waves*); the conditions obtained by writing equations (10.4) in the sense of distributions read (27)

$$
\begin{aligned}
&N'^{\lambda} - N^{\lambda} \quad \text{is tangent to } \Sigma \\
&H'_{\lambda} - H_{\lambda} \quad \text{is normal to } \Sigma \qquad\qquad (10.16) \\
&h'^{2} - h^{2} = [u'h'+uh][p'-p] \ ;
\end{aligned}
$$

if one adds that the discontinuity S'-S is positive, one obtains the shock equations of Rankine-Hugoniot in their relativistic form.

XI. WEAKLY DISSIPATIVE MOTIONS

The generating function ϕ can be put in general, taking into account condition (8.8), in the form

$$
\begin{aligned}
&\phi = \overset{\circ}{T}{}^{\mu\nu}\gamma_{\mu\nu} + \phi \\
&\overset{\circ}{T}{}^{\mu\nu} = \frac{\partial F}{\partial g_{\mu\nu}} \ \text{and} \ \left.\frac{\partial \phi}{\partial \gamma_{\mu\nu}}\right|_{\gamma = 0} = 0
\end{aligned} \qquad (11.1)
$$

ϕ is called the dissipation function as one can show that

$$\hat{\text{div}}\, S = \frac{\partial\phi}{\partial\gamma_{\mu\nu}}\gamma_{\mu\nu}. \qquad (11.2)$$

The quadratic approximation of ϕ in terms of γ is what is called the weakly dissipative approximation of the model

$$\phi = \frac{1}{2}[\lambda a^2 + E_{ij}\chi^i\chi^j + F_{ij,im}k^{ij}k^{lm} + 2aB_i\chi^i + 2aL_{ij}k^{ij} + 2R_{i,jL}\chi^i k^{ji}] \qquad (11.3)$$

(λ,E,F,B,L,R) are the dissipation coefficients; they are functions of (q,β,h) and total 55.

We recover the thermal conduction tensor E and viscosity tensor F; their components satisfy of course the Onsager symmetry relations :

$$E_{ij} = E_{ji}$$
$$F_{ij,hl} = F_{ji,hl} = F_{ij,lh} = F_{lh,ij} . \qquad (11.4)$$

To these coefficients, the model adds :

- λ, which we shall call the thermal susceptibility;
- the tensors B_i, L_{ij}, $R_{i,jk}$ which couple the effects of conduction and susceptibility, viscosity and susceptibility, conduction and viscosity , respectively.

Their components satisfy the symmetry relations

$$L_{ij} = L_{ji}$$
$$R_{i,jk} = R_{i,kj} . \qquad (11.5)$$

The reader is referred to (7) for the expression of the momentum-energy tensor and the equations of notion.

Furthermore, the limit $\lambda = 0$, $L = 0$, $R = 0$ and $B = 0$ lead in the Newtonian approximation to the Fourier heat equations and the Navier viscosity equations. Note that the conservation equations $\partial_\mu T^{\mu\nu} = 0$ lead, taking into account the convexity relations:

$$\lambda > 0$$
$$\lambda E - B \otimes B > 0 \qquad (11.6)$$

a system of partial differential equations of the elliptic kind. The elliptic character of these equations seems inevitable in as much as we have set ourselves at a macroscopic level. Taking into account derivatives of higher order of the kinematic variables (e.g. capillarity) would change of course the nature of the system of equations; one should not therefore give any fundamental interpretation to the fact that the system be elliptic or hyperbolic.

NOTES

(1) Unless mentioned explicitly, all functions considered in this paper will be taken to be C^∞ ; in particular, $(\vec{r},t) \to \vec{F}$.

(2) i.e. $\sigma_V(dy)(\delta y) = 0, \forall\, \delta y$: this is the generalization of d'Alembert principle.

(3) A *density on a manifold* is a function f defined on the frames R and satisfying $f(RM) = f(R)|\det(M)|$ for any matrix M; on a symplectic manifold, there exists a density f_o, the *Liouville density* such that $f_o(R) = 1$ for any canonical frame. One can define the integral of a density with compact support on a manifold independently of any coordinate system; this allows one to identify each field of densities with a measure.

