

EXAMPLE OF SINGULAR REDUCTION IN SYMPLECTIC DIFFEOLOGY

PATRICK IGLESIAS-ZEMMOUR

ABSTRACT. We present an example of symplectic reduction in diffeology where the space involved is infinite dimensional and the reduction is singular. This example is a mix of two cases that are not handled by ordinary symplectic geometry. We show that, in this infinite dimensional example, the singularities are distributed in such a way that the symplectic form, restricted to a generic level of the moment map, passes to the reduced space.

INTRODUCTION

We equip the space of periodic functions from \mathbf{R} to \mathbf{C} with a homogeneous symplectic structure. Then, we push this structure forward on the space of Fourier coefficients. We give an autonomous description of the diffeology we obtain that way. Next, we equip the infinite product of toruses with a *tempered diffeology*, introduced for the purpose of the example. Equipped with this diffeology, the infinite torus acts smoothly by automorphisms on the symplectic space of Fourier coefficients. We compute the moment map of the action of the infinite torus. Given a sequence of rationally independent real numbers we induce the real line in the infinite torus as an *irrational solenoid* and consider the induced action on the space of Fourier coefficients. This action is not free and generates infinitely many singular orbits. We consider then a generic level subspace of the moment map of the solenoid (they are all isomorphic to the infinite sphere) and we reduce it by the action of the solenoid. We call that quotient an *infinite quasi-projective space*. We show eventually that, in spite of the presence of singular orbits, the restriction of the symplectic form passes to the quotient and equip every infinite quasi-projective space with a *parasymplectic structure*, that is, a closed 2-form.

This is a very informative example. It shows that, not only is diffeology a very versatile and flexible approach of differential geometry, but it offers moreover the right framework to deal with singularities, even in infinite dimensional situation where all the difficulties are met. For the basic constructions and the vocabulary of diffeology used here we recommend the lecture of the book [PIZ13].

A few words on Symplectic Diffeology. For the sake of simplicity I decided to call *parasymplectic space* any diffeological space equipped with a closed 2-form. In diffeology there may be a few different ways to understand the word *symplectic*¹. The definition I present here uses the two main tools at our disposal: the group of automorphisms of the

Date: March 8, 2015.

¹For example, a symplectic space could also be defined as a parasymplectic space generated by symplectic plots, as it has been suggested once by Yael Karshon. That would include, for example, the cone orbifold $\mathcal{Q}_m = \mathbf{C}/\mathbf{Z}_m$, which is excluded by the definition of this article.

parasymplectic structure and the universal moment map². We recall that the moment maps are defined for any parasymplectic space and do not need any particular conditions on the parasymplectic form. Actually what we really need is the pseudogroup of local automorphisms. Then we can distinguish four cases for a parasymplectic space:

- (+ +) The local automorphisms are transitive and the universal moment map has discrete prevalues³. For manifolds, that is a characterisation of symplectic forms⁴. In this case, we shall say that the diffeological space, equipped with such a parasymplectic form, is **symplectic**⁵.
- (+ −) The local automorphisms are transitive but the the prevalues of the moment map are not discrete. For manifolds, that is a characterization of presymplectic forms. We shall say in this case that the parasymplectic form is **presymplectic**.
- (− +) The local automorphisms are not transitive but the moment map has discrete prevalues. That is the case of some classic pictures like cone orbifolds for examples; or forms, symplectic everywhere except in a few points, where it vanishes. We have no special vocabulary yet for this kind of space.
- (− −) The local automorphisms are not transitive and the prevalues of the moment map are not discrete. That is the general case of parasymplectic spaces. In this case, as well as the previous one, we can still analyse the space in terms of orbits of the pseudogroup of automorphisms and the various level of the universal moment map in each orbit.

Thanks to Elisa Prato who invited me to Firenze for discussing the subject, and also to Fiamma Bataglia. The discussion I had with them led me to develop this example as an infinite dimensional version of one of their cases, see [EP01]. It would be certainly interesting to develop more seriously the diffeological version of their general constructions. Thanks also to the anonymous referee for his careful reading of the manuscript, for his advices and corrections that contributed to improve it.

REVIEW ON THE MOMENT MAPS OF A PARASYMPLECTIC FORM

Let (X, ω) be a parasymplectic space. Let G be a diffeological group, with a smooth action $g \mapsto g_X$ on X , preserving ω , that is, $g_X^*(\omega) = \omega$ for all $g \in G$. We denote by \mathcal{G}^* the space of *momenta* of G , defined as the left-invariant differential 1-forms on G ,

$$\mathcal{G}^* = \{\varepsilon \in \Omega^1(G) \mid L(g)^*(\varepsilon) = \varepsilon, \text{ for all } g \in G\}.$$

To understand the essential nature of the moment map, which is a map from X to \mathcal{G}^* , it is good to consider the simplest case, and use it then as a guide to extend this simple construction to the general case.

²The *moment maps* relative to a closed 2-form are described in [PIZ10] or in chapter 9 of [PIZ13]. The *universal moment map* is the moment map for the group of all the automorphisms of the 2-form.

