EVERY SYMPLECTIC MANIFOLD IS A (LINEAR) COADJOINT ORBIT

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Abstract. We prove that every symplectic manifold is a coadjoint orbit of the group of automorphisms of its integration bundle, acting linearly on its space of momenta, for any group of periods of the symplectic form. This result generalizes the Kirilov-Kostant-Souriau theorem when the symplectic manifold is homogeneous under the action of a Lie group and the symplectic form is integral.

Introduction

It is well known since Kostant, Souriau and Kirillov [Kos70] [Sou70] [Kir74] that a symplectic manifold (X, ω), homogeneous under the action of a Lie group, is isomorphic — up to a covering — to a possibly affine coadjoint orbit.

It is less known that any symplectic manifold[1] is isomorphic to a coadjoint orbit of its group of symplectomorphisms (or Hamiltonian diffeomorphisms), possibly affine [PIZ16]. This has been established, in particular, in the rigorous framework of diffeology and uses essentially the notion of Moment Map for that category [PIZ10]. But this theorem still seems to lack something. Although this is not a fundamental flaw, we would like to get rid of the affine action, defined by a twisted cocycle of the automorphisms. We would prefer to identify the symplectic manifold with an ordinary coadjoint orbit, that is an orbit of the usual linear coadjoint action.[2] This can be achieved by passing to a central extension of the group of automorphisms.

We recall that we are no longer in the classical framework but in diffeology and we shall see that the difficulty to absorb this cocycle in an extension of the automorphisms disappears in this category by the capacity to treat irrational tori. The fundamental element is the *integration bundle* existing for any symplectic manifold, as it has been established in the paper "La Trilogie du Moment" [PIZ95]. This is a principal fiber bundle over the manifold, with group the *torus of periods* of the symplectic form, quotient of the real line by the *group of periods*, *i.e.* the integrals of the 2-form on every 2-cycle. This principal

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[1]The manifolds are assumed to be connected, Hausdorff and second countable.

[2]Thanks to François Ziegler who first suggested to make this improvement.

bundle comes equipped with a connection form, with curvature the symplectic form. Of course, the torus of periods is almost never a manifold, but it is still a non-trivial diffeological group [DI83] [IZL90]. We establish first the following

THEOREM 1. — *Let* (X, ω) *be a symplectic manifold. Let* P_ω *be its group of periods and* $T_\omega = \mathbf{R}/P_\omega$ *be its torus of periods. Let* $\pi : Y \to X$ *be an integration bundle equipped with a connection form* λ *with curvature* ω*. Let* $\mathrm{Aut}(Y, \lambda)$ *be the identity component of the group of automorphisms of the integration structure. Then, the kernel of the projection* $\mathrm{pr} : \mathrm{Aut}(Y, \lambda) \to \mathrm{Diff}(X, \omega)$ *is reduced to the action of the torus* T_ω*, and its image is the group* $\mathrm{Ham}(X, \omega)$ *of Hamiltonian diffeomorphisms. In other words, we get an exact sequence of homomorphisms, which is a central extension:*

$$\mathbf{1} \longrightarrow T_\omega \longrightarrow \mathrm{Aut}(Y, \lambda) \longrightarrow \mathrm{Ham}(X, \omega) \longrightarrow \mathbf{1}.$$

The group of Hamiltonian diffeomorphisms is defined precisely in terms of diffeology in [PIZ13, §9.15]. Then, denoting by $\mathscr{A}^*$ the space of momenta of the group $\mathrm{Aut}(Y, \lambda)$, that is, its space of left-invariant 1-forms, we prove the following theorem which reveals the universal model of symplectic manifolds.

THEOREM 2. — *The Moment Map* $\mu_Y : Y \to \mathscr{A}^*$ *of the action of* $\mathrm{Aut}(Y, \lambda)$ *on* $(Y, d\lambda)$*, is equivariant with respect to the coadjoint action:* $\mu_Y(\varphi(y)) = \mathrm{Ad}_*(\varphi)(\mu_Y(y))$ *for all* $\varphi \in \mathrm{Aut}(Y, \lambda)$*, and invariant by* T_ω*. Its projection* $\mu_X : X \to \mathscr{A}^*$ *is injective and identifies* X *with the orbit* $\mathscr{O}_\lambda = \mu_Y(Y) = \mu_X(X)$*. Therefore, every symplectic manifold is a coadjoint orbit of a linear action of a diffeological group, at least* $\mathrm{Aut}(Y, \lambda)$*.*

NOTE 1. — The idea that every symplectic manifold is a coadjoint orbit of its group of symplectomorphisms (or Hamiltonian diffeomorphisms) is not new. It appeared already at an early age of symplectic mechanics, a few decades ago. It is mentioned for example, in a functional analysis context, by Marsden & Weinstein in their paper on Vlasov equation [MW82, Note 3, p. 398]. It was taken up later by Omohundro, a Weinstein student, in his book on geometric perturbation theory in physics [Omo86, p. 364]. What was already original in the first paper [PIZ16] was the rigorous diffeology framework in which the result was proved, the role of the universal moment map, the identification of Souriau's cocycle of the action of the automorphisms and the affine rather than linear action if the cocycle is not trivial. What is original in this paper is that the affine coadjoint orbit is made linear anyway, by absorbing the cocycle into a central extension of the group of Hamiltonian diffeomorphisms thanks to the integration bundle.

NOTE 2. — We follow the vocabulary introduced in a previous work. We call *parasymplectic form* any closed 2-form on a diffeological space; and a *parasymplectic space* any diffeological space equipped with a parasymplectic form, which we denote in general by (X, ω). We refer to the textbook [PIZ13] for all generic constructions in diffeology.

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Review of Diffeological Constructions

This *universal model for symplectic manifolds* builds on a few major constructions established in previous works:

(1) The construction of the *moment map* for any parasymplectic form, on any diffeological space for any smooth action of any diffeological group preserving the parasymplectic form, in "Moment Map in Diffeology" [PIZ10].
(2) The general construction of the group of *Hamiltonian diffeomorphisms* that follows from this construction *op. cit.* §9.2.
(3) The *integration bundle* of any parasymplectic form on manifold, in [PIZ95].
(4) The realisation of *every symplectic manifolds as an affine coadjoint orbits of the group of symplectomorphisms* [PIZ16].

In what follows, we recall the basics of these main constructions which can be found and are detailed in the three references given above.

1. The Moment Maps for Parasymplectic Spaces — First of all, let G be a diffeological group. We denote by $\mathscr{G}^*$ its space of its *momenta*, that is, the space of the left-invariant differential 1-forms on G,

$$\mathscr{G}^* = \{\varepsilon \in \Omega^1(\mathrm{G}) \mid \mathrm{L}(g)^*(\varepsilon) = \varepsilon, \text{ for all } g \in \mathrm{G}\}.$$

Now, let (X, ω) be a parasymplectic space with a *parasymplectic action* of G on X. That is, a smooth morphism $\rho : \mathrm{G} \to \mathrm{Diff}(\mathrm{X}, \omega)$, denoted by $g \mapsto g_{\mathrm{X}}$, where $\mathrm{Diff}(\mathrm{X}, \omega) \subset \mathrm{Diff}(\mathrm{X})$ is the group of automorphisms of ω, equipped with the functionnal diffeology. Hence, $g_{\mathrm{X}}^*(\omega) = \omega$ for all $g \in \mathrm{G}$.

To understand the essential nature of the moment map, which is a map from X to $\mathscr{G}^*$, it is good to consider the simplest case, and use it then as a guide to extend this simple construction to the general case.

