Every Symplectic Manifold is a (Linear) Coadjoint Orbit

Patrick Iglesias-Zemmour
The Hebrew University of Jerusalem, Israel

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The Program

- The case of Lie-group homogeneous symplectic manifolds.
- The homogeneous action of Hamiltonian diffeomorphisms.
- Diffeology the right framework.
- Irrational tori.
- The moment map in diffeology.
- Symplectic Manifolds Revisited
- Every symplectic manifold is a (affine) coadjoint orbit
- The integration bundle of a (para)sympletic manifold.
- Every symplectic manifold is a (linear) coadjoint orbit.

The case of Lie-group homogeneous symplectic manifolds

Theorem [JM Souriau][1]

Let (M, ω) be a symplectic manifold and G be a Lie group acting transitively on M, preserving ω. Assume the action to be Hamiltonian and let $\mu : M \to \mathcal{G}^*$ be the moment map. Then, its image $\mathcal{O} = \mu(M)$ is a coadjoint orbit (maybe affine) and μ is a covering from M over $\mathcal{O}$.

[1] Jean-Marie Souriau. *Structure des systèmes dynamiques*. Dunod, Paris, 1970.

Example of Lie-group Homogeneous Symplectic Manifolds

Consider the action of $\mathrm{SL}(2, \mathbf{R})$ on the cylinder $\mathbf{R}^2 - \{0\}$. This action is transitive and Hamiltonian for the standard symplectic form $\mathrm{Surf} = dx \wedge dy$. The moment map is given by:

$$\mu(z) : \sigma \mapsto \frac{1}{2}\,\mathrm{Surf}(z, \sigma z),$$

for all σ in the Lie algebra of $\mathrm{SL}(2, \mathbf{R})$ (traceless 2×2 matrices). Clearly $\mu(z) = \mu(-z)$. The moment map μ is a two-folds covering.

The Action of Hamiltonian Diffeomorphisms.

Theorem (W. Boothby[2])

Let (M, ω) *be a symplectic manifold. The group of Hamiltonian diffeomorphisms* $\mathrm{Ham}(M, \omega)$ *is transitive on* M*.*

- Actually if $\dim(M) = 2n$, then H_ω is n-transitive.

Question

In what sense (M, ω) could be regarded as a coadjoint orbit of $\mathrm{Ham}(M, \omega)$?

[2] William M. Boothby. *Transitivity of the automorphisms of certain geometric structures.* Trans. Amer. Math. Soc. vol. 137, pp. 93–100, 1969.

Diffeology, a Good Framework

A diffeology on a set X declares <u>which parametrizations are smooth</u>, as long as they satisfy <u>three natural conditions</u> satisfied by ordinary smooth maps.

$$\mathcal{D} = \{\mathrm{P} : \mathrm{U} \to \mathrm{X} \mid \text{Conditions}\},\ \mathrm{U} \in \mathcal{T}\mathsf{op}(\mathbf{R}^n),\ n \in \mathbf{N}.$$

Together with smooth maps, the diffeological spaces form a category {Diffeology} that is stable by all set-theoretic operations: sum, product, subset, quotient. The category is *complete* and *cocomplete*.

Moreover, the set of smooth maps $\mathrm{C}^\infty(\mathrm{X}, \mathrm{X}')$ has a natural functional diffeology that makes the category *Cartesian closed*.

The Irrational Torus T_α.

The development of diffeology, whose axiomatics was posed by J.-M. Souriau in 1980[3] began integrating singular objects with the example of the *irrational torus* in 1983[4]:

$$T_\alpha = T^2/\Delta_\alpha \simeq \mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z}), \quad \text{with} \quad \Delta_\alpha = (e^{2i\pi t}, e^{2i\pi\alpha t}),$$

and $\alpha \in \mathbf{R} - \mathbf{Q}$.

The main fact was that, as a quotient space, the irrationl torus is not trivial. In particular, two tori T_α and T_β are diffeomorphic iff $\alpha \sim \beta \bmod \mathrm{GL}(2, \mathbf{Z})$.

