EVERY SYMPLECTIC MANIFOLD IS A COADJOINT ORBIT

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Abstract. Because symplectic structures have no local invariants, they have a huge group of automorphisms, always big enough to be transitive. We show here, that a symplectic manifold is always a coadjoint orbit of its group of symplectomorphisms, in a sense involving affine action and holonomy. In other words, coadjoint orbits are the universal models for symplectic manifolds. To establish that fact, and there is no heuristic here, the main tool is the Moment Map for Diffeological Spaces. We shall see by the way, that for a homogeneous presymplectic manifold, the characteristics are the connected components of the preimages of the universal moment map.

Introduction

At the end of the sixties, last century, coming from different points of view, Kostant, Kirillov and Souriau showed that a symplectic manifold (M, ω), homogeneous under the action of a Lie group, is isomorphic — up to a covering — to a coadjoint orbit [Kos70] [Sou70] [Kir74]. Souriau's proof was based on the moment map which he introduced during the same period. Now, the group of automorphisms $\mathrm{Diff}(\mathrm{M}, \omega)$ of a connected symplectic manifold[1] (M, ω), is transitive on M. It is then tempting to look for an analogous of the Kostant-Kirillov-Souriau (KKS) theorem, relative to $\mathrm{Diff}(\mathrm{M}, \omega)$, even if this groups is not, strictly speaking, a Lie group.

This is what we present in this paper: considering a symplectic manifold (M, ω) and its group of symplectomorphisms $\mathrm{Diff}(\mathrm{M}, \omega)$ as diffeological object, we show that the *universal moment map* [Piz10] identifies the manifold M with a coadjoint orbit, linear or affine, of its group of symplectomorphisms, for an extended version of the moment map involving possibly the holonomy of the symplectic form:

Theorem 1. — *Let* (M, ω) *be a Hausdorff symplectic Manifold. Then the universal moment map* $\mu_\omega \colon \mathrm{M} \to \mathscr{G}^*_\omega/\Gamma_\omega$ *is a diffeomorphism onto its image, equipped with the quotient diffeology of the group of symplectomorphisms.*

The space of *momenta* $\mathscr{G}^*_\omega$, the holonomy group Γ_ω, the universal moment map μ_ω, are defined always, for every diffeological space equiped with a closed 2-form, independently of their specific nature. Their definitions are recalled in the first section below.

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[1] The group of *symplectomorphisms*.

The idea that *every symplectic manifold is a coadjoint orbit* of its group of symplectomorphisms (or Hamiltonian diffeomorphisms), is not new. It appeared already at an early age of symplectic mechanics, a few decades ago. It is mentionned for example, in a functional analysis context, by Marsden & Weinstein in their paper on Vlasov equation [MW82, Note 3, p. 398], Taken up later by Omohundro, Weinstein's student, in his book on geometric perturbation theory in physics [Omo86, p. 364].

This is why it may be necessary to emphasise what makes our statement original compared with the previous approaches of the subject. It is obviously the gain in technicalities by using diffeology versus functional analysis, but not only. It is the role of the moment map in diffeology, for diffeological groups preserving a closed 2-form, which is at the center of this construction. Let us be more specific: there is a general consent to regard, by analogy, the moment map of $\mathrm{Ham}(\mathrm{M}, \omega)$ on a symplectic manifold as the mapping that associates with each point m in M, the Dirac distribution at m. That comes from the commonly accepted identification of the "Lie algebra" of the group of Hamiltonian diffeomorphisms with the algebra of smooth real functions (modulo constants). In our approach there is no freedom of choice, our results are founded on a precise axiomatic that pre-exists the various heuristics, which are made in general to force fitting the sentence into a box. The diffeology framework turns, by its own internal logic, heuristics into theorems.

In particular, there is no need of any presumptive Lie algebra. The moment map takes its values directly in the vector space of left-invariant 1-form on the group of automorphisms — or its quotient by the holonomy group — and these differential forms are defined categorically, and not by duality with some supplemental, unnecessary[2], "tangent space". It is worth mentioning too that, building a heuristic for the moment map of the whole group of symplectomorphisms is less easy than for the group of hamiltonian diffeomorphisms. The diffeology way, being at the same time conceptually more satisfactory, is easier and a rest for the mind.

Let us add also that, not only the universal moment map identifies each point of the manifold with some *momentum* of the goup of symplectomorphisms, but it pushes forward the smooth structure of the manifold onto the coadjoint orbit, for the quotient diffeology of the group of symplectomorphisms.

It is worth also mentioning that the theorem remains true replacing the group $\mathrm{Diff}(\mathrm{M}, \omega)$ by the subgroup $\mathrm{Ham}(\mathrm{M}, \omega)$ of Hamiltonian diffeomorphisms, which is the biggest group of automorphisms that has no holonomy. In this case the moment map takes its values in $\mathscr{H}^*_\omega$, the space of momenta of $\mathrm{Ham}(\mathrm{M}, \omega)$.

By the way, and not less meaningful, we prove in (art. 6) a second theorem:

[2]That does not mean that there will no case in the future of diffeology where some kind of tangent space will be useful, but in this case it is not only distracting but wrong.

THEOREM 2. — *The characteristics of a closed* 2*-form* ω *defined on a Hausdorff manifold* M*, homogeneous under the action of* $\mathrm{Diff}(\mathrm{M}, \omega)$*, are the connected components of the preimages of the universal moment map* μ_ω.

In that case, μ_ω integrates the characteristic distribution $m \mapsto \ker(\omega_m)$. This result will actually apply to any homogeneous action of a diffeological group, and in particular to the group of Hamiltonian diffeomorphisms.

Again, this gives a new interpretation of a symplectic 2-form — in opposition with presymplectic — as a homogeneous 2-form whose levels of the moment map are (diffeologically) discrete[3].

We give two examples: in (art. 7) we compute a classical moment map using the techniques of diffeology, and in (art. 8) we compute the universal holonomy for the 2-torus.

VOCABULARY. — For the sake of unification we shall call *parasymplectic* a general closed 2-form, without any other condition but to be smooth. It can be defined on a manifold or on a diffeological space. A space equipped with a parasymplectic form will be called a *parasymplectic space*.

Also, a parasymplectic form ω, on a diffeological space X, will be said to be *presymplectic* if its pseudogroup of local automorphisms $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{X}, \omega)$ is transitive on X. This is an interpretation of the (presymplectic) Darboux theorem for manifolds, regarded as a definition in diffeology.

THANKS. — I am grateful to the Hebrew University of Jerusalem Israel, who invited me, and where I spent the wonderful time in which I elaborated the first version of this article, a few years ago now.

[3]Precisely: such that there is an homogeneous action of a diffeological group with discrete level of its moment map.

Review on the Moment Maps of a Parasymplectic Form

Let G be a diffeological group, we denote by $\mathscr{G}^*$ its space of *momenta*[4] of G, that is, the left-invariant differential 1-forms on G,

$$\mathscr{G}^* = \{\varepsilon \in \Omega^1(\mathrm{G}) \mid \mathrm{L}(g)^*(\varepsilon) = \varepsilon, \text{ for all } g \in \mathrm{G}\}.$$

Now, let (X,ω) be a parasymplectic space with a smooth action of G[5], $g \mapsto g_{\mathrm{X}}$ on X, preserving ω, that is, $g_{\mathrm{X}}^*(\omega) = \omega$ for all $g \in \mathrm{G}$. To understand the essential nature of the moment map, which is a map from X to $\mathscr{G}^*$, it is good to consider the simplest case, and use it then as a guide to extend this simple construction to the general case.

The Simplest Case. Consider the case where X is a manifold, and G is a Lie group. Let us assume that ω is exact $\omega = d\alpha$, and that α is also invariant by G. Regarding ω, the *moment map*[6] of the action of G on X is the map

$$\mu : \mathrm{X} \to \mathscr{G}^* \quad \text{defined by} \quad \mu(x) = \hat{x}^*(\alpha),$$

where $\hat{x} : \mathrm{G} \to \mathrm{X}$ is the *orbit map* $\hat{x}(g) = g_{\mathrm{X}}(x)$.

As we can see, there is no obstacle, in this simple situation, to generalize, *mutatis mutandis*, the moment map to a diffeological group acting by symmetries on a diffeological parasymplectic space. But, as we know, not all closed 2-forms are exact, and even if they are exact, they do not necessarily have an invariant primitive. We shall see now, how we can generally come to a situation, so close to the simple case above, that modulo some minor subtleties we can build a good moment map in all cases.