The Simplest Case. Consider the case where X is a manifold, and G is a Lie group. Let us assume that ω is exact, $\omega = d\alpha$, and that α is also invariant by G. Then, the *moment map*[3] of the action of G on X is the map

$$\mu : \mathrm{X} \to \mathscr{G}^* \quad \text{defined by} \quad \mu(x) = \hat{x}^*(\alpha),$$

where $\hat{x} : \mathrm{G} \to \mathrm{X}$ is the *orbit map* $\hat{x}(g) = g_{\mathrm{X}}(x)$.

As we can see, there is no obstacle, in this simple situation, to generalize, *mutatis mutandis*, the Moment Map to a diffeological group acting by symmetries on a diffeological parasymplectic space. However, as we know, not all closed 2-forms are exact and, even if they are exact they do not necessarily have an invariant primitive. We shall see now how we can reduce the general case to the simple particular situation by passing to the spaces of paths: $\mathrm{Paths}(\mathrm{X}) = \mathrm{C}^\infty(\mathbf{R}, \mathrm{X})$.

[3]Precisely, one moment map, since they are defined up to a constant.

The General Case. We consider a connected parasymplectic diffeological space (X, ω), and a diffeological group G acting on X and preserving ω. Let $\mathscr{K}$ be the Chain-Homotopy Operator, defined in [PIZ13, §6.83]. We recall that

$$\mathscr{K} : \Omega^k(\mathrm{X}) \to \Omega^{k-1}(\mathrm{Paths}(\mathrm{X}))$$

is a linear operator which satisfies the property

$$d \circ \mathscr{K} + \mathscr{K} \circ d = \hat{1}^* - \hat{0}^*, \qquad (\heartsuit)$$

where $\hat{t}(\gamma) = \gamma(t)$, with $t \in \mathbf{R}$ and $\gamma \in \mathrm{Paths}(\mathrm{X})$. Then, the differential 1-form $\mathscr{K}\omega$, defined on Paths(X), is related to ω by $d[\mathscr{K}\omega] = (\hat{1}^* - \hat{0}^*)(\omega)$, and $\mathscr{K}\omega$ is invariant by G (*op. cit.* §6.84). Considering $\bar{\omega} = (\hat{1}^* - \hat{0}^*)(\omega)$ and $\bar{\alpha} = \mathscr{K}\omega$, we are in the simple case: $\bar{\omega} = d\bar{\alpha}$ with $\bar{\alpha}$ invariant. We can apply the construction above and define then the *Moment Map of Paths* by

$$\Psi : \mathrm{Paths}(\mathrm{X}) \to \mathscr{G}^* \quad \text{with} \quad \Psi(\gamma) = \hat{\gamma}^*(\mathscr{K}\omega),$$

and $\hat{\gamma} : \mathrm{G} \to \mathrm{Paths}(\mathrm{X})$ is the orbit map $\hat{\gamma}(g) = g_{\mathrm{X}} \circ \gamma$ of a path γ. The moment of paths is additive with respect to the concatenation [PIZ10, §4.4],

$$\Psi(\gamma \vee \gamma') = \Psi(\gamma) + \Psi(\gamma').$$

This paths Moment Map Ψ is equivariant by G, acting by composition on Paths(X), and by coadjoint action on $\mathscr{G}^*$. Next, the *Holonomy* of the action of G on X is defined by

$$\Gamma = \{\Psi(\ell) \mid \ell \in \mathrm{Loops}(\mathrm{X})\} \subset \mathscr{G}^*,$$

the *Two-Points Moment Map* is defined by pushing Ψ forward on $\mathrm{X} \times \mathrm{X}$,

$$\psi \colon \mathrm{X} \times \mathrm{X} \to \mathscr{G}^*/\Gamma \quad \text{with} \quad \psi(x, x') = \mathrm{class}(\Psi(\gamma)),$$

where γ is a path connecting x to x', and where class denotes the projection from $\mathscr{G}^*$ onto its quotient $\mathscr{G}^*/\Gamma$. The holonomy Γ is the obstruction for the action of G to be *Hamiltonian*. The additivity of Ψ becomes the Chasles' cocycle condition

$$\psi(x, x') + \psi(x', x'') = \psi(x, x'').$$

Let $\mathrm{Ad} : \mathrm{G} \to \mathrm{Diff}(\mathrm{G})$ be the *adjoint action*, $\mathrm{Ad}(g) \colon k \mapsto gkg^{-1}$. That induces on $\mathscr{G}^*$ a linear *coadjoint action*

$$\mathrm{Ad}_* : \mathrm{G} \to \mathrm{L}(\mathscr{G}^*) \quad \text{with} \quad \mathrm{Ad}_*(g) : \varepsilon \mapsto \mathrm{Ad}(g)_*(\varepsilon) = \mathrm{Ad}(g^{-1})^*(\varepsilon).$$

Next, the group Γ is made of closed forms, invariant by the linear coadjoint action. Thus, the coadjoint action passes to the quotient $\mathscr{G}^*/\Gamma$ and we denote the quotient action in the same way:

$$\mathrm{Ad}_*(g) : \mathrm{class}(\varepsilon) \mapsto \mathrm{class}(\mathrm{Ad}_*(g)(\varepsilon)).$$

The 2-points Moment Map ψ is equivariant for the quotient coadjoint action. Note that the quotient $\mathscr{G}^*/\Gamma$ is in all cases a diffeological Abelian group.

Now, because X is connected, there exists always a map

$$\mu : \mathrm{X} \to \mathscr{G}^*/\Gamma \quad \text{such that} \quad \psi(x,x') = \mu(x') - \mu(x).$$

The solutions of this equation are given by

$$\mu(x) = \psi(x_0, x) + c,$$

where $x_0 \in \mathrm{X}$ is an arbitrary point and $c \in \mathscr{G}^*/\Gamma$ is any constant. This map is *a priori* no longer equivariant with respect to Ad_* on $\mathscr{G}^*/\Gamma$. Its lack of equivariance defines a 1-cocycle θ of G with values in $\mathscr{G}^*/\Gamma$:

$$\mu(g(x)) = \mathrm{Ad}_*(g)(\mu(x)) + \theta(g),$$

with

$$\theta(g) = \psi(x_0, g(x_0)) - \Delta c(g) \quad \text{and} \quad \Delta(c) : g \mapsto \mathrm{Ad}_*(g)(c) - c$$

is the coboundary due to the constant c in the choice of μ. The cocycle θ defines then a new action of G on $\mathscr{G}^*/\Gamma$, that is, a quotient *affine action* :

$$\mathrm{Ad}_*^\theta(g) : \tau \mapsto \mathrm{Ad}_*(g)(\tau) + \theta(g) \quad \text{for all} \quad \tau \in \mathscr{G}^*/\Gamma.$$

The Moment Map μ is then equivariant with respect to this affine action:

$$\mu(g(x)) = \mathrm{Ad}_*^\theta(g)(\mu(x)).$$

Note that, in particular, if G is transitive on X, then the image of the Moment Map μ is an affine coadjoint orbit in $\mathscr{G}^*/\Gamma$.

This construction extends the Moment Map for {Manifolds} introduced by Souriau in the sixties [Sou70] to the category {Diffeology}.

The group of all automorphisms of a parasymplectic space is naturally a diffeological group, denoted by $\mathrm{Diff}(\mathrm{X},\omega)$ or by G_ω. The constructions above give the space of momenta $\mathscr{G}^*_\omega$, the *universal*[4] *path moment map* Ψ_ω, the *universal holonomy* Γ_ω, the *universal two-points moment map* ψ_ω, the *universal moment maps* μ_ω, and the *universal Souriau's cocycles* θ_ω.