[3] Jean-Marie Souriau. *Groupes différentiels.* Lecture notes in mathematics, 836:91–128, 1980.

[4] Paul Donato and Patrick Iglesias. *Exemple de groupes difféologiques : flots irrationnels sur le tore.* Preprint CPT-83/P.1524. Centre de Physique Théorique, Marseille, 1983. Published in C. R. Acad. Sci.

Differential Forms in Diffeology.

Definition

A differential k-form α on a Diffeological space X is a map that associates, with every plot $P : U \to X$, a k-smooth form $\alpha(P)$ on U such that

$$\alpha(P \circ F) = F^*(\alpha(P)),$$

for all smooth parametrization F in U

The set of differential k-forms on X is denoted by:

$$\Omega^k(X).$$

Note. Let $\alpha' \in \Omega^k(X')$ and $f \in C^\infty(X, X')$, then $f^*(\alpha') \in \Omega^k(X)$ with $f^*(\alpha')(P) = \alpha'(f \circ P)$.

The Space of Momenta of a Diffeological Group.

Let G be a diffeological group G. Let $\mathcal{G}^*$ be *the space of momenta* of G, that is, the space of left-invariant 1-forms on G

$$\mathcal{G}^* = \{\epsilon \in \Omega^1(\mathrm{G}) \mid \mathrm{L}(g)^*(\epsilon) = \epsilon\},$$

where

$$\mathrm{L}(g) : g' \mapsto gg'$$

denotes the left multiplication. The *adjoint action* of G on itself is defined by:

$$\mathrm{Ad}(g) : g' \mapsto gg'g^{-1}.$$

On $\mathcal{G}^*$ the adjoint action gives the *coadjoint action*:

$$\mathrm{Ad}_*(g)(\epsilon) = \mathrm{Ad}(g^{-1})^*(\epsilon).$$

An orbit of the coadjoint action is a *coadjoint orbit*.

Moment Maps: The Chain-Homotopy Operator.

There exists a *Chain-Homotopy Operator*[5]

$$\mathrm{K} : \Omega^{\mathsf{p}}(\mathrm{X}) \to \Omega^{\mathsf{p}-1}(\mathrm{Paths}(\mathrm{X}))$$

defined for any diffeological space X that satisfies

$$\mathsf{d} \circ \mathrm{K} + \mathrm{K} \circ \mathsf{d} = \hat{1}^* - \hat{0}^*,$$

where

$$\hat{\mathsf{t}}(\gamma) = \gamma(\mathsf{t})$$

for all $\gamma \in \mathrm{Paths}(\mathrm{X})$.

[5] P.I-Z. *The Moment Maps in Diffeology*, Memoir of the AMS, vol.972, 2010.

The Paths Moment Map.

Let G be a diffeological group acting on a diffeological space X preserving a closed 2-form ω.

• The 1-form $K\omega$ on Paths(X) is invariant by the action of G on paths.

• For all $\gamma \in \text{Paths}(X)$, the pullback of $K\omega$ by the orbit map is a left-invariant 1-form on Paths(X).

$$\hat{\gamma}^*(K\omega) \in \mathcal{G}^* \quad \text{with} \quad \hat{\gamma} : g \mapsto g(\gamma).$$

Definition

The *path moment map* is defined as

$$\Psi : \text{Paths}(X) \to \mathcal{G}^* \quad \text{with} \quad \Psi(\gamma) = \hat{\gamma}^*(K\omega).$$

The Two-Points Moment Map.

The path moment map Ψ passes onto the product $\mathrm{X} \times \mathrm{X}$, modulo some subgroup in $\mathcal{G}^*$, as the *two-points moment map* ψ.

$$
\begin{array}{ccc}
\mathrm{Paths}(\mathrm{X}) & \xrightarrow{\ \Psi\ } & \mathcal{G}^* \\
{\scriptstyle \mathrm{ends}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{class}} \\
\mathrm{X} \times \mathrm{X} & \xrightarrow[\ \psi\ ]{} & \mathcal{G}^*/\Gamma
\end{array}
$$

With

$$\Gamma = \{\Psi(\ell) \mid \ell \in \mathrm{Loops}(\mathrm{X})\}$$

The subgroup $\Gamma \subset \mathcal{G}^*$ is the obstruction for the action of G to be *Hamiltonian*.