³We call *prevalues* of a function, the preimages of its values.

⁴For manifold the universal moment map is actually injective.

⁵A stronger case is when the group of automorphisms is transitive, in this case the space is symplectic homogeneous.

The Simplest Case. Consider the case where X is a manifold, and G is a Lie group. Let us assume that ω is exact $\omega = d\alpha$, and that α is also invariant by G . Regarding ω , the *moment map*⁶ of the action of G on X is the map

$$\mu : X \rightarrow \mathcal{G}^* \quad \text{defined by} \quad \mu(x) = \hat{x}^*(\alpha),$$

where $\hat{x} : G \rightarrow X$ is the *orbit map* $\hat{x}(g) = g_X(x)$.

As we can see, there is no obstacle, in this simple situation, to generalize, *mutatis mutandis*, the moment map to a diffeological group acting by automorphisms on a diffeological parasymplectic space⁷. But, as we know, not all closed 2-forms are exact, and even if they are exact, they do not necessarily have an invariant primitive. We shall see now, how we can generally come to a situation, so close to the simple case above, that modulo some minor subtleties we can build a good moment map in all cases.

The General Case. We consider a connected parasymplectic diffeological space (X, ω) , and a diffeological group G acting on X and preserving ω . Let \mathcal{K} be the Chain-Homotopy Operator, defined in [PIZ13, 6.83]. The differential 1-form $\mathcal{K}\omega$, defined on $\text{Paths}(X)$, is related to ω by $d[\mathcal{K}\omega] = (\hat{1}^* - \hat{0}^*)(\omega)$, and $\mathcal{K}\omega$ is invariant by G . Considering $\bar{\omega} = (\hat{1}^* - \hat{0}^*)(\omega)$ and $\bar{\alpha} = \mathcal{K}\omega$, we are in the simple case: $\bar{\omega} = d\bar{\alpha}$ and $\bar{\alpha}$ invariant by G . We can apply the construction above and define then the *Moment Map of Paths* by

$$\Psi : \text{Paths}(X) \rightarrow \mathcal{G}^* \quad \text{with} \quad \Psi(\gamma) = \hat{\gamma}^*(\mathcal{K}\omega),$$

and $\hat{\gamma} : G \rightarrow \text{Paths}(X)$ is the orbit map $\hat{\gamma}(g) = g_X \circ \gamma$ of a path γ . The moment of paths is additive with respect to the concatenation,

$$\Psi(\gamma \vee \gamma') = \Psi(\gamma) + \Psi(\gamma').$$

This paths moment map Ψ is equivariant by G , acting by composition on $\text{Paths}(X)$, and by coadjoint action on \mathcal{G}^* . Next, defining the *Holonomy* of the action of G on X by

$$\Gamma = \{\Psi(\ell) \mid \ell \in \text{Loops}(X)\} \subset \mathcal{G}^*,$$

the *Two-Points Moment Map* is defined by pushing Ψ forward on $X \times X$,

$$\phi(x, x') = \text{class}(\Psi(\gamma)) \in \mathcal{G}^*/\Gamma,$$

where γ is a path connecting x to x' , and where class denotes the projection from \mathcal{G}^* onto its quotient \mathcal{G}^*/Γ . The holonomy Γ is the obstruction for the action of G to be *Hamiltonian*. The additivity of Ψ becomes the Chasles' cocycle condition

$$\phi(x, x') + \phi(x', x'') = \phi(x, x'').$$

The group Γ is invariant by the coadjoint action. Thus, the coadjoint action passes to the quotient \mathcal{G}^*/Γ , and ϕ is equivariant. Because X is connected, there exists always a map

$$\mu : X \rightarrow \mathcal{G}^*/\Gamma \quad \text{such that} \quad \phi(x, x') = \mu(x') - \mu(x).$$

⁶Precisely, one moment map, since they are defined up to a constant.

⁷Actually, we shall be in this simple situation further, with the action of the infinite torus on the space of rapidly decreasing sequences.

Up to an arbitrary constant, μ is given by $\mu(x) = \phi(x_0, x)$, where x_0 is a chosen point in X . But this map is *a priori* no longer equivariant. Its variance introduce a 1-cocycle θ of G with values in \mathcal{G}^*/Γ such that

$$\mu(g(x)) = \text{Ad}_*(g)(\mu(x)) + \theta(g),$$

with $\theta(g) = \phi(x_0, g(x_0))$, modulo a coboundary due to the constant in the choice of μ . This construction extends to the category $\{\text{Diffeology}\}$, the moment map for manifolds introduced by Souriau in [Sou70]. The remarkable point is that none of the constructions brought up above involves differential equations, and there is no need of considering a putative Lie algebra either. That is a very important point. The momenta appear as invariant 1-forms on the group, naturally, without intermediary, and the moment map as a map in the space of momenta.