The General Case. We consider a connected parasymplectic diffeological space (X,ω), and a diffeological group G acting on X and preserving ω. Let $\mathscr{K}$ be the Chain-Homotopy Operator, defined in [Piz13, §6.83]. We recall that $\mathscr{K}$ is a linear operator from $\Omega^k(\mathrm{X})$ to $\Omega^{k-1}(\mathrm{Paths}(\mathrm{X}))$ which satisfies the property

$$d \circ \mathscr{K} + \mathscr{K} \circ d = \hat{1}^* - \hat{0}^*,$$

where $\hat{t}(\gamma) = \gamma(t)$, with $t \in \mathbf{R}$ and $\gamma \in \mathrm{Paths}(\mathrm{X})$. Then, the differential 1-form $\mathscr{K}\omega$, defined on Paths(X), is related to ω by $d[\mathscr{K}\omega] = (\hat{1}^* - \hat{0}^*)(\omega)$, and $\mathscr{K}\omega$ is invariant by G. Considering $\bar{\omega} = (\hat{1}^* - \hat{0}^*)(\omega)$ and $\bar{\alpha} = \mathscr{K}\omega$, we are in the simple case: $\bar{\omega} = d\bar{\alpha}$ and $\bar{\alpha}$ invariant by G. We can apply the construction above and define then the *Moment Map of Paths* by

$$\Psi : \mathrm{Paths}(\mathrm{X}) \to \mathscr{G}^* \quad \text{with} \quad \Psi(\gamma) = \hat{\gamma}^*(\mathscr{K}\omega),$$

and $\hat{\gamma} : \mathrm{G} \to \mathrm{Paths}(\mathrm{X})$ is the orbit map $\hat{\gamma}(g) = g_{\mathrm{X}} \circ \gamma$ of a path γ. The moment of paths is additive with respect to the concatenation,

$$\Psi(\gamma \vee \gamma') = \Psi(\gamma) + \Psi(\gamma').$$

[4] I chose to call *momentum* (plur. momenta) the elements of $\mathscr{G}^*$.

[5] A smooth action of a diffeological group G on a diffeological space X is a smooth morhism $\rho : \mathrm{G} \to \mathrm{Diff}(\mathrm{X})$, where Diff(X) is equipped with the functionnal diffeology.

[6] Precisely, one moment map, since they are defined up to a constant.

This paths moment map Ψ is equivariant by G, acting by composition on Paths(X), and by coadjoint action on $\mathscr{G}^*$. Next, defining the *Holonomy* of the action of G on X by

$$\Gamma = \{\Psi(\ell) \mid \ell \in \mathrm{Loops(X)}\} \subset \mathscr{G}^*,$$

the *Two-Points Moment Map* is defined by pushing Ψ forward on $\mathrm{X} \times \mathrm{X}$,

$$\psi(x, x') = \mathrm{class}(\Psi(\gamma)) \in \mathscr{G}^*/\Gamma,$$

where γ is a path connecting x to x', and where class denotes the projection from $\mathscr{G}^*$ onto its quotient $\mathscr{G}^*/\Gamma$. The holonomy Γ is the obstruction for the action of G to be *Hamiltonian*. The additivity of Ψ becomes the Chasles' cocycle condition

$$\psi(x, x') + \psi(x', x'') = \psi(x, x'').$$

Let $\mathrm{Ad} : \mathrm{G} \to \mathrm{Diff(G)}$ be the *adjoint action*, $\mathrm{Ad}(g) \colon k \mapsto g k g^{-1}$. That induces on $\mathscr{G}^*$ a linear *coadjoint action*

$$\mathrm{Ad}_* : \mathrm{G} \to \mathrm{L}(\mathscr{G}^*) \quad \text{with} \quad \mathrm{Ad}_*(g) : \varepsilon \mapsto \mathrm{Ad}(g)_*(\varepsilon) = \mathrm{Ad}(g^{-1})^*(\varepsilon).$$

Next, the group Γ is made of closed forms, invariant by the linear coadjoint action. Thus, the coadjoint action passes to the quotient $\mathscr{G}^*/\Gamma$, and we denote the quotient action the same way:

$$\mathrm{Ad}_*(g) : \mathrm{class}(\varepsilon) \mapsto \mathrm{class}(\mathrm{Ad}_*(g)(\varepsilon)).$$

The 2-points moment map ψ is equivariant for the quotient coadjoint action. Note that the quotient $\mathscr{G}^*/\Gamma$ is a legit diffeological Abelian group[7]

Now, because X is connected, there always exists a map

$$\mu : \mathrm{X} \to \mathscr{G}^*/\Gamma \quad \text{such that} \quad \psi(x, x') = \mu(x') - \mu(x).$$

The solutions of this equation are given by

$$\mu(x) = \psi(x_0, x) + c,$$

where $x_0 \in \mathrm{X}$ is an arbitrary point and $c \in \mathscr{G}^*/\Gamma$ is any constant. But this map is *a priori* no longer equivariant with respect to Ad_* on $\mathscr{G}^*/\Gamma$. Its variance introduces a 1-cocycle θ of G with values in $\mathscr{G}^*/\Gamma$ such that

$$\mu(g(x)) = \mathrm{Ad}_*(g)(\mu(x)) + \theta(g),$$

with

$$\theta(g) = \psi(x_0, g(x_0)) - \Delta c(g), \quad \text{and} \quad \Delta(c) : g \mapsto \mathrm{Ad}_*(g)(c) - c$$

is the coboundary due to the constant c in the choice of μ. The cocycle θ defines then a new action of G on $\mathscr{G}^*/\Gamma$, that is, a quotient *affine action* :

$$\mathrm{Ad}_*^\theta(g) : \tau \mapsto \mathrm{Ad}_*(g)(\tau) + \theta(g) \quad \text{for all} \quad \tau \in \mathscr{G}^*/\Gamma.$$

[7]For the quotient of the functional diffeology of $\mathscr{G}^* \subset \Omega^1(\mathrm{G})$ by Γ. In particular, for Lie groups, it is always a product $\mathbf{R}^k \times \mathrm{T}^\ell$, $k, \ell \in \mathbf{N}$.

The moment map μ is then equivariant with respect to this affine action:

$$\mu(g(x)) = \mathrm{Ad}_*^{\theta}(g)(\mu(x)).$$

Note that, in particular, if G is transitive on X, then the image of the moment map μ is an affine coadjoint orbit in $\mathscr{G}^*/\Gamma$.

This construction extends to the category {Diffeology}, the moment map for manifolds introduced by Souriau in [Sou70]. The remarkable point is that none of the constructions brought up above involves differential equations, and there is no need of considering a putative Lie algebra either. That is a very important point. The momenta appear as invariant 1-forms on the group, naturally, without intermediary, and the moment map as a map in the space of momenta.

The group of all automorphisms of a parasymplectic space is denoted by $\mathrm{Diff}(X,\omega)$ or by G_ω, it is a legitimate diffeological group. The constructions above give the space of momenta $\mathscr{G}^*_\omega$, the *universal path moment map* Ψ_ω, the *universal holonomy* Γ_ω, the *universal two-points moment map* ψ_ω, the *universal moment maps* μ_ω, and the *universal Souriau's cocycles* θ_ω.

The group of *Hamiltonian diffeomorphisms* is denoted by $\mathrm{Ham}(X,\omega)$ or by H_ω, it is the biggest group that has no holonomy [Piz10]. Its space of momenta and the universal moment maps objects associated are denoted by: $\mathscr{H}^*_\omega$, $\bar{\Psi}_\omega$, $\bar{\psi}_\omega$, $\bar{\mu}_\omega$, and $\bar{\theta}_\omega$.

A *parasymplectic action* of a diffeological group G is any smooth morphism $h : G \to G_\omega$. For a Hamiltonian action, h will be with values in H_ω. The various moment maps objects associated with the actions of G, are naturally subordinate to their universal counterparts.

The Universal Moment Maps of a Symplectic Manifold

In this section we established the particular expression of the universal moment map, and associated objects, for a parasymplectic manifold.