2. The Case of a Symplectic Manifold — Let (X,ω) be a connected parasymplectic manifold. The value of the paths Moment Map Ψ_ω at the point $p \in \mathrm{Paths}(\mathrm{X}) = \mathrm{C}^\infty(\mathbf{R},\mathrm{X})$, evaluated on the n-plot $\mathrm{F} : \mathrm{U} \to \mathrm{G}_\omega$ is explicitely given by [PIZ10, §10.1]

$$\Psi_\omega(p)(\mathrm{F})_r(\delta r) = \int_0^1 \omega_{p(t)}(\dot p(t), \delta p(t))\, dt, \qquad (\diamondsuit)$$

where $r \in \mathrm{U}$ and $\delta r \in \mathbf{R}^n$, δp denotes the lifting in the tangent space TX of the path p, defined by

$$\delta p(t) = [\mathrm{D}(\mathrm{F}(r))(p(t))]^{-1} \frac{\partial \mathrm{F}(r)(p(t))}{\partial r}(\delta r) \quad \text{for all} \quad t \in \mathbf{R}. \qquad (\heartsuit)$$

In that case we have the folowing theorem, see [PIZ16] for example:

[4] The adjective "universal" relates to the group G_ω [PIZ10, §9].

THEOREM (P.I-Z). *Let* X *be a connected Hausdorff manifold. A parasymplectic form* ω *on* X *is symplectic if and only if the following two conditions are satisfied:*

(1) *The manifold* X *is homogeneous under the action of* G_ω.
(2) *The universal Moment Map* $\mu_\omega : X \to \mathscr{G}^*_\omega/\Gamma_\omega$ *is injective.*

Hence, the Moment Map identifies the manifold X *with a,* a priori *affine,* $(\Gamma_\omega, \theta_\omega)$-*coadjoint orbit* $\mathscr{O}_\omega$ *of* G_ω,

$$\mu_\omega(X) = \mathscr{O}_\omega \subset \mathscr{G}^*_\omega/\Gamma_\omega.$$

The situation is summarized by the diagram

$$\begin{tikzcd} & G_\omega \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_{\mathscr{O}}"] & \\ X \arrow[rr, "\mu_\omega"'] & & \mathscr{O}_\omega \end{tikzcd} \qquad (\clubsuit)$$

where $\pi_X(\varphi) = \varphi(x_0)$, $\pi_{\mathscr{O}}(\varphi) = \mathrm{Ad}^\theta_*(\varphi)(\mu_\omega(x_0))$, for all $\varphi \in G_\omega$ and $x_0 \in X$ is any base point. The projections π_X is a subduction [Boo69, Don84], $\mathscr{O}_\omega$ is equipped with the pushforward diffeology of G_ω by $\pi_{\mathscr{O}}$, and μ_ω is then a diffeomorphism.

3. HAMILTONIAN DIFFEOMORPHISMS — In the construction of the Moment Map of a parasymplectic action of a diffeological group G on (X, ω), the holonomy group Γ is the obstruction of the action of G to be *Hamiltonian*.

DEFINITION. *A parasymplectic action of a diffeological group* G *on* (X, ω) *is said to be Hamiltonian if* $\Gamma = \{0\}$.

Hence, the moment maps have their values in $\mathscr{G}^*$. We get then the following theorem [PIZ10, §9.2]

THEOREM (P.I-Z). *Let* (X, ω) *be a connected parasymplectic diffeological space. There exists a largest connected subgroup* $\mathrm{Ham}(X, \omega) \subset \mathrm{Diff}(X, \omega)$ *whose action is Hamiltonian, that is, whose holonomy is trivial. The elements of* $\mathrm{Ham}(X, \omega)$ *are called* Hamiltonian diffeomorphisms. *An action* ρ *of a diffeological group* G *on* X *is Hamiltonian if and only if, restricted to the identity component of* G, ρ *takes its values in* $\mathrm{Ham}(X, \omega)$.

The group $\mathrm{Ham}(X, \omega)$ is precisely built as follows. Let G°_ω be the identity component of $G_\omega = \mathrm{Diff}(X, \omega)$. Let $\pi : \widetilde{G}^\circ_\omega \to G^\circ_\omega$ be the universal covering. Since the universal holonomy Γ_ω is made of closed momenta [PIZ10, §4.7], every $\gamma \in \Gamma_\omega$ defines a unique homomorphism $\mathbf{k}(\gamma)$ from $\widetilde{G}^\circ_\omega$ to $\mathbf{R}$ such that $\pi^*(\gamma) = d[\mathbf{k}(\gamma)]$ [PIZ10, §3.11]. Let

$$\widehat{\mathbf{H}}_\omega = \bigcap_{\gamma \in \Gamma_\omega} \ker(\mathbf{k}(\gamma)), \quad \text{then} \quad \mathrm{Ham}(X, \omega) = \pi(\widehat{\mathbf{H}}^\circ_\omega),$$

where $\widehat{\mathbf{H}}^\circ_\omega \subset \mathbf{H}_\omega$ is its identity component. The space of momenta and the universal moment maps objects associated to $H_\omega = \mathrm{Ham}(X, \omega)$ are denoted by: $\mathscr{H}^*_\omega$, $\bar{\Psi}_\omega$, $\bar{\psi}_\omega$, $\bar{\mu}_\omega$, and $\bar{\theta}_\omega$.

4. The Integration Bundle of a Parasymplectic Form — Let (X, ω) be a connected, Hausdorff and second countable parasymplectic manifold. Let P_ω its *group of periods*, that is,

$$P_\omega = \left\{ \int_\sigma \omega \;\middle|\; \sigma \in H_2(X, \mathbf{Z}) \right\} \subset \mathbf{R}.$$

Since X is second countable, P_ω is diffeogically discrete, that is, the diffeology induced by the standard diffeology of $\mathbf{R}$ is the discrete diffeology. The plots are locally constant. Let

$$T_\omega = \mathbf{R}/P_\omega$$

be its *torus of periods*. Except in the case where the group of periods has only one generator, the torus of periods is not a manifold but nevertheless, a non trivial diffeological group. See for example the paper on the irrational torus $T_\alpha = \mathbf{R}/\mathbf{Z} + \alpha\mathbf{Z}$ [DI83], where $\alpha \notin \mathbf{Q}$.

Then, we get the following theorem in [PIZ95, Theorem 1.5].

Theorem (P.I-Z). *Let* (X, ω) *be a second countable Hausdorff parasymplectic manifold. There exists always a* T_ω*–principal fiber bundle* $\pi : Y \to X$ *equipped with a connection* 1*-form* λ *with curvature* ω*. That is,* $\pi^*(\omega) = d\lambda$*. Such* integration bundles *are classified by the extension group* $\mathrm{Ext}(H_1(X, \mathbf{Z}), P_\omega)$*.*

A connection 1-form λ on Y is a T_ω invariant calibrated 1-form, that is, $\tau_Y^*(\lambda) = \lambda$ for all $\tau \in T_\omega$ and for all $y \in Y$, $\hat{y}^*(\lambda) = \theta$, where $\hat{y} : T_\omega \to Y$ is the orbit map $\hat{y}(\tau) = \tau_Y(y)$; and θ is the canonical 1-form on T_ω defined by $\mathrm{class}^*(\theta) = dt$, with $\mathrm{class} : \mathbf{R} \to T_\omega$ the projection, see [PIZ13, §8.37].

Exact Sequence of Automorphisms

For a symplectic manifold, the transition from an affine orbit to a linear orbit[5] needs to absorb the Souriau cocycle somewhere. We do it by building an extension of the group of Hamiltonian diffeomorphisms, associated with the integration bundle of the symplectic form (§4).