Note that the group of automorphisms $G_\omega = \text{Diff}(X, \omega)$ is a legitimate diffeological group. The above constructions apply and give rise to universal objects: *universal momenta* \mathcal{G}_ω^* , *universal path moment map* Ψ_ω , *universal holonomy* Γ_ω , *universal two-points moment map* ϕ_ω , *universal moment maps* μ_ω , *universal Souriau's cocycles* θ_ω . A *parasymplectic action* of a diffeological group G is a smooth morphism $h : G \rightarrow G_\omega$, and the objects, associated with G , introduced by the above moment maps constructions, are naturally subordinate to their universal counterparts.

DIFFEOLGY ON FOURIER COEFFICIENTS OF PERIODIC FUNCTIONS

We denote by $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$ the space of 1-periodic functions defined on \mathbf{R} with values in \mathbf{C} , that is,

$$\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C}) = \{f \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{C}) \mid f(x+1) = f(x)\}.$$

We equip this space with the functional diffeology. Let us recall that a plot for this diffeology is a parametrization $P : U \rightarrow \mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$ such that the map $(r, t) \mapsto P(r)(t)$, from $U \times \mathbf{R}$ to \mathbf{C} , is smooth.

Let $f \in \mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$, we denote by $(f_n)_{n \in \mathbf{Z}}$ its sequence of Fourier coefficients, that is,

$$f_n = \int_0^1 f(x) e^{-2i\pi n x} dx, \text{ for all } n \in \mathbf{Z}.$$

We recall that the Fourier series converges to f , uniformly on $[0, 1]$ [Vil68, §2 Thm. 1]. We denote by

$$f(x) = \sum_{n \in \mathbf{Z}} f_n e^{2i\pi n x} \quad \text{the limit} \quad f(x) = \lim_{N \rightarrow \infty} \sum_{n=-N}^{+N} f_n e^{2i\pi n x}.$$

Let j be the map from $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$ to the set of series of complex numbers $\text{Maps}(\mathbf{Z}, \mathbf{C})$, that associates a smooth periodic function with its Fourier coefficients,

$$j : \mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C}) \rightarrow \text{Maps}(\mathbf{Z}, \mathbf{C}) \quad \text{with} \quad j(f) = (f_n)_{n \in \mathbf{Z}}.$$

That map j is injective, and we denote by \mathcal{E} its image, that is, the subspace of $\text{Maps}(\mathbf{Z}, \mathbf{C})$ made of the Fourier coefficients of smooth periodic functions,

$$\mathcal{E} = \{(f_n)_{n \in \mathbf{Z}} = j(f) \mid f \in \mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})\}.$$

We know that the subspace \mathcal{E} is made exactly of all rapidly decreasing sequences of complex numbers (*op. cit.*),

$$(f_n)_{n \in \mathbb{Z}} \in \mathcal{E} \quad \text{iff, for all } p \in \mathbb{N}, \quad n^p f_n \xrightarrow{|n| \rightarrow \infty} 0.$$

1. Functional Diffeology on Fourier Coefficients. A parametrizations $P : r \mapsto (f_n(r))_{n \in \mathbb{Z}}$ in \mathcal{E} is a plot, for the pushforward of the functional diffeology on $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$, if and only if:

- (1) The functions $f_n : \text{dom}(P) \rightarrow \mathbf{C}$ are smooth.
- (2) For all closed ball $\overline{\mathcal{B}} \subset \text{dom}(P)$, for every $k \in \mathbb{N}$, for all $p \in \mathbb{N}$, there exists a positive number $M_{k,p}$ such that, for all integer $n \neq 0$,

$$\left| \frac{\partial^k f_n(r)}{\partial r^k} \right| \leq \frac{M_{k,p}}{|n|^p} \quad \text{for all } r \in \mathcal{B}. \quad (\diamond)$$

NOTE 1. In other words, the parametrization $r \mapsto (f_n(r))_{n \in \mathbb{Z}}$ is a plot of this diffeology if the functions f_n are smooth and their derivatives are uniformly rapidly decreasing, what we denote also by

$$n^p \frac{\partial^k f_n(r)}{\partial r^k} \xrightarrow{|n| \rightarrow \infty} 0.$$

NOTE 2. By compactness, it is enough that, for every point $r_0 \in \text{dom}(P)$, there exists a ball \mathcal{B}' centered at r_0 such that (\diamond) holds to ensure that (\diamond) holds on every closed ball $\overline{\mathcal{B}} \subset \text{dom}(P)$.

Proof. Let us check, first of all, that the condition (\diamond) defines a diffeology on the space \mathcal{E} of rapidly decreasing sequences of complex numbers.

COVERING AXIOM — If $f_n(r) = f_n$ is constant in r , for all n , then the condition (\diamond) is trivially satisfied for $k > 0$, and for $k = 0$ it means that the series is rapidly decreasing, what it is.

LOCALITY AXIOM — According to Note 2, (\diamond) is a local condition.