1. The Moment Maps for Parasymplectic Manifolds — Let M be a connected manifold equipped with a closed 2-form ω. The value of the paths moment map Ψ_ω at the point $p \in \mathrm{Paths}(M) = C^\infty(\mathbf{R}, M)$, evaluated on the n-plot $F : U \to G_\omega$ is given by

$$\Psi_\omega(p)(F)_r(\delta r) = \int_0^1 \omega_{p(t)}(\dot{p}(t), \delta p(t))\, dt \qquad (\diamondsuit)$$

where $r \in U$ and $\delta r \in \mathbf{R}^n$, δp denotes the lifting in the tangent space TM of the path p, defined by

$$\delta p(t) = [D(F(r))(p(t))]^{-1} \frac{\partial F(r)(p(t))}{\partial r}(\delta r) \quad \text{for all} \quad t \in \mathbf{R}. \qquad (\heartsuit)$$

Note 1 — Let us remind that if a differential 1-form is defined by its values on all the plots, it is however characterized by the values it takes on the 1-plots. Moreover, any momentum of a diffeological group is characterized by its values on the 1-plots pointed at the identity. Thus, in order to characterize $\Psi(p)$, it is sufficient, in the formula above, to consider F as a 1-plot pointed at the identity, $F(0) = \mathbf{1}_M$, to choose $r = 0$ and $\delta r = 1$.

Note 2 — The same formula ($\diamondsuit$) gives the paths moment map associated with the group of Hamiltonian diffeomorphisms. For any plot F in $H_\omega \subset G_\omega$ and any path p in M we have

$$\bar{\Psi}_\omega(p)(F)_r(\delta r) = \Psi_\omega(p)(F)_r(\delta r).$$

Now, since by construction the holonomy of H_ω is trivial, this expression gives also the values of the 2-points moment map and we have, for any pair $m, m' \in M$

$$\bar{\psi}_\omega(m, m')(F) = \bar{\Psi}_\omega(p)(F),$$

where p is a path in M such that $m = p(0)$ and $m' = p(1)$. And, we get also the values of the moment maps

$$\bar{\mu}_\omega : m \mapsto \bar{\psi}_\omega(m_0, m) + c,$$

where m_0 is any base point of M and some $c \in \mathscr{H}_\omega^*$. ▶

Proof. By definition, $\Psi_\omega(p)(F) = \hat{p}^*(\mathscr{K}\omega)(F) = \mathscr{K}\omega(\hat{p} \circ F)$, where $\hat{p}$ is the orbit map $\phi \mapsto \phi \circ p$, from G_ω to Paths(M). The expression of the Chain-Homotopy operator $\mathscr{K}$, given in [Piz10], applied to the plot $\hat{p} \circ F : r \mapsto F(r) \circ p$ of Paths(M) gives

$$(\mathscr{K}\omega)(\hat{p} \circ F)_r(\delta r) = \int_0^1 \omega\left[\begin{pmatrix} s \\ u \end{pmatrix} \mapsto (\hat{p} \circ F)(u)(s+t)\right]_{\binom{s=0}{u=r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ \delta r \end{pmatrix} dt.$$

But $(\hat{p} \circ F)(u)(s+t) = F(u)(p(s+t))$, let us denote temporarily by Φ_t the plot $(s, u) \mapsto F(u)(p(s+t))$, then $F(u)(p(s+t))$ writes $\Phi_t(s, u)$. Thus, by definition of differential forms, the integrand

$$(\mathscr{I}) = \omega\left[\begin{pmatrix} s \\ r \end{pmatrix} \mapsto \Phi_t(s, r)\right]_{\binom{0}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ \delta r \end{pmatrix}$$

of the right term of this expression writes:

$$\begin{aligned}
(\mathscr{I}) &= \Phi_t^*(\omega)_{\binom{0}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ \delta r \end{pmatrix} \\
&= \omega_{\Phi_t\binom{0}{r}}\left(D(\Phi_t)_{\binom{0}{r}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, D(\Phi_t)_{\binom{0}{r}}\begin{pmatrix} 0 \\ \delta r \end{pmatrix}\right) \\
&= \omega_{F(r)(p(t))}\left(\frac{\partial}{\partial s}\Big\{F(r)(p(s+t))\Big\}_{s=0}, \frac{\partial}{\partial r}\Big\{F(r)(p(t))\Big\}(\delta r)\right).
\end{aligned}$$

But,

$$\begin{aligned}
\frac{\partial}{\partial s}\Big\{F(r)(p(s+t))\Big\}_{s=0} &= D(F(r))(p(t))\left(\left.\frac{\partial p(s+t)}{\partial s}\right|_{s=0}\right) \\
&= D(F(r))(p(t))(\dot{p}(t)).
\end{aligned}$$

Then, using this expression and the fact that, for all r in U, $\mathrm{F}(r)^*(\omega) = \omega$, we have:

$$\begin{aligned}
(\mathscr{I}) &= \omega_{\mathrm{F}(r)(p(t))}\Big(\mathrm{D}(\mathrm{F}(r))(p(t))(\dot p(t)), \frac{\partial \mathrm{F}(r)(p(t))}{\partial r}(\delta r)\Big) \\
&= \omega_{p(t)}\Big(\dot p(t), [\mathrm{D}(\mathrm{F}(r))(p(t))]^{-1} \frac{\partial \mathrm{F}(r)(p(t))}{\partial r}(\delta r)\Big) \\
&= \omega_{p(t)}(\dot p(t), \delta p(t)).
\end{aligned}$$

Therefore,

$$\Psi_\omega(p)(\mathrm{F})_r(\delta r) := \mathscr{K}\omega(\hat p \circ \mathrm{F})_r(\delta r) = \int_0^1 \omega_{p(t)}(\dot p(t), \delta p(t))\, dt,$$

with δp given by (♡), is the expression announced above. □

2. The case of symplectic manifolds — Let (M, ω) be a Hausdorff symplectic manifold. Let m_0 and m_1 be two points of M connected by a path p, $m_0 = p(0)$ and $m_1 = p(1)$. Let $f \in \mathrm{C}^\infty(\mathrm{M}, \mathbf{R})$ with compact support. Let

$$\mathrm{F} : t \mapsto e^{t\,\mathrm{grad}_\omega(f)}$$

be the exponential of the symplectic gradient of the f. Then, F is a 1-parameter group of H_ω and the Hamiltonian moment map $\bar\Psi_\omega$, computed at the path p, evaluated to the 1-plot F, is the constant 1-form

$$\bar\Psi_\omega(p)(\mathrm{F}) = [f(m_1) - f(m_0)] \times dt,$$

where dt is the standard 1-form of $\mathbf{R}$. ▶

Proof. Let us remark that, in our case, the lifting δp defined by (♡) of (art. 1) writes simply, with $\xi = \mathrm{grad}_\omega(f)$,

$$\delta p(t) = [\mathrm{D}(e^{s\xi})(p(t))]^{-1} \frac{\partial e^{s\xi}(p(t))}{\partial s}(\delta s) = \xi(p(t)) \times \delta s,$$

where s and δs are real numbers. Then, the expression (♢) of (art. 1) becomes

$$\begin{aligned}
\Psi_\omega(p)(\mathrm{F})_s(\delta s) &= \int_0^1 \omega_{p(t)}(\dot p(t), \xi(p(t))\, dt \times \delta s \\
&= \int_0^1 \omega_{p(t)}(\dot p(t), \mathrm{grad}_\omega(f)(p(t))\, dt \times \delta s \\
&= \int_0^1 df\Big(\frac{dp(t)}{dt}\Big) dt \times \delta s \\
&= [f(p(1)) - f(p(0))] \times \delta s
\end{aligned}$$

We remind that, by definition, $\mathrm{grad}_\omega(f) = -\omega^{-1}(df)$. Now, it is clear that for all loop ℓ of M, $\Psi_\omega(\ell)(\mathrm{F}) = 0$, thus, F is a plot of H_ω. And therefore, $\bar\Psi_\omega(p)(\mathrm{F}) = \Psi_\omega(p)(\mathrm{F}) = [f(m_1) - f(m_0)] \times dt$. □

The Universal Model for Symplectic Manifolds

In this section we show that every symplectic manifold is a coadjoint orbit of its group of automophisms.

3. Symplectic manifolds — Let M be a connected Hausdorff manifold. A closed 2-form ω on M is symplectic if and only if:

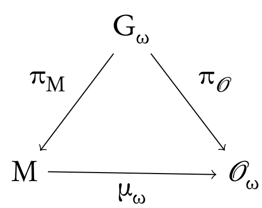
1. The manifold M is homogeneous under the action of G_ω.
2. The universal moment map $\mu_\omega : M \to \mathscr{G}^*_\omega/\Gamma_\omega$ is injective.