The The One-Points Moment Map.

We assume the space X to be connected. Then, the equation

$$\psi(x, x') = \mu(x') - \mu(x)$$

has always a solution. They are all of the type:

$$\mu(x) = \psi(x_o, x) + c$$

where x_o is some base point in X and $c \in \mathcal{G}^*/\Gamma$ is some constant. The map

$$\mu : X \to \mathcal{G}^*/\Gamma$$

is the *one point moment map*.

Equivariance of Moment Map.

The moment maps satisfy a few important properties:

- $\Psi(\gamma \vee \gamma') = \Psi(\gamma) + \Psi(\gamma')$.
- $\Psi \circ g = \mathrm{Ad}_*(g) \circ \Psi$.
- $\psi(x, x') + \psi(x', x'') = \psi(x, x'')$.
- $\psi(g(x), g(x')) = \mathrm{Ad}_*(g)\big(\psi(x, x')\big)$.
- $\mu(g(x) = \mathrm{Ad}_*(g)(\mu(x)) + \theta(g)$.
- $\theta(g) = \psi(x_0, g(x_o))) - \Delta c(g)$.

The map $\theta : G \to \mathcal{G}^*/\Gamma$ is a cocycle

$$\theta(gg') = \mathrm{Ad}_*(g)(\theta(g')) + \theta(g),$$

and

$$\Delta c(g) = \mathrm{Ad}_*(g)(c) - c$$

is a coboundary.

Affine Coadjoint Orbits.

Every cocycle $\theta \in Z^1(G, \mathcal{G}^*/\Gamma)$ defines an affine action of G on $\mathcal{G}^*/\Gamma$ with:

$$\mathrm{Ad}_*^\theta(g)([\epsilon]) = \mathrm{Ad}_*(g)([\epsilon]) + \theta(g)$$

- The orbits of this action are *affine coadjoint orbits*.
- The one-point moment map is affine equivariant:

$$\mu \circ g = \mathrm{Ad}_*^\theta(g) \circ \mu.$$

- If G is transitive on X, then $\mu(X)$ is an affine coadjoint orbit.

The Universal Moment Maps.

The group $G_\omega = \mathrm{Diff}(X, \omega)$ is a diffeolgical group, and what have been said previously apply to it. That gives the universal moment maps:

$$\Psi_\omega, \psi_\omega, \mu_\omega, \Gamma_\omega, \theta_\omega.$$

Proposition

There exists a maximal subgroup $\mathrm{Ham}(X, \omega) \subset \mathrm{Diff}(X, \omega)$ *whose holonomy* Γ_H *vanishes*[6]*. This is the group of Hamiltonian diffeomorphisms, denoted also by* H_ω*. And we have for this group:*

$$\Psi_H, \psi_H, \mu_H, \Gamma_H = \{0\}, \theta_H.$$

[6] *op. cit. The Moment Maps in Diffeology*, §9.2.

Symplectic Manifolds.

A *symplectic manifold* is a manifold M equiped with a *symplectic form* ω. That is,

$$\omega \in \Omega^2(M),\ d\omega = 0 \text{ and } \ker(\omega) = 0.$$

Theorem [P.I-Z][7]

Let ω be a closed 2-form on a Hausdorff manifold M. Then, ω is symplectic if and only if:

1. The group G_ω is transitive on M.
2. The moment map $\mu_\omega : M \to \mathcal{G}^*_\omega / \Gamma_\omega$ is injective.

[7] P.I-Z. *Every symplectic manifold is a coadjoint orbit.* Frontiers of Fundamental Physics 14 (FFP14) – Mathematical Physics, 224 (DOI: https://doi.org/10.22323/1.224.0141):1–17, September 2016.

The Extreme Situations.