SMOOTH COMPATIBILITY AXIOM — Let $P : (r \mapsto f_n(r))_{n \in \mathbb{Z}}$ satisfying (\diamond) and $F : s \mapsto r$ be a smooth parametrization in the domain of P . We have, for all $k > 0$,

$$\frac{\partial^k f_n(s)}{\partial s^k} = \sum_{\ell=1}^k \frac{\partial^\ell f_n(r)}{\partial r^\ell} \cdot Q_{k,\ell} \left(\frac{\partial r}{\partial s}, \dots, \frac{\partial^k r}{\partial s^k} \right),$$

where the $Q_{k,\ell}$ are polynomials. Now, since $s \mapsto r$ is smooth, the partial derivatives are bounded on every ball, and because the $Q_{k,\ell}$ are polynomials in the partial derivatives, as a function of s , they are bounded operators on every ball. Let then $s_0 \in \text{dom}(F)$, $r_0 = F(s_0)$ and \mathcal{B} be a ball centered at r_0 such that the condition (\diamond) is satisfied. Let $\mathcal{B}' \subset F^{-1}(\mathcal{B})$ be a ball centered at s_0 , we have for all $s \in \mathcal{B}'$,

$$\begin{aligned} \left| \frac{\partial^k f_n(s)}{\partial s^k} \right| &\leq \sum_{\ell=1}^k \left| \frac{\partial^\ell f_n(r)}{\partial r^\ell} \right| \left| Q_{k,\ell} \left(\frac{\partial r}{\partial s}, \dots, \frac{\partial^k r}{\partial s^k} \right) \right| \\ &\leq \sum_{\ell=1}^k \frac{M_{\ell,p}}{|n|^p} \sup_{s \in \mathcal{B}'} \left| Q_{k,\ell} \left(\frac{\partial r}{\partial s}, \dots, \frac{\partial^k r}{\partial s^k} \right) \right|, \end{aligned}$$

where the $M_{\ell,p}$ are the constants of the inequality (\diamond) for the ball \mathcal{B} . Let then

$$m_{k,\ell} = \sup_{s \in \mathcal{B}'} \left| Q_{k,\ell} \left(\frac{\partial r}{\partial s}, \dots, \frac{\partial^k r}{\partial s^k} \right) \right| \quad \text{and} \quad M'_{k,p} = \sum_{\ell=1}^k m_{k,\ell} M_{\ell,p},$$

we get, for all $s \in \mathcal{B}'$,

$$\left| \frac{\partial^k f_n(s)}{\partial s^k} \right| \leq \frac{M'_{k,p}}{|n|^p}.$$

Hence, thanks to Note 2, $P \circ F$ still satisfies the condition (\diamond) . Therefore, this condition defines a diffeology on the set \mathcal{E} of rapidly decreasing sequences of complex numbers.

Let us show now that the diffeology defined on \mathcal{E} by (\diamond) is the pushforward by j of the functional diffeology of $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$.

A.— Let us prove, first of all, that j is smooth, where $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$ is equipped with the functional diffeology and \mathcal{E} with the diffeology defined by (\diamond) . Let $P : r \rightarrow f_r$ be a plot of $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$ defined on some real domain U , the composite $j \circ P$ writes

$$j \circ P : r \mapsto (f_n(r))_{n \in \mathbf{Z}} \quad \text{with} \quad f_n(r) = \int_0^1 f_r(x) e^{-2i\pi n x} dx.$$

Clearly, since $(r, x) \mapsto f_r(x)$ is smooth, the $f_n : r \mapsto f_n(r)$ are smooth, and we have

$$\frac{\partial^k f_n(r)}{\partial r^k} = \int_0^1 \frac{\partial^k f_r(x)}{\partial r^k} e^{-2i\pi n x} dx.$$

For all nonzero integer n , after p integrations by parts, we get

$$\frac{\partial^k f_n(r)}{\partial r^k} = \frac{1}{(2i\pi n)^p} \int_0^1 \frac{\partial^p}{\partial x^p} \left(\frac{\partial^k f_r(x)}{\partial r^k} \right) e^{-2i\pi n x} dx.$$

Therefore, defining for every ball $\mathcal{B} \subset \text{dom}(P)$ the number

$$M_{k,p} = \frac{1}{(2\pi)^p} \sup_{r \in \mathcal{B}} \sup_{x \in [0,1]} \left| \frac{\partial^p}{\partial x^p} \frac{\partial^k}{\partial r^k} f_r(x) \right|,$$

we have

$$\left| \frac{\partial^k f_n(r)}{\partial r^k} \right| \leq \frac{M_{k,p}}{|n|^p} \quad \text{for all } r \in \mathcal{B}.$$

Hence, $j \circ P$ is a plot of \mathcal{E} . Therefore j is smooth.

B.— Conversely, remember that j is injective, then let us show that $j^{-1} : \mathcal{E} \rightarrow \mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$ is smooth. Let $P : r \mapsto (f_n(r))_{n \in \mathbf{Z}}$ be a parametrization in \mathcal{E} satisfying the condition (\diamond) , and let

$$j^{-1} \circ P : r \mapsto [x \mapsto f_r(x)].$$

The parametrization $r \mapsto f_r$ is given by

$$f_r(x) = \lim_{N \rightarrow \infty} S_N(r, x) \quad \text{with} \quad S_N(r, x) = \sum_{n=-N}^{+N} f_n(r) e^{2i\pi n x}.$$