Hence, the moment map identifies M with a $(\Gamma_\omega, \theta_\omega)$-coadjoint orbit $\mathscr{O}_\omega$ of G_ω,

$$\mu_\omega(M) = \mathscr{O}_\omega \subset \mathscr{G}^*_\omega/\Gamma_\omega.$$

Remember that $\mathscr{G}^*_\omega/\Gamma_\omega$ is regarded here as an Abelian diffeological group.

We can replace the group of automorphisms $\mathrm{Diff}(M,\omega)$ by the group of Hamiltonian diffeomorphisms H_ω, and the the universal moment map μ_ω by the universal Hamiltonian moment map $\bar\mu_\omega : M \to \mathscr{H}^*_\omega$. Also, the Hamiltonian moment map $\bar\mu_\omega$ identifies M with a $\bar\theta_\omega$-coadjoint orbit $\bar{\mathscr{O}}_\omega$ of H_ω, $\bar\mu_\omega(M) = \bar{\mathscr{O}}_\omega \subset \mathscr{H}^*_\omega$. This is what we summarize by the sentence: *Every symplectic manifold is a coadjoint orbit.*



On this diagram: on the left $M \simeq G_\omega/\mathrm{St}(x_0)$, where x_0 is any point in M, and $\pi_M : \phi \mapsto \phi(x_0)$ is a principal fibration[8] with group the stabilizer $\mathrm{St}(x_0) \subset G_\omega$. On the right, $\mathscr{O}_\omega \simeq G_\omega/\mathrm{St}(\mu_\omega(x_0))$, where $\mathrm{St}(\mu_\omega(x_0))$ is the stabilizer for the affine coadjoint action on $\mathscr{G}^*_\omega/\Gamma_\omega$, with respect to the universal cocycle θ_ω. The Moment Map μ_ω being then a diffeomorphism.

Example. — These two examples show how the two conditions above are necessary. The space $(\mathbf{R}^2, dx \wedge dy)$ satisfies theses condition and is symplectic. However, if the space $(\mathbf{R}^2, (x^2+y^2)dx \wedge dy)$ has still an injective universal moment map, as one can check it easily, its group of automrphisms is not transitive, since $(0,0)$ is fixed. And of course, this space is not symplectic. ▶

Proof. A) Let us assume that ω is symplectic, that is, nondegenerate. Then, the group G_ω is transitive on M [Boo69]. Moreover, for every $m \in M$, the orbit map $\hat m : \phi \mapsto \phi(m)$ is a subduction [Don84]. Thus, the image of moment moment map μ_ω is one orbit $\mathscr{O}_\omega$ of the affine coadjoint action of G_ω on $\mathscr{G}^*_\omega/\Gamma_\omega$, associated with the cocycle θ_ω. Hence,

[8] In the category {Diffeology}.

for the orbit $\mathcal{O}_\omega$, equipped with the quotient diffeology of G_ω, the moment map μ_ω is a subduction.

Now, let m_0 and m_1 be two different points of M such that $\mu_\omega(m_0) = \mu_\omega(m_1)$, that is, $\psi_\omega(m_0, m_1) = \mu_\omega(m_1) - \mu_\omega(m_0) = 0$ with $m_1 \neq m_0$. Since M is connected, there exists $p \in \mathrm{Paths}(M)$ such that $p(0) = m_0$ and $p(1) = m_1$. Thus, $\psi_\omega(m_0, m_1) = \mathrm{class}(\Psi_\omega(p))$, and $\psi_\omega(m_0, m_1) = 0$ is equivalent to $\mathrm{class}(\Psi_\omega(p)) = 0$, that is, $\Psi_\omega(p) \in \Gamma_\omega$. Then, by definition of Γ_ω, there exists a loop ℓ in M such that $\Psi_\omega(p) = \Psi_\omega(\ell)$. Without loss of generality, we can choose $\ell(0) = \ell(1) = m_0$. Since M is Hausdorff there exists a smooth real function $f \in C^\infty(M, \mathbf{R})$, with compact support, such that $f(m_0) = 0$ and $f(m_1) = 1$. Let us denote by ξ the symplectic gradient field associated to f and by F the exponential of ξ. Thanks to (art. 2), we have $\Psi_\omega(p)(F) = [f(m_1) - f(m_0)]dt = dt$, on the one hand, and on the other hand $\Psi_\omega(p)(F) = \Psi_\omega(\ell)(F) = [f(m_0) - f(m_0)]dt = 0$. But $dt \neq 0$, therefore $\psi_\omega(m_0, m_1) \neq 0$. But, $\psi_\omega(m_0, m_1) = \mu_\omega(m_1) - \mu_\omega(m_0)$, then $\mu_\omega(m_1) \neq \mu_\omega(m_0)$ and the moment map μ_ω is injective. Therefore, μ_ω is an injective subduction on $\mathcal{O}_\omega$, that is, a diffeomorphism.

For the group H_ω the proof is somewhat simpler.

A') Let us assume that ω is symplectic. We know that the group of Hamiltonian diffeomorphisms is transitive. The orbit map $\hat{m}$ restricted to the group H_ω is still a subduction [Don84]. Thus, M is homogeneous under the action of H_ω. Now let m_0 and m_1 be two different points of M. Let p be a path connecting m_0 to m_1, thus $\bar{\mu}_\omega(m_1) - \bar{\mu}_\omega(m_0) = \bar{\Psi}_\omega(p)$. Since M is Hausdorff there exists a smooth real function $f \in C^\infty(M, \mathbf{R})$ with compact support such that $f(m_0) = 0$ and $f(m_1) = 1$. Let us denote by ξ the symplectic gradient field associated to f and by F the exponential of ξ. Thus, $\bar{\Psi}_\omega(p)(F) = dt$ by (art. 2). Hence, $(\bar{\mu}_\omega(m_1) - \bar{\mu}_\omega(m_0))(F) = dt \neq 0$ and $\bar{\mu}_\omega(m_0) \neq \bar{\mu}_\omega(m_0)$. Therefore μ_ω is injective, that is, an injective subduction on $\mathcal{O}_\omega$, and thus a diffeomorphism.

The proof of the converse proposition is the same considering G_ω or H_ω.

B) — B') Let us assume that M is an homogeneous space of G_ω and μ_ω is injective. Let us notice first that, since G_ω is transitive, the rank of ω is constant. Thus, $\dim(\ker(\omega_m)) = \mathrm{const}$. Now, let us assume that ω is degenerate, $\dim(\ker(\omega_m)) \neq 0$. Since $m \mapsto \ker(\omega_m)$ is a smooth foliation, for any point m of M there exists a smooth path p of M such that $p(0) = m$ and for t belonging to a small interval around $0 \in \mathbf{R}$, $\dot{p}(t) \neq 0$ and $\dot{p}(t) \in \ker(\omega_{p(t)})$ for all t in this interval. Then, we can re-parametrize the path p and assume now that p is defined on the whole $\mathbf{R}$ and satisfies $p(0) = m$, $p(1) = m'$ with $m \neq m'$, and $\dot{p}(t) \in \ker(\omega_{p(t)})$ for all t. Now, since $\dot{p}(t) \in \ker(\omega_{p(t)})$ for all t, using the expression ($\diamondsuit$) given in (art. 1), we get $\Psi_\omega(p) = 0_{\mathscr{G}^*_\omega}$ and thus $\mu_\omega(m) = \mu_\omega(m')$. But this is a contradiction since $m \neq m'$ and we have assumed that μ_ω is injective. Hence, the kernel of ω is reduced to $\{0\}$. Therefore, ω is a non degenerate closed 2-form, that is, symplectic. $\square$

4. THE HOMOGENEOUS CASE — Let (M,ω) be a connected symplectic manifold. Assume that M is homogeneous under a subgroup $\mathrm{G} \subset \mathrm{H}_\omega$. Then, the moment map μ associated with G, as defined in the first section, is a covering onto its image.