1. On presymplectic manifolds, the moment map μ_ω is not injective: it is constant on the characteristics of ω.

2. On $\mathbf{R}^2$ with $\omega = (x^2 + y^2)\,dx \wedge dy$, μ_ω is injective but the group G_ω is not transitive.

Sketch of Proof I.

For a path p and a plot F in G:

$$\Psi_\omega(p)(F)_r(\delta r) = \int_0^1 \omega_{p(t)}(\dot{p}(t), \delta p(t))\, dt$$

where $r \in U$ and $\delta r \in \mathbf{R}^n$, δp denotes the lifting in the tangent space TM of the path p, defined by

$$\delta p(t) = [D(F(r))(p(t))]^{-1} \frac{\partial F(r)(p(t))}{\partial r}(\delta r) \quad \text{with} \quad t \in \mathbf{R}.$$

When F is the exponential of the symplectic gradient of a smooth function f

$$F : t \mapsto e^{t\mathrm{grad}_\omega(f)}$$

we have, with $m_t = p(t)$:

$$\Psi_\omega(p)(F) = [f(m_1) - f(m_0)] \times dt,$$

Sketch of Proof II.

Let $m_0 \neq m_1$ be two points of M such that

$$\mu_\omega(m_0) = \mu_\omega(m_1),$$

that is,

$$\psi_\omega(m_0, m_1) = \mu_\omega(m_1) - \mu_\omega(m_0) = 0 \text{ with } m_1 \neq m_0.$$

Since M is connected, there exists $p \in \text{Paths}(M)$ such that $p(0) = m_0$ and $p(1) = m_1$. Thus,

$$\psi_\omega(m_0, m_1) = \text{class}(\Psi_\omega(p)),$$

and

$$\psi_\omega(m_0, m_1) = 0 \;\Leftrightarrow\; \text{class}(\Psi_\omega(p)) = 0,$$

that is,

$$\Psi_\omega(p) \in \Gamma_\omega.$$

Sketch of Proof III.

Then, by definition of Γ_ω,

$$\exists \ell \in M \text{ such that } \Psi_\omega(p) = \Psi_\omega(\ell).$$

We can choose $\ell(0) = \ell(1) = m_0$.
Since M is Hausdorff there exists a smooth real function

$$f \in C^\infty(M, \mathbf{R}),$$

with compact support, such that

$$f(m_0) = 0 \quad \text{and} \quad f(m_1) = 1.$$

Let ξ be the symplectic gradient field associated to f and by F the exponential of ξ.

Sketch of Proof IV.

On the one hand:

$$\Psi_\omega(p)(F) = [f(m_1) - f(m_0)]dt = dt,$$

and on the other hand:

$$\Psi_\omega(p)(F) = \Psi_\omega(\ell)(F) = [f(m_0) - f(m_0)]dt = 0;$$

but $dt \neq 0$, therefore,

$$\psi_\omega(m_0, m_1) \neq 0.$$

Since,

$$\psi_\omega(m_0, m_1) = \mu_\omega(m_1) - \mu_\omega(m_0),$$

we get:

$$m_0 \neq m_1 \ \Rightarrow \ \mu_\omega(m_1) \neq \mu_\omega(m_0).$$

The moment map μ_ω is injective. □

Every Symplectic Manifolds is a Coadjoint Orbit.

Theorem (P.I-Z[8])

Let (M, ω) *be a Haussdorf second countable manifold. The one-point moment map* $\mu_{\mathrm{H}} : \mathrm{M} \to \mathcal{H}^*$ *of the group of Hamiltonian diffeomorphisms is injective. Then,* μ_{H} *identifies* M *with its image* $\mu_{\mathrm{H}}(\mathrm{M}) \subset \mathcal{H}^*$*. Since the group of Hamiltonian diffeomorphisms is transitive on* M*, its image* $\mu_{\mathrm{H}}(\mathrm{M})$ *is an affine coadjoint orbit of the group of Hamiltonian diffeomorphisms. Therefore, Every Symplectic Manifolds is a Coadjoint Orbit, maybe affine.*

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[8] P.I-Z. *Every symplectic manifold is a coadjoint orbit.* Frontiers of Fundamental Physics 14 (FFP14) – Mathematical Physics, 224(DOI: https://doi.org/10.22323/1.224.0141):1–17, September 2016.