The S_N are smooth for all N , we want to show that the limit $(r, x) \mapsto f_r(x)$ is smooth too. For all $k, p, N \in \mathbf{N}$, let us introduce the series of partial derivatives

$$S_N^{k,p}(r, x) = \frac{\partial^p}{\partial x^p} \frac{\partial^k}{\partial r^k} S_N(r, x) = \sum_{n=-N}^{+N} (2i\pi n)^p \frac{\partial^k f_n(r)}{\partial r^k} e^{2i\pi n x}.$$

The functions $S_N^{k,p}$ are smooth, with respect to the pair (r, x) , for all N, k, p . Let us denote by $|S_N^{k,p}(r, x)|$ the series of moduli of the terms of $S_N^{k,p}$

$$|S_N^{k,p}(r, x)| = \sum_{n=-N}^{+N} |2\pi n|^p \left| \frac{\partial^k f_n(r)}{\partial r^k} \right|.$$

Then, let $\overline{\mathcal{B}} \subset \text{dom}(P)$ be some closed ball. Thanks to the hypothesis, applying (2) to $p+2$, we have, for all $r \in \mathcal{B}$,

$$\sum_{n=-N}^{+N} |2\pi n|^p \left| \frac{\partial^k f_n(r)}{\partial r^k} \right| \leq c + 2 \sum_{n=1}^N (2\pi)^p \frac{M_{k,p+2}}{n^2},$$

where c , corresponding to $n=0$, is some constant. Hence, for all $r \in \mathcal{B}$, $x \in [0, 1]$ and $N \in \mathbf{N}$

$$|S_N^{k,p}(r, x)| \leq K$$

where $K = c + 2(2\pi)^p M_{k,p+2}(\pi^2/6)$. Next, since the series $|S_N^{k,p}(r, x)|$ is increasing and upper bounded, it is convergent, and therefore the series $S_N^{k,p}(r, x)$ is also convergent. Moreover, according to what preceded: the sequence of smooth functions $S_N^{k,p}$ converges uniformly on $\mathcal{B} \times [0, 1]$ to some function $S_\infty^{k,p}$ when $N \rightarrow \infty$. Now, according to a classical result of differential calculus, see for example (an obvious improvement of) [Don00, Thm. 3.10], the map $(r, x) \mapsto f_r(x) = \lim_{N \rightarrow \infty} S_N(r, x)$ is smooth, and the $S_\infty^{k,p}$ are the partial derivatives

$$S_\infty^{k,p}(r, x) = \frac{\partial^p}{\partial x^p} \frac{\partial^k}{\partial r^k} f_r(x).$$

Thus, the parametrization $r \mapsto [x \mapsto f_r(x)]$ is a plot of $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$ such that $j(r \mapsto f_r) = (f_n(r))_{n \in \mathbf{Z}}$. Therefore, j is a diffeomorphism from $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$, equipped with the functional diffeology, to \mathcal{E} , equipped with the diffeology defined by (\diamond) . In other words, the diffeology defined on \mathcal{E} by (\diamond) is the pushforward of the functional diffeology on $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$. \square

THE INFINITE TORUS

We denote by T^∞ the group of infinite sequences $Z = (z_n)_{n \in \mathbf{Z}}$, with $z_n \in U(1)$, that is,

$$T^\infty = \prod_{n \in \mathbf{Z}} U(1).$$

We consider the natural \mathbf{C} -linear action of T^∞ on \mathcal{E} ,

$$(z_n)_{n \in \mathbf{Z}} \cdot (Z_n)_{n \in \mathbf{Z}} = (z_n Z_n)_{n \in \mathbf{Z}}.$$

Indeed, multiplying by a number of modulus 1 transforms a rapidly decreasing sequence of complex numbers into another, and every element $z = (z_n)_{n \in \mathbb{Z}} \in T^\infty$ is invertible,

$$(z_n)_{n \in \mathbb{Z}}^{-1} = (\bar{z}_n)_{n \in \mathbb{Z}},$$

where \bar{z} denotes the conjugate. Moreover, for every plot $r \mapsto (Z_n(r))_{n \in \mathbb{Z}}$ in \mathcal{E} , for all $p \in \mathbb{N}$,

$$\left| \frac{\partial^p z_n Z_n(r)}{\partial r^p} \right| = \left| \frac{\partial^p Z_n(r)}{\partial r^p} \right|.$$

Obviously, the same holds for the inverse. Hence, the action of $(z_n)_{n \in \mathbb{Z}}$ is smooth as well as its inverse, and then $(z_n)_{n \in \mathbb{Z}}$ acts on \mathcal{E} by diffeomorphism. Thus, we got a monomorphism

$$\eta : T^\infty \rightarrow \text{GL}^\infty(\mathcal{E}) = \text{GL}(\mathcal{E}) \cap \text{Diff}(\mathcal{E}).$$

2. The Tempered Diffeology on the Infinite Torus. We shall say that a parametrization $\zeta : r \mapsto (z_n(r))_{n \in \mathbb{Z}}$ in T^∞ is *tempered* if the z_n are smooth and if for every $k \in \mathbb{N}$, for every r_0 in the domain of the parametrization, there exist a closed ball $\bar{\mathcal{B}} \subset \text{dom}(\zeta)$ centered at r_0 , a polynomial P_k and an integer N such that

$$\forall r \in \bar{\mathcal{B}}, \forall n > N, \quad \left| \frac{\partial^k z_n(r)}{\partial r^k} \right| \leq P_k(n). \quad (\heartsuit)$$