For G a Lie group, this is the Souriau's theorem [Sou70] on homogeneous symplectic manifolds, but proved the diffeology way. It is illustrated by the example of (art. 7). ▶

Proof. Let p be a path in M such that $\mu \circ p = \text{const}$. Then, $\Psi(p) = 0_{\mathscr{G}^*}$, where Ψ is the paths moment map of G. Thus, for any integer n and any n-plot F in G, we have $\Psi(p)(\mathrm{F})_r(\delta r) = 0$, for all $r \in \mathrm{dom}(\mathrm{F})$ and all $\delta r \in \mathbf{R}^n$. Using the expression of Ψ given in (art. 3) part B, we get $\int_0^1 \omega_{p(t)}(\dot p(t), \delta p(t))\, dt = 0$. Considering the 1-parameter family of paths $p_s : t \mapsto p(st)$, the derivative of $\Psi(p_s)(\mathrm{F})_r(\delta r) = 0$, with respect to s at $s = s_0$, gives $\omega_x(u, \delta x) = 0$, with $x = p(s_0)$, $u = \dot p(s_0) \in \mathrm{T}_x\mathrm{M}$ and

$$\delta x = [\mathrm{D}(\mathrm{F}(r))(x)]^{-1} \frac{\partial \mathrm{F}(r)(x)}{\partial r}(\delta r) \in \mathrm{T}_x\mathrm{M}.$$

Now, let $v \in \mathrm{T}_x\mathrm{M}$ be any vector, and let γ be a path in M such that $\gamma(0) = x$ and $v = \dot\gamma(0)$. Since M is assumed to be homogeneous under G, there exists a smooth path $r \mapsto \mathrm{F}(r)$ in G such that $\mathrm{F}(r)(x) = \gamma(r)$, with $\mathrm{F}(0) = \mathbf{1}_\mathrm{G}$. Thus, for this F and for $r = 0$, $\delta x = v$. Therefore, for all $v \in \mathrm{T}_x\mathrm{M}$, $\omega_x(u, v) = 0$, that is, $u \in \ker(\omega_x)$. But ω is symplectic, then $u = 0$. Hence, $\dot p(s_0) = 0$, for all s_0. Therefore, the path p is constant. $p(t) = x$ for all t. Thus, the preimages of the values of the moment map μ are (diffeologically) discrete. Thanks to the double homogeneity: G over M, and by equivariance, G over the (possibly affine) coadjoint orbit $\mathscr{O} = \mu(\mathrm{M})$, μ is a covering onto its image. □

The Presymplectic Case

Considering a parasymplectic form ω on a manifold M, one says that ω is *presymplectic* if the dimension of the kernel of ω is constant over M. On a presymplectic manifold, the characteristic distribution $x \mapsto \ker(\omega_x)$ is integrable, that is a consequence of a Fröbenius theorem, the integral submanifolds are called *characteristics* of ω. By definition they are connected.

5. PRESYMPLECTIC SPACES AND THE NŒTHER–SOURIAU THEOREM — For a presymplectic manifold The Darboux theorem (M,ω) states that M is locally diffeomorphic, at each point, to $(\mathbf{R}^{2k} \times \mathbf{R}^\ell, \omega_{\mathrm{st}})$, where ω_{st} is the standard symplectic form on the factor $\mathbf{R}^{2k}$ and vanishes on the factor $\mathbf{R}^\ell$. This implies in particular that the pseudo group $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{M},\omega)$ of local automorphisms[9] is transitive. Conversely, if $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{M},\omega)$ is transitive, then the kernel of ω is constant and ω is presymplectic. That suggest a definition in diffeology:

[9] See [Piz13, §2.1] for local maps and local diffeomorphisms in general.

DEFINITION. — *We shall say that a parasymplectic form* ω*, defined on a diffeological space* X*, is* presymplectic *if its pseudogroup of local symmetries* $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{X}, \omega)$ *is transitive.*

Let us come back to the case of a manifold M:

PROPOSITION. — *The Nœther–Souriau theorem, applied to the whole group* G_ω *(which is not a Lie group* stricto sensu*), states that the universal moment map* μ_ω *is constant on the characteristic of* ω*.*

By functoriality, this proposition applies to any group of automorphisms. ▶

Proof. Then, the proposition is an immediate consequence of the explicit formula of (art. 1). If a path p connects m to m' and $\dot{p}(t) \in \ker(\omega_{p(t)})$, for all t, then for every n-plot F of G_ω, for every $r \in \mathrm{dom}(\mathrm{F})$, for every $\delta r \in \mathbf{R}^n$, we have

$$\Psi_\omega(p)(\mathrm{F})_r(\delta r) = \int_0^1 \omega(\dot{p}(t), \delta p(t))\, dt = 0.$$

Thus, $0 = \Psi_\omega(p)$, but $\psi_\omega(m, m') = \mathrm{class}(\Psi_\omega(p)) \in \mathscr{G}^*_\omega/\Gamma_\omega$. And since $\psi_\omega(m, m') = \mu_\omega(m') - \mu_\omega(m)$, we have $\mu_\omega(m) = \mu_\omega(m')$. □

6. PRESYMPLECTIC HOMOGENEOUS MANIFOLDS — Let M be a connected Hausdorff manifold, and let ω be a parasymplectic form on M. Let $\mathrm{G} \subset \mathrm{G}_\omega$ be a connected subgroup. If M is a homogeneous space[10] of G, then the characteristics of ω are the connected components of the preimages of the moment maps μ.

NOTE. — In particular, if M is a homogeneous space of G_ω, and thus of its identity component, then the characteristics of ω are the connected components of the preimages of the values of a universal moment map μ_ω. This justifies *a posteriori* a general definition for the characteristics of a homogeneous presymplectic diffeological space, as the connected components of the preimages of the universal moment map.

Also, from a pure linguistic point of view, *motion* and *moment* (in French: *mouvement* and *moment*) share the same Latin etymology: *momentum*[11]. And in symplectic mechanics, a motion of a dynamical system appears as an integral curve of a presymplectic structure, see [Sou70]. This theorem shows how the universal moment map confounds definitely these two apparently different objects. ▶

Proof. The Souriau-Nœther theorem states that if m and m' are on the same characteristic of ω, then $\mu(m) = \mu(m')$ (art. 5). We shall prove the converse in a few steps.

(a) Let us consider first the case when the holonomy of Γ is trivial, $\Gamma = \{0\}$. Let us assume m and m' connected by a path p such that $\mu(p(t)) = \mu(m)$ for all t. Then, let $s \mapsto p_s$ be defined by $p_s(t) = p(st)$, for all s and t. We have $\mu(p_s(1)) = \mu(p_s(0))$, that is, $\Psi(p_s) = 0_{\mathscr{G}^*}$, for all s. Thus, for all n-plots F of G, for all $r \in \mathrm{dom}(\mathrm{F})$, all $\delta r \in \mathbf{R}^n$

[10] That means that the orbit map $\hat{x}\colon \mathrm{G} \to \mathrm{M}$, defined by $\hat{x}(g) = g(x)$, is a subduction.

[11] See for example the Merriam-Webster dictionnary,
`http://www.merriam-webster.com/dictionary/moment`.

and all s, $\Psi(p_s)(\mathrm{F})_r(\delta r)=0$. That is, after a change of variable $t\mapsto st$ and noticing that $\delta p_s(t)=\delta p(st)$ (art. 1, ♡),

$$\Psi(p_s)(\mathrm{F})_r(\delta r)=\int_0^1 \omega_{p_s(t)}(\dot p_s(t),\delta p_s(t))\,dt=\int_0^s \omega_{p(t)}(\dot p(t),\delta p(t))\,dt=0$$

Hence, after derivation:

$$\frac{\partial}{\partial s}\Psi(p_s)(\mathrm{F})_r(\delta r)=\omega_{p(s)}(\dot p(s),\delta p(s))=0.$$

Next, let $v\in \mathrm{T}_{p(t)}(\mathrm{M})$, then v is the speed of some path c in M at the point $p(t)$, that is,

$$c(0)=p(t) \quad \text{and} \quad \left.\frac{dc(s)}{ds}\right|_{s=0}=v.$$

Since M is assumed to be homogeneous under the action of G, there exists a smooth path $s\mapsto \mathrm{F}(s)$ in G, centered at the identity, $\mathrm{F}(0)=\mathbf{1}_{\mathrm{M}}$, such that $\mathrm{F}(s)(p(t))=c(s)$. Then, for $s=0$ and $\delta s=1$, we get from above,

$$\delta p(t)=\mathbf{1}_{\mathrm{T}_{p(t)}\mathrm{M}}\left.\frac{d\mathrm{F}(s)(p(t))}{ds}\right|_{s=0}=\left.\frac{dc(s)}{ds}\right|_{s=0}=v.$$

Hence, for every $v\in \mathrm{T}_{p(t)}\mathrm{M}$, $\omega(\dot p(t),v)=0$, *i.e.* , $\dot p(t)\in\ker(\omega_{p(t)})$ for all t. Therefore, the connected components of the preimages of the values of the moment map μ of the group G are the characteristics of ω.