(4) By construction, ρ is a function defined on U; it is thus lifted on V through a first integral of the equations of motion; if one chooses an identification of the various phase spaces, this implies that ρ is a solution of Liouville's equation.

(5) A vector field F defined on a Hausdorff manifold U can be associated to the differential equation $\frac{dx}{ds} = F(x)$; the solution of this equation, which equals x_o for $S = 0$, is written as $\exp(sF)(x_o)$; if it exists $\forall\, x_o \in U$ and $\forall\, S \in \mathbb{R}$, F is said to be complete, then $S \mapsto \exp(sF)$ is a morphism of the group $(\mathbb{R},+)$ in the group of diffeomorphisms of U. If U is symplectic, and if $X \mapsto U$ is C^∞ on U, then the *symplectic gradient* of the dynamical variable U is the vector field F defined by $\sigma(\delta x)(F(x)) = \delta U, \forall\, \delta$; the associated equation is the Hamilton equation; $\exp(sF)$ preserves the symplectic form σ and is therefore called a symplectomorphism.

(6) With the usual sign conventions, E must be replaced by $-E$.

(7) More precisely, these transformations (4.1) define a subgroup of the galilean group which is not an invariant subgroup.

(8) This is the Lie group, of dimension 10, generated by the isometries of $\mathbb{R}^3$: time-translations and galilean transformations $\vec{r} \to \vec{r} + \vec{a}t$, $\vec{v} \to \vec{v} + \vec{a}$.

(9) The group of isometries in Minkowski space, also of dimension 10.

(10) Q is an element of $\mathcal{G}^*$, which generalizes the usual concept of "heat".

(11) In non-quantum statistical mechanics, the *spectrum* of a dynamical variable u in a statistical state μ is the *image* by $x \mapsto u$ of the probability law μ; it is a probability law of $\mathbb{R}$ (or of $\mathcal{G}^*$ in the case of the moment).

(12) Another method which avoids certain topological difficulties, appeals to the *prequantization* algorithm (see (23)).

(13) The trial variable $x \mapsto \delta g$ being a covariant tensor field, T_x is called a distribution-tensor (**contravariant**).

(14) Z is said to be a Killing vector if exp(sZ) is an isometry $\forall$ s.

(15) The easiest way is to suppose that the support of T is *compact in space*, i.e. its intersection with any time slice $t_o \leqslant t \leqslant t_1$, being the line in an arbitrary Lorentz frame, is compact. This condition is satisfied by the T_x we have considered for a particle (provided it is not a tachyon !).

(16) The connected component of the neutral element of a Lie group is an invariant subgroup; the quotient by this subgroup is the *component group*. In the case of the Poincaré group, the component group is the Klein group (4 elements) which is abelian. Elements which "respect the orientation of time" make up the union of two components; they form thus an invariant subgroup called the orthochrone group.

(17) Usual entropy is the product by k of the quantity used here. Temperature units can always be chosen so as to make k=1.

(18) One will note that this argument relies implicitly on an approximation; on the one hand, one uses special relativity to construct the Poincaré moments; on the other hand one considers the $g_{\nu\rho}$ as variables as they give by differentiation the $T^{\nu\rho}$ through the Einstein equations. This approximation amounts to taking G to be small and to neglecting the gravitational self-interaction of the system; this is customary in thermodynamics.

(19) This current is defined by a 3-form on the reference manifold and lifts to space-time through a vector S by $vol_4(S)$, where vol_4 is the riemannian volume.

(20) $F = - Z/\beta$; Z is the specific Planck potential ; $\theta^\mu = \beta U^\mu$ $U^\mu U_\mu = 1$.

(21) ϕ must be understood as a function with density value.

(22) covariant, in the sense of densities, is the exact term.

(23) ϕ and F are still to be taken in the sense of densities.

(24) Definition which generalizes beyond equilibrium formula (8.1) established for Gibbs states.

(25) Because of the relativistic equivalence between specific mass and specific energy (c=1), h can be interpreted as the *enthalpy per particle*.

(26) This pfaffian is defined by $\frac{1}{2}\,\Omega \wedge \Omega = pf(\Omega)\,vol$, where vol represents the riemannian volume form defined through an orientation of V_4. π is the relativistic equivalent of the vortex potential of Ertel (5,6).

(27) Dasked variables are taken after the shock.

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