The tempered parametrizations form a group diffeology on T^∞ . Equipped with the tempered diffeology, the action of the group T^∞ on \mathcal{E} is smooth. Moreover, for all $N \in \mathbb{N}$, the injection $\iota_N : T^N \rightarrow T^\infty$ defined as follows is an induction:

$$\iota_N(\zeta) = Z \quad \text{with} \quad \begin{cases} Z_n = \zeta_n & \text{if } n \in \{1, \dots, N\}, \\ Z_n = 1 & \text{otherwise.} \end{cases}$$

Proof. Let us show first that the condition (\heartsuit) defines a diffeology, actually a sub-diffeology of the product diffeology on the infinite product of toruses $T^\infty = \prod_{n \in \mathbb{Z}} T$.

COVERING AXIOM — If $\zeta = (z_n)_{n \in \mathbb{Z}}$ is constant, then $|\partial^k z_n(r)/\partial r^k|$ is equal to 1 for $k = 0$ and equal to 0 for $k > 0$. Thus we chose $P_0(n) = 1$ and $P_k(n) = 0$ for $k \neq 0$, and the condition (\heartsuit) is satisfied.

LOCALITY AXIOM — The condition (\heartsuit) is local in the variable r , by construction.

SMOOTH COMPATIBILITY AXIOM — Let $\zeta : (r \mapsto z_n(r))_{n \in \mathbb{Z}}$ satisfying (\heartsuit) and $F : s \mapsto r$ be a smooth parametrization in the domain of ζ . We have, for all $k > 0$,

$$\frac{\partial^k z_n(s)}{\partial s^k} = \sum_{\ell=1}^k \frac{\partial^\ell z_n(r)}{\partial r^\ell} \cdot Q_{k,\ell} \left(\frac{\partial r}{\partial s}, \dots, \frac{\partial^k r}{\partial s^k} \right),$$

where the $Q_{k,\ell}$ are polynomials. Therefore,

$$\left| \frac{\partial^k z_n(s)}{\partial s^k} \right| \leq \sum_{\ell=1}^k \left| \frac{\partial^\ell z_n(r)}{\partial r^\ell} \right| \left| Q_{k,\ell} \left(\frac{\partial r}{\partial s}, \dots, \frac{\partial^k r}{\partial s^k} \right) \right|.$$

Since the function $s \mapsto r$ is smooth, the partial derivatives are bounded on every ball, and these polynomials are locally absolutely bounded operators. Let

$$M_{k,\ell} = \sup_{r \in \bar{\mathcal{B}}} \left| Q_{k,\ell} \left(\frac{\partial r}{\partial s}, \dots, \frac{\partial^k r}{\partial s^k} \right) \right|.$$

Next, let N_ℓ satisfying (\heartsuit) for $k = \ell$, and $N' = \sup_{\ell=1\dots k} N_\ell$, then for all $n > N'$,

$$\left| \frac{\partial^k z_n(s)}{\partial s^k} \right| \leq \sum_{\ell=1}^k M_{k,\ell} \left| \frac{\partial^\ell z_n(r)}{\partial r^\ell} \right| \leq \sum_{\ell=1}^k M_{k,\ell} P_\ell(n).$$

Hence, the partial derivatives of the z_n with respect to r are still upper bounded by polynomials, and the axiom of compatibility is satisfied.

Therefore, the condition (\heartsuit) defines a diffeology on the set T^∞ . Let us check now that (\heartsuit) defines a group diffeology. First of all, let $z : r \mapsto (z_n(r))_{n \in \mathbb{Z}}$ and $z' : r \mapsto (z'_n(r))_{n \in \mathbb{Z}}$ be two plots of T^∞ , the derivatives of the n -th term $r \mapsto z_n(r)z'_n(r)$, of the product zz' , writes

$$\frac{\partial^k z_n(r)z'_n(r)}{\partial r^k} = \sum_{\ell=0}^k \binom{k}{\ell} \frac{\partial^{k-\ell} z_n(r)}{\partial r^{k-\ell}} \cdot \frac{\partial^\ell z'_n(r)}{\partial r^\ell}.$$

Hence,

$$\begin{aligned} \left| \frac{\partial^k z_n(r)z'_n(r)}{\partial r^k} \right| &\leq \sum_{\ell=0}^k \binom{k}{\ell} \left| \frac{\partial^{k-\ell} z_n(r)}{\partial r^{k-\ell}} \right| \left| \frac{\partial^\ell z'_n(r)}{\partial r^\ell} \right| \\ &\leq \sum_{\ell=0}^k \binom{k}{\ell} P_{k-\ell}(n) P'_\ell(n). \end{aligned}$$