(b) Let us consider the general case. Let $\tilde{\mathrm{M}}$ be the universal covering of M, $\pi:\tilde{\mathrm{M}}\to\mathrm{M}$ be the projection, and let $\tilde\omega=\pi^*(\omega)$. Let $\hat{\mathrm{G}}$ be the group of automorphisms of $\tilde{\mathrm{M}}$ over G, defined by

$$\hat{\mathrm{G}}=\{\hat g\in \mathrm{Diff}(\tilde{\mathrm{M}},\tilde\omega)\mid \exists g\in \mathrm{G} \text{ and } \pi\circ\hat g=g\circ\pi\}.$$

Let $\rho:\hat{\mathrm{G}}\to\mathrm{G}$ be the morphism $\hat g\mapsto g$. By construction, the group $\hat{\mathrm{G}}$ is an extension of G by the homotopy group $\pi_1(\mathrm{M})$. Let us show that the following sequence of morphisms is exact:

$$\mathbf{1}_{\tilde{\mathrm{M}}}\longrightarrow\pi_1(\mathrm{M})\longrightarrow\widehat{\mathrm{G}}\xrightarrow{\rho}\mathrm{G}\longrightarrow\mathbf{1}_{\mathrm{M}}.$$

We shall prove now a few lemmas presented as short propositions.

b1.— *The morphism* ρ *is surjective.* Indeed, let $g\in\mathrm{G}$. Consider $g\circ\pi:\tilde{\mathrm{M}}\to\mathrm{M}$. Since $\tilde{\mathrm{M}}$ is simply connected, thanks to the monodromy theorem, there exists a smooth lifting $\hat g:\tilde{\mathrm{M}}\to\tilde{\mathrm{M}}$, that is, $\pi\circ\hat g=g\circ\pi$. Let fix a point $m\in\mathrm{M}$ and a point $\tilde m\in\tilde{\mathrm{M}}$ over m. Let $m'=g(m)$, the lifting $\hat g$ is unique after chosing $\tilde m'=\hat g(\tilde m)$ in $\pi^{-1}(m')$. Now, let $\hat g^{-1}$ be the lifting of g^{-1} defined by $\hat g^{-1}(\tilde m')=\tilde m$. Hence, $\hat g^{-1}\circ\hat g$ is a lifting of $\mathbf{1}_{\mathrm{M}}$, fixing $\tilde m$. But, $\mathbf{1}_{\tilde{\mathrm{M}}}$ also lifts $\mathbf{1}_{\mathrm{M}}$, fixing $\tilde m$. Thus, $\hat g^{-1}\circ\hat g=\mathbf{1}_{\tilde{\mathrm{M}}}$, and $\hat g^{-1}=(\hat g)^{-1}$. Therefore, $\hat g$ is a diffeomorphism satisfying $\pi\circ\hat g=g\circ\pi$, it preserves then $\tilde\omega=\pi^*(\omega)$: it belongs to $\hat{\mathrm{G}}$. We proved that ρ is surjective.

b2.— *The kernel of* ρ *is exactly* $\pi_1(\mathrm{M})$. First of all, $\tilde{\mathrm{M}}$ is a $\pi_1(\mathrm{M})$-principal bundle over M, the action of $\pi_1(\mathrm{M})$ preserves $\tilde{\omega} = \pi^*(\omega)$. Thus, $\pi_1(\mathrm{M}) \subset \hat{\mathrm{G}}$. Now, by the monodromy theorem, $\ker(\rho)$ corresponds to the various liftings of $\mathbf{1}_{\mathrm{M}}$. But these liftings are uniquely defined by their images of a base point $\tilde{m} \in \pi^{-1}(m)$, and these points are just the $k(\tilde{m})$ with $k \in \pi_1(\mathrm{M})$. Thus, $\ker(\rho) = \pi_1(\mathrm{M})$. That achieves to prove that the morphisms sequence above is exact.

b3.— *The morphism* ρ *is smooth.* The group $\hat{\mathrm{G}}$ is equipped with the functional diffeology. The morphism ρ is smooth if and only if, for all plot $\mathrm{P} : \mathrm{U} \to \hat{\mathrm{G}}$, the parametrisation $\rho \circ \mathrm{P}$, with values in G, is smooth. By definition of the functional diffeology, P is a plot in $\hat{\mathrm{G}}$ means that $\tilde{\mathrm{ev}}\colon (r, \tilde{m}) \to \mathrm{P}(r)(\tilde{m})$ is smooth. And, $\rho \circ \mathrm{P}$ is a plot in G means that $\mathrm{ev}\colon (r, m) \mapsto \rho(\mathrm{P}(r))(m)$ is smooth. Consider then the commutative diagram:

$$
\begin{array}{ccc}
(r, \tilde{m}) & \xrightarrow{\ \tilde{\mathrm{ev}}\ } & \mathrm{P}(r)(\tilde{m}) \\
{\scriptstyle \mathbf{1}_{\mathrm{U}} \times \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\
(r, m) & \xrightarrow[\ \mathrm{ev}\ ]{} & \pi(\mathrm{P}(r)(\tilde{m})) = \rho(\mathrm{P}(r))(m)
\end{array}
$$

Since the arrow $\mathbf{1}_{\mathrm{U}} \times \pi$ is a subduction and $\tilde{\mathrm{ev}}$ and π are smooth, ev is smooth. Therefore, ρ is a smooth morphism.

b4.— *The morphism* ρ *is a subduction.* We have seen that ρ is smooth and surjective. It remains to see that the plots of G lift locally into plots of $\hat{\mathrm{G}}$, according to criterion [Piz13, §1.31]. Consider a plot $r \mapsto g_r$, that is, a parametrisation such that $(r, m) \mapsto g_r(m)$ is smooth. Let us choose a parameter r_0, a point $m_0 \in \mathrm{M}$, a point $\tilde{m}_0 \in \pi^{-1}(m_0)$, and a point $\tilde{m}'_0 \in \pi^{-1}(g_{r_0}(m_0))$. Let us restrict the parametrisation to a small ball around r_0. Thanks again to the monodromy theorem, the map $(r, \tilde{m}) \mapsto g_r(\pi(\tilde{m}))$ has a unique smooth lifting $(r, \tilde{m}) \mapsto \tilde{m}'_r$ into $\tilde{\mathrm{M}}$ such that, $\pi(\tilde{m}'_r) = g_r(m)$ and $\tilde{m}'_{r_0} = \tilde{m}'_0$. Let $\hat{g}_r\colon \tilde{m} \mapsto \tilde{m}'_r$. By construction $\pi \circ \hat{g}_r = g_r \circ \pi$, and the map $\hat{g}_r$ has an inverse, by lifting the same way $r \mapsto g_r^{-1}$, mapping $\tilde{m}'_0$ to $\tilde{m}_0$. Now, since $(r, \tilde{m}) \mapsto \hat{g}_r(\tilde{m})$ and $(r, \tilde{m}) \mapsto \hat{g}_r^{-1}(\tilde{m})$ are smooth, we deduce two things: first of all, the maps $\hat{g}_r$ and $\hat{g}_r^{-1}$ are smooth, that is, $\hat{g}_r \in \hat{\mathrm{G}}$, and then $r \mapsto \hat{g}_r$ is a plot of $\hat{\mathrm{G}}$, and thus a (local) smooth lifting of $r \mapsto g_r$. Hence, ρ is a subduction. Moreover now, as quotient space, $\mathrm{G} \simeq \hat{\mathrm{G}}/\pi_1(\mathrm{M})$, and since the subgroup $\pi_1(\mathrm{M})$ is discrete, ρ is a covering. Note however that $\hat{\mathrm{G}}$ may be not connected.

b5.— *The action of* $\hat{\mathrm{G}}$ *on* $\tilde{\mathrm{M}}$ *is homogeneous.* Let us choose two points $m \in \mathrm{M}$ and $\tilde{m} \in \pi^{-1}(m)$. Let $\mathrm{pr}_m\colon \mathrm{G} \to \mathrm{M}$ be the orbit map of m, with respect to G, $\mathrm{pr}_m(g) = g(m)$. By hypothesis, pr_m is a subduction. And let $\mathrm{pr}_{\tilde{m}}\colon \hat{\mathrm{G}} \to \tilde{\mathrm{M}}$, be the be the orbit map of $\tilde{m}$, $\mathrm{pr}_{\tilde{m}}(\hat{g}) = \hat{g}(\tilde{m})$. We will check that $\mathrm{pr}_{\tilde{m}}$ is also a subduction. Consider the diagram:

$$\begin{array}{ccc} \hat{\mathrm{G}} & \xrightarrow{\;\mathrm{pr}_{\tilde{m}}\;} & \tilde{\mathrm{M}} \\ \rho \downarrow & & \downarrow \pi \\ \mathrm{G} & \xrightarrow[\;\mathrm{pr}_{m}\;]{} & \mathrm{M} \end{array}$$