5. The Central Extension of Hamiltonian Diffeomorphisms — Let (X, ω) be a symplectic manifold. Let P_ω be its group of periods and $T_\omega = \mathbf{R}/P_\omega$ be its torus of periods. Let $\pi : Y \to X$ be a T_ω-principal integration fiber bundle, and λ be its connection form. Let $\mathrm{Aut}(Y, \lambda)$ be the group of automorphisms of (Y, λ), that is,

$$\mathrm{Aut}(Y, \lambda) = \{\varphi \in \mathrm{Diff}(Y) \mid \varphi^*(\lambda) = \lambda \text{ and } \exists f \in \mathrm{Diff}(X),\ \pi \circ \varphi = f \circ \pi\}.$$

Actually we reduce $\mathrm{Aut}(Y, \lambda)$ to its identity component[6]. Then, the diffeomorphism f belongs naturally to the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(X, \omega)$. The

[5]We recall that we say "linear orbit" as a shortcut for "orbit of a linear action".

[6]We keep the same notation for the sake of simplicity.

mapping $\eta\colon \varphi \mapsto f$ is then a surjective homomorphism. Its kernel is the torus of periods T_ω, and η is a central extension. This is summarized by the exact sequence:

$$\mathbf{1} \longrightarrow T_\omega \longrightarrow \mathrm{Aut}(Y,\lambda) \xrightarrow{\eta} \mathrm{Ham}(X,\omega) \longrightarrow \mathbf{1} \quad .$$

Note. — The integration bundles of a parasymplectic form being classified by the group $\mathrm{Ext}(H_1(X,\mathbf{Z}),P_\omega)$, the theorem above applies to each of them indifferently. This had been noticed in the special case of an integral form where $P_\omega = a\mathbf{Z}$, for any $a \in \mathbf{R}$, in particular by Bertram Kostant. In this case, the integration bundle is called the prequantization bundle. In the general case P_ω is dense in $\mathbf{R}$ and the integration bundle is not a manifold.

It is remarkable too, that all this construction is purely diffeologial, involves only differential forms and does not need tangent vectors or integration of vector fields. That aspect of diffeology had been already underlined in the construction of the Moment Map, in particular in [PIZ10].

Proof. Let us begin by fixing our notation. The action of an element $\tau \in T_\omega$ on $y \in Y$ will be denoted indifferently by

$$\tau \cdot y \quad \text{or by} \quad \tau_Y(y).$$

Now, let $\varphi \in \mathrm{Aut}(Y,\lambda)$ and $f = \eta(\varphi)$. Since $f \circ \pi = \pi \circ \varphi$, $\varphi^*(\lambda) = \lambda$ and $\pi^*(\omega) = d\lambda$, π being a subduction, we get $f \in \mathrm{Diff}(X,\omega)$.

(A) Let us to prove that $\ker(\eta) = T_\omega$, acting on Y by $\tau\colon y \mapsto \tau \cdot y$. Let $\varphi \in \ker(\eta)$, that is, $\pi \circ \varphi = \pi$. Then, for all $y \in Y$, there exists a unique $\tau(y) \in T_\omega$ such that $\varphi(y) = \tau(y)\cdot y$.

(a) Let us first check that $\tau\colon Y \to T_\omega$ is smooth. Let $r \mapsto y_r$ by a plot in Y, the composite with φ gives the plot $r \mapsto \tau(y_r)\cdot y_r$. We need to prove that $r \mapsto \tau(y_r)$ itself is smooth. The pullback of $\pi\colon Y \to X$ by the plot $r \mapsto x_r = \pi(y_r)$ is localy trivial, then we can restrict these plots to a ball B above which the pullback

$$[r \mapsto x_r]^*(Y) = \{(r,y) \in B \times Y \mid \pi(y) = x_r\}$$

is trivial. Any T_ω-principal bundle isomorphism F from this pullback to the product $B \times T_\omega$ writes $F(r,y) = (r, t(r)(y))$, and the smooth map t with values in T_ω satisfies the equivariance $t(r)(\tau \cdot y) = \tau \cdot t(r)(y)$. Thus, $r \mapsto t(r)(y_r)$ is smooth as well as $r \mapsto t(r)(\tau(y_r)(y_r)) = \tau(y_r)\cdot t(r)(y_r)$. Hence, $r \mapsto \tau(y_r)$ is smooth. Therefore, the function τ is smooth.

(b) Let us prove now that the function τ is constant. The invariance $\varphi^*(\lambda) = \lambda$ implies $\lambda(r \mapsto \tau(y_r)\cdot y_r) = \lambda(r \mapsto y_r)$, for all plots $r \to y_r$. That is, thanks to the *partial derivatives formula* [PIZ13, §8.37 ♣]

$$\begin{aligned} \lambda(r \mapsto y_r) &= \lambda(r \mapsto \tau(y_r)\cdot y_r) \\ &= \lambda(r \mapsto y_r) + \tau^*(\theta)(r \mapsto y_r), \end{aligned}$$

where θ is the canonical 1-form on T_ω. Thus, $\tau^*(\theta)=0$. Then τ is constant. Hence, $\ker\eta = T_\omega$.

(B) Let us prove that η takes its values in $\mathrm{Ham}(X,\omega)$. That is, that the holonomy group of $\mathrm{Aut}(Y,\lambda)$ vanishes when acting on (X,ω) through the action

$$\varphi_X(x) = f(x) \quad \text{with} \quad f = \eta(\varphi).$$

We shall denote by $\mathscr{A}^*$ the space of momenta of $\mathrm{Aut}(Y,\lambda)$. The Moment Map Ψ_X of the action of $\mathrm{Aut}(Y,\lambda)$ on (X,ω) is given, according to previous notations by:

$$\Psi_X : \mathrm{Paths}(X) \to \mathscr{A}^* \quad \text{with} \quad \Psi(\gamma) = \hat{\gamma}^*(\mathscr{K}_X(\omega)),$$

where $\hat{\gamma} : \varphi \mapsto f \circ \gamma$ is the orbit map. Let us prove now that $\Psi_X(\ell) = \hat{\ell}^*(\mathscr{K}_X(\omega)) = 0$, for all $\ell \in \mathrm{Loops}(X)$.

Let us recall, first of all, that the principal fiber bundle $\pi : Y \to X$ induces, in particular, a subduction of loops spaces:

$$\pi_* : \mathrm{Loops}(Y) \to \mathrm{Loops}(X) \quad \text{by pushforward} \quad \pi_*(\underline{\ell}) = \pi \circ \underline{\ell}\,,$$

see [PIZ13, §8.32] and [PIZ19]. That is, every plot $r \mapsto \ell_r$ in $\mathrm{Loops}(X)$ has a local smooth lifting $r \mapsto \underline{\ell}_r$, everywhere, in $\mathrm{Loops}(Y)$. Note that we shall underline the paths in Y, to distinguish them from paths in X. Now, let ℓ and $\underline{\ell}$ such that $\pi \circ \underline{\ell} = \ell$. We have $\hat{\ell}(\varphi) = f \circ \ell = f \circ \pi \circ \underline{\ell} = \pi \circ \varphi \circ \underline{\ell} = \pi \circ \hat{\underline{\ell}}(\varphi)$, that is, $\hat{\ell} = \pi_* \circ \hat{\underline{\ell}}$. Thus,

$$\hat{\ell}^*(\mathscr{K}_X(\omega)) = (\pi \circ \hat{\underline{\ell}})^*(\mathscr{K}_X(\omega)) = \hat{\underline{\ell}}^*\Big(\big(\pi_*\big)^*\big(\mathscr{K}_X(\omega)\big)\Big).$$

Then, let us recall the variance[7] of the chain-homotopy operators $\mathscr{K}_X$ and $\mathscr{K}_Y$, relative to X and Y [PIZ13, §6.84], summarized by the commutative diagram:

$$\begin{array}{ccc}
\Omega^k(Y) & \xrightarrow{\mathscr{K}_Y} & \Omega^{k-1}(\mathrm{Paths}(Y)) \\
{\scriptstyle \pi^*}\uparrow & & \uparrow{\scriptstyle (\pi_*)^*} \\
\Omega^k(X) & \xrightarrow{\mathscr{K}_X} & \Omega^{k-1}(\mathrm{Paths}(X))
\end{array}$$