Thus, the partial derivatives of the product $z_n z'_n$ are still upper bounded by a polynomial. For the inverse mapping $z^{-1} = (\bar{z}_n)_{n \in \mathbb{Z}}$, we have, since $\bar{z}_n(r)z_n(r) = 1$,

$$\frac{\partial^k z_n(r)\bar{z}_n(r)}{\partial r^k} = 0,$$

that is,

$$z_n \frac{\partial^k \bar{z}_n(r)}{\partial r^k} + \sum_{\ell=0}^{k-1} \binom{k}{\ell} \frac{\partial^{k-\ell} z_n(r)}{\partial r^{k-\ell}} \cdot \frac{\partial^\ell \bar{z}_n(r)}{\partial r^\ell} = 0.$$

Therefore,

$$\frac{\partial^k \bar{z}_n(r)}{\partial r^k} = -\frac{1}{z_n(r)} \sum_{\ell=0}^{k-1} \binom{k}{\ell} \frac{\partial^{k-\ell} z_n(r)}{\partial r^{k-\ell}} \cdot \frac{\partial^\ell \bar{z}_n(r)}{\partial r^\ell}.$$

Thus, if the partial derivatives $\partial^\ell \bar{z}_n(r)/\partial r^\ell$ are absolutely bounded by a polynomial for all $\ell < k$, then the k -th derivative $\partial^k \bar{z}_n(r)/\partial r^k$ is also absolutely bounded by a polynomial. And since $|\partial \bar{z}_n(r)/\partial r| = |\partial z_n(r)/\partial r|$, by recursion, $\partial^k \bar{z}_n(r)/\partial r^k$ is upper bounded by a polynomial. Therefore, T^∞ , equipped with the tempered diffeology is a diffeological group.

For the last sentence, the injection $\iota_N : T^N \rightarrow T^\infty$ is clearly smooth since, for all plots ζ in T^N , the derivatives of the components Z_n of the composite $\iota_N \circ \zeta : r \mapsto (Z_n(r))_{n \in \mathbb{Z}}$ vanish for n outside $\{1, \dots, N\}$, hence

$$n^p \frac{\partial^k Z_n(r)}{\partial r^k} = 0 \quad \text{if } |n| > N.$$

Conversely if $\zeta : r \mapsto (Z_n(r))_{n \in \mathbb{Z}}$ is a plot of T^∞ with values in $\iota_N(T^N)$, then the components $r \mapsto Z_n(r)$ are smooth, by definition of the diffeology on T^∞ , and hence $\iota_N^{-1} \circ \zeta$ is smooth. Therefore, ι_N is an induction. \square

3. Induced Solenoids — A sequence $\alpha = (\alpha_n)_{n \in \mathbf{Z}}$ of positive numbers, is independent over \mathbf{Q} , if for every finitely supported sequence $(q_n)_{n \in \mathbf{Z}}$ of rational numbers $q_n \in \mathbf{Q}$,

$$\sum_{n \in \mathbf{Z}} q_n \alpha_n = 0 \quad \Rightarrow \quad q_n = 0 \quad \text{for all } n.$$

In the following we will consider such sequences with $|\alpha_n| \leq 1$. Then, the map

$$\iota : \mathbf{R} \mapsto \mathbf{T}^\infty, \quad \text{defined by } \iota(t) = \left(e^{2i\pi\alpha_n t} \right)_{n \in \mathbf{Z}},$$

is obviously injective, but it is moreover an induction, that is, a diffeomorphism onto its image equipped with the subset diffeology. We call the image $\iota(\mathbf{R}) \subset \mathbf{T}^\infty$, an *irrational solenoid*.

Proof. Let us show first that such irrational sequences $\alpha = (\alpha_n)_{n \in \mathbf{Z}}$ exist. Consider a transcendental number α . We define

$$\alpha_0 = 1 \quad \text{and} \quad \alpha_m = \alpha^m - [\alpha^m] \quad \text{for } m \neq 0,$$

where the bracket denotes the integral part. That defines α_n for all $n \in \mathbf{Z}$. Now, for a finitely supported sequence of rational numbers q_n , we have

$$\begin{aligned} \sum_{n \in \mathbf{Z}} q_n \alpha_n &= q_0 + \sum_{\substack{n \in \mathbf{Z} \\ n \neq 0}} q_n (\alpha^n - [\alpha^n]) \\ &= q_0 - \sum_{\substack{n \in \mathbf{Z} \\ n \neq 0}} q_n [\alpha^n] + \sum_{\substack{n \in \mathbf{Z} \\ n \neq 0}} q_n \alpha^n, \end{aligned}$$

with only a finite part of this sum is not zero. Then, if $\sum_{n \in \mathbf{Z}} q_n \alpha_n = 0$ we multiply both sides by α^ℓ , with ℓ big enough to get an algebraic equation in α , with all powers positive. But we assumed α transcendental, then all the coefficients are 0, what implies $q_n = 0$, for all $n \in \mathbf{Z}$.