All the arrows are subductions and ρ and π are covering. Let $r \mapsto \tilde{m}_r$ be a plot in $\tilde{\mathrm{M}}$, and let $m_r = \pi(\tilde{m}_r)$. Since pr_m is a subduction, there exists locally a smooth lifting $r \mapsto g_r$ in G such that $g_r(m) = m_r = \pi(\tilde{m}_r)$. Now, there is a smooth lifting $r \mapsto \hat{g}'_r$ in $\hat{\mathrm{G}}$ such that $\rho(\hat{g}'_r) = g_r$. Thus, $\pi(\tilde{m}_r) = m_r = g_r(m) = \rho(\hat{g}'_r)(m) = \rho(\hat{g}'_r)(\pi(\tilde{m})) = \pi(\hat{g}'_r(\tilde{m}))$. Hence, $r \mapsto \tilde{m}_r$ and $\hat{g}'_r(\tilde{m})$ are two smooth lifting in $\tilde{\mathrm{M}}$ of $r \mapsto m_r$. Restricted to a small ball, these two liftings differ only from a constant element k of $\pi_1(\mathrm{M})$, that is $\tilde{m}_r = k(\hat{g}'_r(\tilde{m}))$, but $r \mapsto \hat{g}_r = k \circ \hat{g}'_r$ is also a smooth lifting of $r \mapsto g_r$. Thus, there always exists locally a smooth lifting $r \mapsto \hat{g}_r$ in $\hat{\mathrm{G}}$ such that $\tilde{m}_r = \hat{g}_r(\tilde{m})$, that is, $\tilde{m}_r = \mathrm{pr}_{\tilde{m}}(\hat{g}_r)$. Therefore, $\mathrm{pr}_{\tilde{m}}$ is a subduction, and the action of $\hat{\mathrm{G}}$ on $\tilde{\mathrm{M}}$ is homogeneous.

b6.— *The characteristics of* ω *are the connected components of the preimages of* μ. First of all, since $\tilde{\mathrm{M}}$ is simply connected, there is no holonomy. The moment map $\tilde{\mu}$ takes its values in the space of momenta of $\hat{\mathrm{G}}$. But $\hat{\mathrm{G}}$ being a covering of G, there is a canonical identification between the spaces of momenta of the two groups. Thus $\tilde{\mu} \colon \tilde{\mathrm{M}} \to \mathscr{G}^*$ [Piz13, 7.13]. And thanks to the variance of the moment map [Piz13, §9.13], we have the commutating diagram:

$$\begin{array}{ccc} \tilde{\mathrm{M}} & \xrightarrow{\;\tilde{\mu}\;} & \mathscr{G}^* \\ \pi \downarrow & & \downarrow \mathrm{class} \\ \mathrm{M} & \xrightarrow[\;\mu\;]{} & \mathscr{G}^*/\Gamma \end{array}$$

Now, consider the characteristic foliation $\ker(\tilde{\omega})$. Since $\tilde{\mathrm{M}}$ is a covering of M, the tangent map $\mathrm{D}(\pi)$ is an isomorphism from $\ker(\tilde{\omega})$ onto $\ker(\omega)$. Therefore, the characteristics of $\tilde{\omega}$, that is, the integral manifolds of the characteristic distribution, maps onto the characteristics of ω, and are connected coverings of their images. Hence, the characteristics of ω are the projections by π of the characteristics of $\tilde{\omega}$. Note that, since $\pi_1(\mathrm{M})$ preserve $\tilde{\omega}$, it exchanges the characteristics of $\tilde{\omega}$, over the characteristics of ω. Now, let $c = \mu(m)$, one has $(\mu \circ \pi)^{-1}(c) = (\mathrm{class} \circ \tilde{\mu})^{-1}(c)$, that is, $\pi^{-1}(\mu^{-1}(c)) = \tilde{\mu}^{-1}(\mathrm{class}^{-1}(c))$. And then, $\mu^{-1}(c) = \pi(\tilde{\mu}^{-1}(\mathrm{class}^{-1}(c)))$. Let $\tilde{m} \in \pi^{-1}(m)$ and $\tilde{c} = \tilde{\mu}(\tilde{m})$, then $\tilde{c} \in \mathrm{class}^{-1}(c)$. Thus, $\mathrm{class}^{-1}(c) = \{\tilde{c} + \gamma \mid \gamma \in \Gamma\}$. Hence,

$$\mu^{-1}(c) = \pi\big(\tilde{\mu}^{-1}\{\tilde{c} + \gamma \mid \gamma \in \Gamma\}\big).$$

But for each $\gamma \in \Gamma$, either $\tilde{\mu}^{-1}(\tilde{c}+\gamma)$ is empty or is a union of characteristics of $\tilde{\omega}$, thanks to previous paragraph a). Then, since $\tilde{\mu}^{-1}(\tilde{c})$ is not empty, $\tilde{\mu}^{-1}\{\tilde{c}+\gamma \mid \gamma \in \Gamma\}$ is a union of characteristics of $\tilde{\omega}$. Its projection by π, that is $\mu^{-1}(c)$, is then a union of characteristics of ω. □

Examples

We give here two simple examples that illustrate the previous constructions of moment maps, using the diffeological framework.

7. The cylinder and $\mathrm{SL}(2,\mathbf{R})$ — This is a classical example for which the moment maps of a transitive Hamiltonian action of a Lie group is a nontrivial covering. I use this example here to show how the algorithm of the moment map in diffeology works in a concrete case. Let us consider the real space $\mathbf{R}^2$ equipped with the standard symplectic form $\mathrm{surf} = dx \wedge dy$, with $(x,y) \in \mathbf{R}^2$. The special linear group $\mathrm{SL}(2,\mathbf{R})$ preserves the standard form ω. Its action on $\mathbf{R}^2$ is effective and has two orbits, the origin $0 \in \mathbf{R}^2$ and the "cylinder" $\mathrm{M} = \mathbf{R}^2 - \{0\}$. The restriction $\omega = \mathrm{surf} \restriction \mathrm{M}$ is still symplectic and invariant by $\mathrm{SL}(2,\mathbf{R})$. Since $\mathbf{R}^2$ is simply connected the holonomy of $\mathrm{SL}(2,\mathbf{R})$ is trivial, so its action is Hamiltonian. And since 0 is a fixed point, the 2-points moment map ψ is exact [Piz10, §6.2, Note 2]. Then, there exists an equivariant moment map $\mu : \mathbf{R}^2 \to \mathfrak{sl}(2,\mathbf{R})^*$ such that $\psi(z,z') = \mu(z') - \mu(z)$, for all $z, z' \in \mathbf{R}^2$ [Piz10]. Moreover, we know an explicit expression for μ. For every $z \in \mathbf{R}^2$, let $p_z = [t \mapsto tz] \in \mathrm{Paths}(\mathbf{R}^2)$ connecting 0 to z. The general expression given in (art. 1) ($\diamondsuit$) and ($\heartsuit$) gives, in the particular case of $p = p_z$ and $\mathrm{F}_\sigma = [s \mapsto e^{s\sigma}]$, with[12] $\sigma \in \mathfrak{sl}(2,\mathbf{R})$, the following:

$$\mu(z)(\mathrm{F}_\sigma) = \frac{1}{2}\mathrm{surf}(z, \sigma z) \times dt.$$

By choosing various σ in $\mathfrak{sl}(2,\mathbf{R})$, we can check that $\mu(z) = \mu(z')$ if and only if $z' = \pm z$. Restricting this construction to M, which is an orbit of $\mathrm{SL}(2,\mathbf{R})$, and thanks to the functoriality of the moment maps [Piz10], the moment map $\mu_{\mathrm{M}} = \mu \restriction \mathrm{M}$ of $\mathrm{SL}(2,\mathbf{R})$ on M is a non trivial double sheets covering onto its image $\mathcal{O} = \mu(\mathrm{M})$. It is possible to complicate this example by considering the universal covering $\tilde{\mathrm{M}}$ of M, equipped with the pullback $\tilde{\omega}$ of ω by the projection $\pi : \tilde{\mathrm{M}} \to \mathrm{M}$. Then, the action of the universal covering $\tilde{\mathrm{SL}}(2,\mathbf{R})$ on $\tilde{\mathrm{M}}$ is still effective homogeneous and Hamiltonian, and the moment map $\tilde{\mu}$ factorizes through π and has the same image $\mathcal{O}$. ▶

8. The linear cylinder — The example of the cylinder is interesting because it shows simply and explicitly what happens when a symplectic form is exact but not its primitive. So, let $\mathrm{M} = \mathbf{R} \times \mathrm{S}^1$ equipped with the 2-form $\omega = d\alpha$, and $\alpha = r \times dz/iz$, where $(r,z) \in \mathbf{R} \times \mathrm{S}^1$ and S^1 is identified with the complex numbers of modulus 1. The manifold M is also a group G, acting by $g_{\mathrm{M}}(r,z) = (r+\rho, \zeta z)$, with $g = (\rho, \zeta)$. Now, for all

[12] $\mathfrak{sl}(2,\mathbf{R})$ denotes the Lie algebra of $\mathrm{SL}(2,\mathbf{R})$, that is, the vector space of real 2×2 traceless matrices.