The Periods of a Closed 2-Form.

Let $\omega \in \Omega^2(\mathrm{M})$ with $d\omega = 0$. Let σ be a singular 2-cycle of M, that is a singular 2-chain such that

$$\partial\sigma = 0.$$

Then the integral

$$\int_\sigma \omega = \int_\Delta \sigma^*(\omega),$$

where Δ is the standard 2-simplex, depends only of $[\sigma] \in \mathrm{H}_2(\mathrm{M}, \mathbf{Z})$, since ω is closed.
That induces a homomorphism from $\mathrm{H}_2(\mathrm{M}, \mathbf{Z})$ to $\mathbf{R}$, whose image is the *group of periods of* ω:

$$\mathrm{P}_\omega = \left\{ \int_\sigma \omega \;\middle|\; [\sigma] \in \mathrm{H}_2(\mathrm{M}, \mathbf{Z}) \right\} \subset \mathbf{R}.$$

The Torus of Periods of a Closed 2-Form.

We consider our manifolds to be Haussdorf and second countable.

• The group P_ω of periods of ω is a (diffeologically) discrete subgroup of $\mathbf{R}$.

• We define the *torus of periods* to be the 1-dimensional diffeological quotient group:

$$T_\omega = \mathbf{R}/P_\omega.$$

When $P_\omega = a\mathbf{Z}$ the 2-form ω is said to be *integral*. Otherwise, we shall say that the 2-form is *irrational*.

Note

When ω is integral the torus of period is a circle of perimeter a. Otherwise T_ω is an irrational torus.

The Integration Bundle.

The general case of closed 2-forms has been solved in the framework of diffeology.[9]:

Theorem

Let M *be manifold and* ω *be any closed* 2*-form. Let* P_ω *be its group of periods and* T_ω *be its torus of periods. Then, there exists a principal principal* T_ω*-bundle* $\pi : \mathrm{Y} \to \mathrm{M}$ *with a connexion form* λ *of curvature* ω*.*

Note

The fact that the torus of periods is not trivial in the diffeology framework is crucial for this theorem. There would be no such theorem otherwise for ω irrational.

[9] Patrick Iglesias. *La trilogie du moment.* Ann. Inst. Fourier, 45(3):825–857, 1995.

Classification of Integration Bundles.

We call $\pi : \mathrm{Y} \to \mathrm{M}$, an *integration bundle*, and the pair (π, λ) an *integration structure* of (M, ω).

The integration structures are classified thanks to the following exact sequence:

$$0 \to \mathrm{Hom}(\mathrm{H}_1(\mathrm{M}, \mathbf{Z}), \mathrm{P}_\omega) \to \mathrm{Hom}(\mathrm{H}_1(\mathrm{M}, \mathbf{Z}), \mathbf{R}) \to$$
$$\to \mathrm{Hom}(\mathrm{H}_1(\mathrm{M}, \mathbf{Z}), \mathrm{T}_\omega) \to \mathrm{Ext}(\mathrm{H}_1(\mathrm{M}, \mathbf{Z}), \mathrm{P}_\omega) \to 0$$

• The group $\mathrm{Hom}(\mathrm{H}_1(\mathrm{M}, \mathbf{Z}), \mathrm{P}_\omega)$ classifies the integration bundles

• The dual group $\mathrm{Hom}(\mathrm{H}_1(\mathrm{M}, \mathbf{Z}), \mathbf{R})$, that is, $\mathrm{H}^1(\mathrm{M}, \mathbf{R})$, classifies the connexion forms.

Hamiltonian diffeomorphisms and Integration Bundle.