We have then:

$$\big(\pi_*\big)^*\big(\mathscr{K}_X(\omega)\big) = \mathscr{K}_Y\big(\pi^*(\omega)\big) = \mathscr{K}_Y(d\lambda).$$

Hence:

$$\hat{\underline{\ell}}^*\Big(\big(\pi_*\big)^*\mathscr{K}_X(\omega)\Big) = \hat{\underline{\ell}}^*\big(\mathscr{K}_Y(d\lambda)\big).$$

Thus:

$$\Psi_X(\ell) = \hat{\ell}^*\big(\mathscr{K}_X(\omega)\big) = \hat{\underline{\ell}}^*\big(\mathscr{K}_Y(d\lambda)\big) = \Psi_Y(\underline{\ell}),$$

[7] The way a quantity varies.

where Ψ_{Y} is the Moment Map for the action of $\mathrm{Aut}(\mathrm{Y}, d\lambda)$ on $(\mathrm{Y}, d\lambda)$. Note that, that could have been deduced directly from [PIZ13, §9.13]. Now, according to the fundamental property of the Chain-Homotopy Operator, we have:

$$\begin{aligned} \hat{\underline{\ell}}^{*}\big(\mathscr{K}_{\mathrm{Y}}(d\lambda)\big)+\hat{\underline{\ell}}^{*}\Big(d\big(\mathscr{K}_{\mathrm{Y}}(\lambda)\big)\Big) &= \hat{\underline{\ell}}^{*}(\hat{1}^{*}(\lambda)-\hat{0}^{*}(\lambda)) \\ &= (\hat{1}\circ\hat{\underline{\ell}})^{*}(\lambda)-(\hat{0}\circ\hat{\underline{\ell}})^{*}(\lambda) \\ &= 0, \end{aligned}$$

because $\underline{\ell}$ is a loop. Therefore,

$$\hat{\underline{\ell}}^{*}\big(\mathscr{K}_{\mathrm{Y}}(d\lambda)\big)=-d\Big(\hat{\underline{\ell}}^{*}\big(\mathscr{K}_{\mathrm{Y}}(\lambda)\big)\Big).$$

For every plot $r\mapsto\varphi_r$ in $\mathrm{Aut}(\mathrm{Y},\lambda)$, for all r in its domain :

$$\hat{\underline{\ell}}^{*}\big(\mathscr{K}_{\mathrm{Y}}(\lambda)\big)(\varphi_r)=\mathscr{K}_{\mathrm{Y}}(\lambda)(\varphi_r\circ\underline{\ell})=\int_{\varphi_{r*}(\underline{\ell})}\lambda=\int_{\underline{\ell}}\varphi_r^{*}(\lambda)=\int_{\underline{\ell}}\lambda.$$

Hence $\hat{\underline{\ell}}^{*}\big(\mathscr{K}_{\mathrm{Y}}(\lambda)\big)$ is constant, its derivative vanishes and therefore

$$\Psi_{\mathrm{X}}(\ell)=\hat{\ell}^{*}\big(\mathscr{K}_{\mathrm{X}}(\omega)\big)=\hat{\underline{\ell}}^{*}\big(\mathscr{K}_{\mathrm{Y}}(d\lambda)\big)=0. \qquad (\diamondsuit)$$

And that complete to prove that $\eta\colon \mathrm{Aut}(\mathrm{Y},\lambda)\to\mathrm{Diff}(\mathrm{X},\omega)$ takes its values in $\mathrm{Ham}(\mathrm{X},\omega)$.

(C) Let us show now that $\mathrm{T}_\omega\subset\mathrm{Aut}(\mathrm{Y},\lambda)$ is central, that is, $\eta\colon\mathrm{Aut}(\mathrm{Y},\lambda)\to\mathrm{Ham}(\mathrm{X},\omega)$ is a central extension. Let $\varphi\in\mathrm{Aut}(\mathrm{Y},\lambda)$. We have seen that $\mathrm{T}_\omega=\ker(\eta)$. Thus, for all $\tau\in\mathrm{T}_\omega$ there exists $\tau'\in\mathrm{T}_\omega$ such that $\tau'=\varphi\circ\tau\circ\varphi^{-1}$. Obviously, $h_\varphi\colon\tau\mapsto\tau'$ defines a group isomorphism of T_ω : $h_\varphi(\tau_1\tau_2)=h_\varphi(\tau_1)h_\varphi(\tau_2)$, and $h_\varphi(\tau)^{-1}=\varphi^{-1}\circ\tau^{-1}\circ\varphi$. But φ is connected to the identity map $\mathbf{1}_{\mathrm{Y}}$ via a smooth path $s\mapsto\varphi_s\in\mathrm{Aut}(\mathrm{Y},\lambda)$, defined on an open interval $\mathscr{I}$ containing $[0,1]$, with $\varphi_0=\mathbf{1}_{\mathrm{Y}}$ and $\varphi_1=\varphi$. That defines a smooth path of isomorphisms $h_{\varphi_s}=\varphi_s\circ\tau\circ\varphi_s^{-1}$. Let us denote h_s for h_{φ_s}. The map $(s,t)\mapsto h_s(\mathrm{class}(t))$ is a plot defined on $\mathscr{I}\times\mathbf{R}$, in T_ω. By the monodromy theorem [PIZ13, §8.25], it has a global lifting $(s,t)\mapsto\mathrm{H}_s(t)$, defined on $\mathscr{I}\times\mathbf{R}$, which is a smooth plot in $\mathbf{R}$. And the lift is unique with $\mathrm{H}_0(0)=0$.

$$\begin{array}{ccc} \mathscr{I}\times\mathbf{R}\ni(s,x) & \xrightarrow{\quad\mathrm{H}\quad} & \mathrm{H}_s(x)\in\mathbf{R} \\ {\scriptstyle \mathbf{1}\times\mathrm{class}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{class}} \\ \mathscr{I}\times\mathrm{T}_\omega\ni(s,\mathrm{class}(x)) & \xrightarrow[h]{} & h_s(\mathrm{class}(x))=\mathrm{class}(\mathrm{H}_s(x))\in\mathrm{T}_\omega \end{array}$$

For every parameter s, the restriction $\mathrm{H}_s\colon\mathbf{R}\to\mathbf{R}$ is a smooth lifting of the isomorphism $h_s\colon\mathrm{T}_\omega\to\mathrm{T}_\omega$. Thus, up to a constant $b_s\in\mathbf{R}$, H_s is a smooth morphism from $\mathbf{R}$ to $\mathbf{R}$. Hence, $\mathrm{H}_s(x)=a_sx+b_s$, where $s\mapsto a_s$ and $s\mapsto b_s$ are smooth, and $a_s\neq0$ since H_s lifts an isomorphism. Now, for all $s\in\mathscr{I}$, all $x,x'\in\mathbf{R}$, $h_s(\mathrm{class}(x+x'))=$