$g \in \mathrm{G}$,

$$g_{\mathrm{M}}^{*}(\alpha) = \alpha + \beta(g) \quad \text{with} \quad \beta(g) = \rho \frac{dz}{iz}, \quad \beta \in \mathrm{C}^{\infty}(\mathrm{G}, \mathrm{Z}_{\mathrm{DR}}^{1}(\mathrm{M})).$$

The form $\beta(g)$ is closed for every $g \in \mathrm{G}$ as it must be. The holonomy group Γ is the subgroup of all $\Psi(\ell) = \hat{\ell}^{*}(\mathscr{K}\omega)$, where ℓ runs over the loops of M (notations [Piz10]). We have,

$$\hat{\ell}^{*}(\mathscr{K}\omega) = \hat{\ell}^{*}(\mathscr{K}d\alpha) = \hat{\ell}^{*}(\hat{1}^{*}\alpha - \hat{0}^{*}\alpha - d[\mathscr{K}\alpha]) = -d[\mathscr{K}\alpha \circ \hat{\ell}],$$

but $\hat{\ell}(g) = g \circ \ell$, thus $\mathscr{K}\alpha \circ \hat{\ell}(g) = \mathscr{K}\alpha(g \circ \ell)$, and then

$$\begin{aligned}
\Psi(\ell) = \hat{\ell}^{*}(\mathscr{K}\omega) &= -d\left[g \mapsto \int_{g\circ\ell} \alpha\right] = -d\left[g \mapsto \int_{\ell} g^{*}(\alpha)\right] \\
&= -d\left[g \mapsto \int_{\ell} \alpha + \int_{\ell} \beta(g)\right] = -d\left[g \mapsto \int_{\ell} \beta(g)\right] \\
&= -d\left[g \mapsto \int_{\ell} \rho \frac{dz}{iz}\right] = -d[g \mapsto 2\pi k \rho] \\
&= -2\pi k \times d\rho,
\end{aligned}$$

where $k \in \mathbf{Z}$ represents the class of the loop ℓ (we know that $\Psi(\ell)$ depends only on the homotopy class of ℓ [Piz10, §4.7 - 2]). Hence, the form $a = d\rho$ is a good closed (even exact) invariant 1-form of G, that is a momenta of G. And,

$$\Gamma = \{2\pi k \times a \mid k \in \mathbf{Z}\} \quad \text{with} \quad a = d\rho.$$

Now, the space $\mathscr{G}^{*}$ of momenta of the Lie group G is generated by $a = d\rho$ and $b = d\zeta/i\zeta$, the quotient $\mathscr{G}^{*}/\Gamma$ is thus equal to $[\mathbf{R}a/2\pi\mathbf{Z}a] \times \mathbf{R}b$ which is equivalent to $\mathrm{S}^{1} \times \mathbf{R}$. ▶

9. The holonomy of the torus — We shall compute the holonomy group Γ_{ω} for the 2-torus $\mathrm{T}^{2} = \mathbf{R}^{2}/\mathbf{Z}^{2}$, equipped with $\omega = \mathrm{class}_{*}(dx \wedge dy)$, the canonical volume form on T^{2}. We denoted by $\mathrm{class} : \mathbf{R}^{2} \to \mathrm{T}^{2}$, the canonical projection.

We know that Γ_{ω} is a homomorphic image of the first homotopy group of T^{2}, that is, $\pi_{1}(\mathrm{T}^{2}) = \mathbf{Z}^{2}$. We choose then a canonical representant of every homotopy class:

$$\ell_{n,m} = [t \mapsto \mathrm{class}(nt, mt)], \quad \text{with } n, m \in \mathbf{Z}.$$

We will show now that the map $j : (n, m) \mapsto \Psi_{\omega}(\ell_{n,m})$ is injective. Since $\Psi_{\omega}(\ell)$ is a closed 1-form on the group $\mathrm{Diff}(\mathrm{T}^{2}, \omega)$ for any loop ℓ [Piz10], it is sufficient, if $(n, m) \neq (0, 0)$, to find a loop γ in $\mathrm{Diff}(\mathrm{T}^{2}, \omega)$ such that $\int_{\gamma} \Psi_{\omega}(\ell_{n,m}) \neq 0$. We have

$$\int_{\gamma} \Psi(\ell) = \int_{0}^{1} \Psi_{\omega}(\ell)(\gamma)_{s}(1)\, ds = \int_{0}^{1} \left[\int_{0}^{1} \omega_{\ell(t)}(\dot{\ell}(t))(\delta\ell(s,t))\, dt\right] ds,$$

with

$$\delta\ell(s,t) = [\mathrm{D}(\gamma(s))(\ell(t))]^{-1}\frac{\partial\gamma(s)(\ell(t))}{\partial s}$$

Consider now two integers $j, k \in \mathbf{Z}$, we check immediately that

$$\gamma(s) = \left[\operatorname{class}\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \operatorname{class}\begin{pmatrix} x + sj \\ y + sk \end{pmatrix}\right]$$

is a loop in $\mathrm{Diff}(\mathrm{T}^2, \omega)$ based at the identity. For that γ, and for $\ell = \ell_{n,m}$, we have:

$$\dot{\ell}_{n,m}(t) = \operatorname{class}_*\begin{pmatrix} n \\ m \end{pmatrix} \quad \text{and} \quad \delta\ell(s,t) = \operatorname{class}_*\begin{pmatrix} j \\ k \end{pmatrix}.$$

Then,

$$\omega_{\ell(t)}(\dot{\ell}(t))(\delta\ell(s,t)) = \det\begin{pmatrix} n & j \\ m & k \end{pmatrix} = nk - mj.$$

Thus,

$$\int_\gamma \Psi_\omega(\ell_{n,m}) = nk - mj.$$

Hence, $j(\ell_{n,m}) = 0$ only for $n = m = 0$. Therefore, j is injective and $\Gamma_\omega \simeq \mathbf{Z}^2$. ▶

References

[Boo69] William M. Boothby. *Transitivity of the automorphisms of certain geometric structures.* Trans. Amer. Math. Soc. vol. 137, pp. 93–100, 1969.

[Don84] Paul Donato *Revêtement et groupe fondamental des espaces différentiels homogènes.* Thèse de doctorat d'état, Université de Provence, Marseille, 1984.

[Kos70] Bertram Kostant. *Orbits and quantization theory*. In Actes, Congrès intern. Math., 1970. vol. 2, pp. 395–400.

[Kir74] Alexandre A. Kirillov. *Elements de la théorie des représentations.* Ed. MIR, Moscou, 1974.

[Piz13] Patrick Iglesias-Zemmour. *Variations of integrals in diffeology.* Canad. J. Math. Vol. 65 (6), 2013 pp. 1255–1286.

[Piz10] Patrick Iglesias-Zemmour *The moment maps in diffeology.* Memoirs of the American Mathematical Society, vol. 207, RI 2010.

[MW82] Jerry Marsden and Alan Weinstein. *The hamiltonian structure of the maxwell-vlasov equations.* Physica D, 4:394–406, 1982.

[Omo86] Stephen Malvern Omohundro. *Geometric Perturbation Theory in Physics.* World Scientific, 1986.

[Sou70] Jean-Marie Souriau. *Structure des systèmes dynamiques.* Dunod Ed., Paris, 1970.

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