Let $\pi : Y \to M$ be any integration bundle, with connexion form λ. Let $\mathrm{Aut}(Y, \lambda)$ be the connected component of the automorphisms of (Y, λ):

$$\mathrm{Aut}(Y,\lambda) = \{\phi \in \mathrm{Diff}(Y) \mid \phi^*(\lambda) = \lambda,\ \exists f \in \mathrm{Diff}(M) \text{ s.t. } \pi \circ \phi = f \circ \pi\}^{o}$$

The groups $\mathrm{Aut}(Y, \lambda)$ and $\mathrm{Ham}(M, \omega)$ are related through the exact sequence of homomorphisms[10]:

$$\mathbf{1} \longrightarrow T_\omega \longrightarrow \mathrm{Aut}(Y,\lambda) \xrightarrow{\eta} \mathrm{Ham}(M,\omega) \longrightarrow \mathbf{1}\ ,$$

which is a central extension.

[10] Paul Donato & Patrick Iglesias-Zemmour. *Every symplectic manifold is a (linear) coadjoint orbit.* Canadian Mathematical Bulletin, 1-16.(2021) doi:10.4153/S000843952100031X.

Lifting the Moment Map to (Y, λ).

Let $\mathcal{A}^*$ be the space of momenta of $\mathrm{Aut}(Y, \lambda)$. The one-point moment map of $\mathrm{Aut}(Y, \lambda)$ relatively to $d\lambda$ is

$$\mu_Y : Y \to \mathcal{A}^* \quad \text{with} \quad \mu_Y(y) = \hat{y}^*(\lambda).$$

- The moment μ_Y descends to $\mu_M : X \to \mathcal{A}^*$, $\mu_Y = \mu_M \circ \pi$.
- μ_Y is equivariant under the coadjoint action of $\mathrm{Aut}(Y, \lambda)$.
- μ_M is injective.
- μ_M defines a diffeomorphism from M onto the coadjoint orbit
$$\mathcal{A}^* \supset \mathcal{O}_\lambda = \mu_Y(Y) = \mu_M(M).$$

Theorem

Every Symplectic Manifold is a (Linear) Coadjoint Orbit.

The Diagram that Explains All.

The following diagram summarize this construction:

$$\begin{tikzcd} Y \arrow[r, "\mu_Y"] \arrow[d, "\pi"'] & \mathcal{A}^* \\ M \arrow[ur, "\mu_M"] \arrow[r, "\mu_H"'] & \mathcal{H}^* \arrow[u, "\eta^*"'] \end{tikzcd}$$

Note. The existence of the integration bundle $\pi : Y \to M$, even when the group of periods is dense in $\mathbf{R}$ and the torus of periods T_ω is not a manifold, is the crucial element to this construciton. This theorem does not exist otherwise. Maybe only the fact that every symplectic manifold is an affine coadjoint orbit of its group of Hamiltonian diffeomorphisms survives.

The Symplectic Structure on the Coadjoint Orbit.

The group $\mathrm{Aut}(Y, \lambda)$ acts on M by:

$$\underline{\phi}(x) = \pi(\phi(y)), \quad \text{with} \quad \phi \in \mathrm{Aut}(Y, \lambda) \quad \text{and} \quad \pi(y) = x.$$

Let $\mathcal{O} \in M$ and $\mathcal{O}_Y \in \pi^{-1}(\mathcal{O})$ let

$$\begin{cases} \hat{\mathcal{O}}_Y : \mathrm{Aut}(Y, \lambda) \to Y & \text{with} \quad \hat{\mathcal{O}}_Y(\phi) = \phi(\mathcal{O}) \\ \hat{\mathcal{O}} \quad : \mathrm{Aut}(Y, \lambda) \to M & \text{with} \quad \hat{\mathcal{O}}(\phi) = \underline{\phi}(\mathcal{O}) \end{cases}$$

Let

$$\epsilon = \mathcal{O}_Y^*(\lambda), \quad \text{then} \quad \epsilon \in \mathcal{A}^* \quad \text{and} \quad d\epsilon = \mathcal{O}^*(\omega).$$

In other word, the exterior derivative $d\epsilon$ of the momenta $\epsilon \in \mathcal{A}^*$, descends on $M \simeq \mathrm{Aut}(Y, \lambda)/\mathrm{Stab}(\mathcal{O})$ as the symplectic form ω.