$h_s(\mathrm{class}(x)) + h_s(\mathrm{class}(x'))$, that is, $\mathrm{class}(\mathrm{H}_s(x+x')) = \mathrm{class}(\mathrm{H}_s(x)) + \mathrm{class}(\mathrm{H}_s(x'))$, *i.e.* $a_s(x+x') + b_s = a_s x + b_s + a_s x' + b_s + p$, and then $b_s \in \mathrm{P}_\omega$ for all $s \in \mathrm{P}_\omega$. Since $s \mapsto b_s$ is smooth and P_ω is (diffeologically) discrete in $\mathbf{R}$, b_s is constant and equal to b_0 which is 0. Thus, $\mathrm{H}_s(x) = a_s x$. Next, $h_s(\mathrm{class}(x)) = \mathrm{class}(\mathrm{H}_s(x))$ implies that, for all $p \in \mathrm{P}_\omega$ there exists $p' \in \mathrm{P}_\omega$ such that $\mathrm{H}_s(x+p) = \mathrm{H}_s(x) + p'$. That is, $a_s(x+p) = a_s x + p'$, and then $a_s p \in \mathrm{P}_\omega$ for all $p \in \mathrm{P}_\omega$. Again, since P_ω is discrete in $\mathbf{R}$, $s \mapsto a_s$ is constant and the lifting H writes $\mathrm{H}_s(x) = ax$, for some number $a \neq 0$. Finally, since H_0 lifts the identity $h_0 = \mathbf{1}_{\mathrm{T}_\omega}$ by a morphism, $\mathrm{H}_0(x) = x$ and $a = 1$. Therefore $\mathrm{H}_s(x) = x$ for all s and $h_s(\tau) = \tau$, that is, $\varphi \circ \tau_\mathrm{Y} = \tau_\mathrm{Y} \circ \varphi$, and the extension $\eta\colon \mathrm{Aut}(\mathrm{Y},\lambda) \to \mathrm{Ham}(\mathrm{X},\omega)$ is central.

(D) Let us show finally that $\eta : \mathrm{Aut}(\mathrm{Y},\lambda) \to \mathrm{Ham}(\mathrm{X},\omega)$ is surjective. Let $f \in \mathrm{Ham}(\mathrm{X},\omega)$. There exists a smooth path $t \mapsto f_t \in \mathrm{Ham}(\mathrm{X},\omega)$, such that $f_0 = 1_\mathrm{X}$ and $f_1 = f$. We define, for all $x \in \mathrm{X}$, the path γ_X in X by $\gamma_x(t) = f_t(x)$. It satisfies $\gamma_x(0) = x$ and $\gamma_x(1) = f(x)$. The map $x \mapsto \gamma_x$ is smooth.

Given any $y \in \mathrm{Y}$ and $x = \pi(y) \in \mathrm{X}$, we will denote by $\underline{\gamma}_y$ the unique horizontal lifting of γ_x with origin y. Moreover, the map $y \mapsto \underline{\gamma}_y$ is smooth and equivariant under the action of T_ω [PIZ13, §8.32]. We define then:

$$\varphi(y) = \underline{\gamma}_y(1) \quad \text{and} \quad \varphi \in \mathrm{C}^\infty(\mathrm{Y}) \quad (\textit{op. cit.})$$

The map φ is a smooth lifting of f, that is, $\pi \circ \varphi = f \circ \pi$. Moreover, the equivariance of $\underline{\gamma}_y$ by T_ω also implies that $\tau_\mathrm{Y} \circ \varphi = \varphi \circ \tau_\mathrm{Y}$ for all $\tau \in \mathrm{T}_\omega$. If φ is equivariant, it has no reason to preserve the contact form λ. We shall show then that there exists a map $\tau \in \mathrm{C}^\infty(\mathrm{Y}, \mathrm{T}_\omega)$ such that

$$\Phi\colon y \mapsto \tau(y) \cdot \varphi(y),$$

which is still a smooth lifting of f and preserves the contact form λ, that is, $\Phi \in \mathrm{Aut}(\mathrm{Y},\lambda)$. Thanks to the partial derivatives formula (*op. cit.*), for any plot $r \mapsto y_r$ of Y, we get:

$$\begin{aligned}
\Phi^*(\lambda)(r \mapsto y_r) &= \lambda(r \mapsto \tau(y_r)\cdot\varphi(y_r)) \\
&= \theta(r \mapsto \tau(y_r)) + \lambda(r \mapsto \varphi(y_r)) \\
&= \tau^*(\theta)(r \mapsto y_r) + \varphi^*(\lambda)(r \mapsto y_r).
\end{aligned}$$

That is, $\Phi^*(\lambda) = \tau^*(\theta) + \varphi^*(\lambda)$. Consider now

$$\Phi^*(\lambda) - \lambda = \tau^*(\theta) + \beta \quad \text{with} \quad \beta = \varphi^*(\lambda) - \lambda.$$

LEMMA 1. *The* 1*-form* β *is the pullback of a closed* 1*-form* ε *on* X*:* $\beta = \pi^*(\varepsilon)$.

◀ Let us check that β is closed: $d(\varphi^*(\lambda) - \lambda) = \varphi^*(d\lambda) - d\lambda = \varphi^*(\pi^*\omega) - \pi^*\omega = \pi \circ \varphi)^*\omega - \pi^*\omega = (f \circ \pi)^*\omega - \pi^*\omega = \pi^* f^*\omega - \pi^*\omega = 0$. Also, β is invariant by T_ω: $\tau^*(\beta) = \tau^*(\varphi^*(\lambda) - \lambda) = (\varphi\circ\tau)^*(\lambda) - \tau^*(\lambda) = (\tau\circ\varphi)^*(\lambda) - \lambda = \varphi^*(\tau^*(\lambda)) - \lambda = \varphi^*(\lambda) - \lambda = \beta$. Moreover, β vanishes vertically. Indeed, let us first remark that $\tau \circ \varphi = \varphi \circ \tau$, for all $\tau \in \mathrm{T}_\omega$, implies $\varphi \circ \hat{y} = \hat{y}'$, for all $y \in \mathrm{Y}$ and $y' = \varphi(y)$. Then, $\hat{y}^*(\beta) = \hat{y}^*(\varphi^*(\lambda) - \lambda) =$

$\hat{y}^*(\varphi^*(\lambda) - \hat{y}^*(\lambda) = (\varphi\circ\hat{y})^*(\lambda) - \hat{y}^*(\lambda) = (\hat{y}')^*(\lambda) - \hat{y}^*(\lambda) = \theta - \theta = 0$. Thus $\lambda' = \lambda + \beta$ is a new connection 1-form, the difference β is then the pullback of a 1-form on X, according to [PIZ13, §8.37, Note]. ▶

Lemma 2. *The* 1*-form* ε *is exact:* $\varepsilon = d\nu$, $\nu \in \mathrm{C}^\infty(\mathrm{X}, \mathbf{R})$.

◀ Indeed, considering the fundamental property of the Chain-Homotopy Operator $\mathscr{K}\circ d + d\circ\mathscr{K} = \hat{1}^* - \hat{0}^*$ (§1 ♡), on the one hand, and the vanishing of the holonomy of the action of $\mathrm{Aut}(\mathrm{Y},\lambda)$ on Y (§5 Proof ◇) on the other, we get,

$$0 = \Psi_{\mathrm{X}}(\ell) = \hat{\ell}^*\big(\mathscr{K}_{\mathrm{X}}(\omega)\big) = \hat{\underline{\ell}}^*\big(\mathscr{K}_{\mathrm{Y}}(d\lambda)\big) = -\hat{\underline{\ell}}^*\Big(d\big(\mathscr{K}_{\mathrm{Y}}(\lambda)\big)\Big) = -d\Big(\hat{\underline{\ell}}^*\big(\mathscr{K}_{\mathrm{Y}}(\lambda)\big)\Big),$$

for all $\ell \in \mathrm{Loops}(\mathrm{X})$ and all $\underline{\ell} \in \mathrm{Loops}(\mathrm{Y})$ over ℓ, because $\hat{1}\circ\underline{\ell}^* = \hat{0}\circ\underline{\ell}^*$. Now, evaluating the Moment Map on the plot $t \mapsto \varphi_t$ connecting 1_{Y} to φ, using $\ell = \pi\circ\underline{\ell}$ and $\varphi_t\circ\underline{\ell} = \varphi_{t*}(\underline{\ell})$, we get:

$$\begin{aligned} d\Big(\hat{\underline{\ell}}^*\big(\mathscr{K}_{\mathrm{Y}}(\lambda)\big)(t\mapsto\varphi_t)\Big) \;&=\; d\Big(\mathscr{K}_{\mathrm{Y}}(\lambda)(t\mapsto\varphi_{t*}(\underline{\ell}))\Big) \;=\; d\left[t\mapsto\int_{\varphi_{t*}(\underline{\ell})}\lambda\right] \\ = \quad d\left[t\mapsto\int_{\underline{\ell}}\varphi_t^*(\lambda)\right] \quad &= \quad \int_{\underline{\ell}}\varphi^*(\lambda) - \int_{\underline{\ell}}\lambda \quad = \quad \int_{\underline{\ell}}\varphi^*(\lambda) - \lambda \\ = \quad \int_{\underline{\ell}}\beta \quad &= \quad \int_{\underline{\ell}}\pi^*\varepsilon \quad = \quad \int_{\ell}\varepsilon. \end{aligned}$$

Thus, for all $\ell \in \mathrm{Loops}(\mathrm{X})$, $\displaystyle\int_{\ell}\varepsilon = 0$. Therefore, according to [PIZ13, §6.89], there exists $\nu \in \mathrm{C}^\infty(\mathrm{X}, \mathbf{R})$ such that $\varepsilon = d\nu$. ▶

We can now complete to prove that $\Phi \in \mathrm{Aut}(\mathrm{Y},\lambda)$. Indeed, let $\underline{\nu} = \nu\circ\pi \in \mathrm{C}^\infty(\mathrm{Y}, \mathbf{R})$. Let us define $\tau \in \mathrm{C}^\infty(\mathrm{Y}, \mathrm{T}_\omega)$ by $\tau = -\mathrm{class}\circ\underline{\nu} = -\mathrm{class}\circ\nu\circ\pi$, where $\mathrm{class}\colon \mathbf{R}\to\mathrm{T}_\omega$. Hence:

$$\tau^*(\theta) = -\pi^*(\nu^*(\mathrm{class}^*(\theta))) = \pi^*(\nu^*(dt)) = -\pi^*(d\nu) = -\pi^*(\varepsilon) = -\beta.$$

Thus $\tau^*(\theta) = -\varphi^*(\lambda) + \lambda$. Therefore:

$$\Phi^*(\lambda) = \tau^*(\theta) + \varphi^*(\lambda) = -\varphi^*(\lambda) + \lambda + \varphi^*(\lambda) = \lambda, \quad \text{and} \quad \Phi \in \mathrm{Aut}(\mathrm{Y},\lambda).$$

So far, we have proved that $\eta : \mathrm{Aut}(\mathrm{Y},\lambda) \to \mathrm{Ham}(\mathrm{X},\omega)$ is surjective, we have to prove then that it is a subduction [PIZ13, §1.46]. For this, we need to check that any plot $\mathrm{P}\colon r\mapsto f_r$ in $\mathrm{Ham}(\mathrm{X},\omega)$, admits a local lifting $\tilde{\mathrm{P}}$ such that $\mathrm{P} =_{\text{locally}} \eta\circ\tilde{\mathrm{P}}$, everywhere. Thanks to the functional diffeology and to both subductions $\pi_*\colon \mathrm{Paths}(\mathrm{Y}) \to \mathrm{Paths}(\mathrm{X})$ [PIZ13, §8.32] and $\pi\colon \mathrm{Y}\to\mathrm{X}$, the map $(r,t,x)\mapsto f_{r,t}(x)$ is smooth and then admits a smooth lifting on Y. Thus, for $x = \pi(y)$, the time $t = 1$ of this lifting defines the smooth family $\varphi_r(y)$ of diffeomorphisms, the shift by $\tau \in \mathrm{C}^\infty(\mathrm{Y}, \mathrm{T}_\omega)$ preserves the smoothness of $r\mapsto\Phi_r \in \mathrm{Aut}(\mathrm{Y},\lambda)$. □

Moment Map of the Universal Extension Bundle Automorphisms

In this section we will show how the sympletic manifold (X, ω) identifies, through the Moment map of the Hamiltonian action of $\mathrm{Aut}(Y, \omega)$ with an orbit of this group for its linear coadjoint action on its space of momenta. We again emphazise the fact that this result generalizes the Kostant-Kirilov-Souriau theorem when the symplectic manifold is homogeneous under the action of a Lie group, and the symplectic form is integral. In the non-integral but homogeneous case, the optimal result in the category of manifolds states that the symplectic manifold is, up to a covering, an affine coadjoint orbit of the group. That result had been extended to the group of all Hamiltonian diffeomorphism in [PIZ16].

6. Sympectic Manifolds As (Linear) Coadjoint Orbits — Let (X, ω) be a symplectic manifold, and as it is described in (§5), let P_ω be its group of periods, $\pi : Y \to X$ be an integration bundle with connection λ, and $\mathrm{Aut}(Y, \lambda)$ be the group of automorphisms of the integration structure.

Let $\mathscr{A}^*$ be the space of Momenta of $\mathrm{Aut}(Y, \lambda)$, that is, the space of left-invariant 1-forms on $\mathrm{Aut}(Y, \lambda)$. The action of $\mathrm{Aut}(Y, \lambda)$ on Y has a moment map, relatively to the parasymplectic form $d\lambda$, given by

$$\mu_Y : Y \to \mathscr{A}^*, \quad \text{with} \quad \mu_Y(y) = \hat{y}^*(\lambda).$$

Then,

(1) The moment μ_Y descends to $\mu_X : X \to \mathscr{A}^*$, $\mu_Y = \mu_X \circ \pi$.
(2) μ_Y is equivariant under the coadjoint action of $\mathrm{Aut}(Y, \lambda)$.
(3) μ_X is injective.
(4) μ_X defines a diffeomorphism from X onto the coadjoint orbit

$$\mathscr{A}^* \supset \mathscr{O}_\lambda = \mu_Y(Y) = \mu_X(X).$$

Therefore the symplectic manifold X inherits the structure of a coadjoint orbit. And this is a universal characterization of symplectic manifolds:

Every Symplectic Manifold is a (Linear) Coadjoint Orbit.

This complements the statement made in [PIZ16] that *Every Symplectic Manifold is a (Affine) Coadjoint Orbit of its group of Symplectomorphisms.*

Note. In case of a transitive action of a Lie subgroup $G \subset G_\omega$, the moment of the action of G is the projection of the moment relative to G_ω.

Proof. Let us begin by checking that μ_Y is constant on each fiber. The action of T_ω is central in $\mathrm{Aut}(Y, \lambda)$, so for any $\tau \in T_\omega$, for all $y \in Y$ and for all $\varphi \in \mathrm{Aut}(Y, \lambda)$ we have: $\widehat{\tau \cdot y}(\varphi) = \varphi(\tau \cdot y) = \tau \cdot \varphi(y) = \tau \cdot (\hat{y}(\varphi))$, hence $\widehat{\tau \cdot y} = \tau \circ \hat{y}$. Thus, $\mu_Y(\tau \cdot y) = (\widehat{\tau \cdot y})^*(\lambda) = (\tau \circ \hat{y})^*(\lambda) = \hat{y}^*(\tau^*(\lambda)) = \hat{y}^*(\lambda) = \mu_Y(y)$.

Now, let us denote, for all φ, ψ in $\mathrm{Aut}(Y,\lambda)$, $R(\varphi)(\psi) = \psi \circ \varphi^{-1}$, the *right action* of the group on its momenta. Then, the equivariance follows from: $\mu_Y(\varphi(y)) = \widehat{\varphi(y)}^*(\lambda) = (\hat{y} \circ R(\varphi^{-1})^*(\lambda) = R(\varphi^{-1})^*(\hat{y}^*(\lambda)) = R(\varphi^{-1})^*(\mu_Y(y)) = R(\varphi^{-1})^* L(\varphi^{-1})^*(\mu_Y(y)) = \mathrm{Ad}(\varphi^{-1})^*(\mu_Y(y)) = \mathrm{Ad}(\varphi)_*(\mu_Y(y))$.

Finally, pushing forward the moment maps [PIZ13, §9.12 & 9.13] leads to the commutative diagram below, where $\bar{\mu}_X$ is the Moment Map for the group $\mathrm{Ham}(X,\omega)$, and $\mathscr{H}^*$ denotes its space of momenta.

$$\begin{tikzcd}
Y \arrow[r, "\mu_Y"] \arrow[d, "\pi"'] & \mathscr{A}^* \\
X \arrow[ur, "\mu_X"] \arrow[r, "\bar{\mu}_X"'] & \mathscr{H}^* \arrow[u, "\eta^*"']
\end{tikzcd}$$

The Moment Map $\bar{\mu}_X$ is known to be injective [PIZ16], as well as η^* since η is a subduction (§5, Proof (C)). Hence, $\mu_X = \eta^* \circ \bar{\mu}_X$ is injective and a subduction on $\mathcal{O}_\lambda = \mu_Y(Y)$. Therefore, μ_X is a diffeomorphism from X to $\mathcal{O}_\lambda = \mu_Y(Y)$, equipped with the quotient diffeology of $\mathrm{Aut}(Y,\lambda)$ by the stabilizer of any point $y \in Y$. $\square$

7. The Case of a Lie Group Action — For the reader who missed that point, let us recall first how the moment map in diffeology [PIZ10] generalizes the definition given originally by Souriau [Sou70]. Let (X,ω) be a symplectic manifold and let G be a connected Lie group acting on X and preserving ω. In that case, the space of momenta $\mathscr{G}^*$ is the dual of the Lie algebra $\mathscr{G}$, represented by the space of invariant vector fields, or equivalently, by the set of one-parameter subgroups. Let $h : s \mapsto \exp(sZ)$ be a one-parameter subgroup, Z belonging to the tangent space at the identity identified with $\mathscr{G}$. Coming back to the expression of the path moment map Ψ, the variation (§2 ♡) becomes, for $\delta s = 1$,

$$\begin{aligned}
\delta p(t) &= [D(\exp(sZ))(p(t))]^{-1} \frac{\partial \exp(sZ)(p(t))}{\partial s} \\
&= \left.\frac{\partial \exp(sZ)(p(t))}{\partial s}\right|_{s=0} \\
&= Z_X(p(t)),
\end{aligned}$$

where Z_X is the infinitesimal action of h on X. Thus, the expression of the path-moment map (§2 ♢) writes

$$\Psi_\omega(p)(h)_s(1) = \int_0^1 \omega_{p(t)}(\dot{p}(t), Z_X(p(t)))\,dt.$$

The classical moment map μ is defined in [Sou70, §11.7], as a solution, at least locally, of the differential equation

$$\omega(Z_X(x)) = -d(\mu(x)\cdot Z).$$

We assume now that μ is defined globally, that is, the action of G is Hamiltonian. Thus

$$\Psi_\omega(p)(h)_s(1) = \int_0^1 \frac{\partial \mu(x)\cdot Z}{\partial x}\bigg|_{x=p(t)} \left(\frac{dp(t)}{dt}\right) dt = \big(\mu(x') - \mu(x)\big)\cdot Z.$$

Hence, $\Psi(p) = \psi(x, x') = \mu(x') - \mu(x)$. Therefore, since this equation has a unique solution, up to a constant, the moment map in diffeology coincides with the classical moment map when X is a manifold and G is a Lie group.

Now, let us assume that the action of G on X is transitive. Souriau proved in [Sou70, §11.38] that the moment map is a covering onto its image. As it is explicitly shown in the example of the "Cylinder and $\mathrm{SL}(2,\mathbf{R})$" [PIZ16, §7], this covering can be non trivial. The group $\mathrm{SL}(2,\mathbf{R})$ acts transitively on the cylinder $\mathbf{R}^2 - \{0\}$ preserving the symplectic form $\mathrm{Surf} = dx \wedge dy$. And the moment map is given by

$$\mu(z)(F_\sigma) = \frac{1}{2}\mathrm{Surf}(z, \sigma z) \times dt,$$

where $z = (x,y) \in \mathbf{R}^2 - \{0\}$, $F_\sigma = [s \mapsto \exp(s\sigma)]$ is the one-parameter group defined by $\sigma \in \mathfrak{sl}(2,\mathbf{R})$, the Lie algebra of $\mathrm{SL}(2,\mathbf{R})$, vector space of real 2×2 traceless matrices. We have clearly $\mu(z) = \mu(-z)$.

So, why is there a discrepancy between the Hamiltonian Lie group situation, where the moment map is a covering but may not be injective, and the full Hamiltonian group, for which the moment map is injective? As we can see in [PIZ16, §3 Proof A'], a key ingredient for the injectivity of the universal moment map is the existence of compactly supported functions that separate points. These are the Hamiltonian functions of the one parameter groups generated by their gradient, against which the moment map is tested. In this example, in particular, the Hamiltonian functions of the one parameter subgroups of $\mathrm{SL}(2\mathbf{R})$ are exactly the functions $f_\sigma : z \mapsto \mu(z)(F_\sigma)$, for all $\sigma \in \mathfrak{sl}(2,\mathbf{R})$, and they do not separate opposite points. So, with a Lie group we may not have enough hamiltonian functions to separate the points of the symplectic manifold; this does not happen with the whole group of Hamiltonian diffeomorphisms.

Conclusion

This paper answers the question of the *ontological* nature of symplectic manifolds, if we can use such a big word. But that question has indeed arised in social networks, for example in mathoverflow.net [Com17]. That is a good justification *a posteriori* of this work.

As we have seen in this construction, for a symplectic manifold, to pass from an orbit of the affine action of the Hamiltonian diffeomorphisms, to an orbit of a linear action needs the integration of the Souriau cocycle. This integration is done by considering the integration bundle of the symplectic manifold, which adds a floor to the construction (art. 2, ♣) and is summarized in the following diagram (♠). We have denoted by G_λ the group of automorphisms of the integration structure, and by $\pi_{\mathcal{O}}$ the subduction from G_λ onto its orbit.

$$
\begin{tikzcd}
& G_\lambda \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_{\mathcal{O}}"] & \\
Y \arrow[rr, "\mu_Y"] \arrow[dr, "\pi"'] & & \mathcal{O}_\lambda \\
& X \arrow[ur, "\mu_X"'] &
\end{tikzcd}
\tag{$\spadesuit$}
$$

It is important to emphasize that the key construction to passing from an affine coadjoint orbit to a linear one is the integration bundle, and this integration bundle always exists only because diffeology deals correctly with irrational tori. The irrationality of the torus of periods is indeed a real chalenge for any general differential frameworks. Think for example of the simple product $S^2 \times S^2$, equipped with the symplectic form $\omega = \mathrm{Surf} \oplus \sqrt{2}\,\mathrm{Surf}$. Its group of periods $P_\omega = \mathbf{Z} + \sqrt{2}\,\mathbf{Z} \subset \mathbf{R}$ is dense and its torus of periods not Hausdorff. It admits however an integration principal bundle with group $T_{\sqrt{2}}$ equipped with a connection form λ of curvature ω.

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