Next, we need to check first of all that ι is smooth. The successive derivatives are simply

$$\frac{\partial^k e^{2i\pi\alpha_n t}}{\partial t^k} = (2i\pi\alpha_n)^k e^{2i\pi\alpha_n t}.$$

Since $0 < \alpha_n \leq 1$, for all $n \in \mathbf{Z}$, we have

$$\left| \frac{\partial^k e^{2i\pi\alpha_n t}}{\partial t^k} \right| \leq (2\pi)^k.$$

Hence, the derivatives are absolutely upper bounded by (constant) polynomials, $P_k(n) = (2\pi)^k$, and therefore ι is a plot in \mathbf{T}^∞ . Conversely, consider a plot in \mathbf{T}^∞ with values in $\iota(\mathbf{R})$, the composite with the projection on the first two components gives a plot in \mathbf{T}^2 with values in the solenoid $(\exp(2i\pi t), \exp(2i\pi\alpha t))_{t \in \mathbf{R}}$. But the injection $t \mapsto (\exp(2i\pi t), \exp(2i\pi\alpha t))$ being an induction [PIZ13, Exercise 31], it follows that ι is also an induction. \square

SYMPLECTIC STRUCTURE ON THE SPACE OF FOURIER COEFFICIENTS

We recall that a differential k -form ε on the diffeological space \mathcal{E} is a map that associates, with every plot P in \mathcal{E} , a k -form $\varepsilon(P)$ on $\text{dom}(P)$ such that, for all smooth parametrization F in $\text{dom}(P)$, one has

$$\varepsilon(P \circ F) = F^*(\varepsilon(P)).$$

4. Symplectic Structure on $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$ — Let Surf be the standard symplectic form on \mathbf{C} , defined by

$$\text{Surf}_z(\delta z, \delta' z) = \frac{1}{2i} [\delta \bar{z} \delta' z - \delta' \bar{z} \delta z],$$

where $z, \delta z, \delta' z \in \mathbf{C}$. For all $x \in \mathbf{R}$, let

$$\hat{x} : \mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C}) \rightarrow \mathbf{C} \quad \text{with} \quad \hat{x}(f) = f(x)$$

be the *evaluation map*. Because \hat{x} is smooth, the pullback $\hat{x}^*(\text{Surf})/\pi$ is a 2-form on $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$, and it is closed because Surf is closed. We denote by ω its mean value

$$\omega = \frac{1}{\pi} \int_0^1 \hat{x}^*(\text{Surf}) dx.$$

Evaluated on a plot $P : r \mapsto f_r$ in $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$, ω is given by

$$\omega(P)_r(\delta r, \delta' r) = \frac{1}{2i\pi} \int_0^1 \left\{ \frac{\partial \overline{f_r(x)}}{\partial r}(\delta r) \frac{\partial f_r(x)}{\partial r}(\delta' r) - \frac{\partial \overline{f_r(x)}}{\partial r}(\delta' r) \frac{\partial f_r(x)}{\partial r}(\delta r) \right\} dx.$$

Since integration is a smooth operation, ω is still a differential closed 2-form. Now, ω is invariant by translation $\text{Tr}_g : f \mapsto f + g$, and $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$ is an homogeneous space of itself. The moment map of this action is given, up to a constant, by:

$$\mu(f) = \frac{1}{2i\pi} d \left[g \mapsto \int_0^1 \bar{f} g - \bar{g} f \right].$$

An easy check shows that μ is injective. Thus $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$ equipped with ω is a *diffeological symplectic space*, in a strong sense. Indeed, $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$ acts transitively on itself by translation. This action preserves ω and the moment map for this action is injective. That falls under the definition we gave for being symplectic in diffeology, as the case $(+ +)$ of the Introduction, and considering [PIZ13, §9.19, Note 3]. By transfer, that applies to the pushforward of ω on \mathcal{E} .

Proof. The 2-form ω is actually exact, we choose the primitive ε defined by

$$\varepsilon(P)_r(\delta r) = \frac{1}{2i\pi} \int_0^1 \overline{f_r(x)} \frac{\partial f_r(x)}{\partial r}(\delta r) dx.$$

We use the notation of the Memoir on the Moment Maps in Diffeology [PIZ10]. Now, since the space $\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C})$ is contractible, we can compute the moment map through the paths moment map applied, for every f , to the path $\gamma_f : t \mapsto tf$. That is,

$$\mu(f) = \Psi(\gamma_f) = \hat{\gamma}_f^*(\mathcal{K} \omega),$$

where \mathcal{K} is Chain-Homotopy Operator. But $\omega = d\varepsilon$ and we recall that

$$d\mathcal{K} + \mathcal{K}d = \hat{1}^* - \hat{0}^*.$$

Therefore

$$\begin{aligned}
\mu(f) &= \hat{\gamma}_f^*(\mathcal{H}(d\varepsilon)) \\
&= \hat{\gamma}_f^*[\hat{1}^*(\varepsilon) - \hat{0}^*(\varepsilon) - d(\mathcal{H}\varepsilon)] \\
&= (\hat{1} \circ \hat{\gamma}_f)^*(\varepsilon) - (\hat{0} \circ \hat{\gamma}_f)^*(\varepsilon) - d[\hat{\gamma}_f^*(\mathcal{H}\varepsilon)]
\end{aligned}$$

But $\hat{\gamma}_f(g) = [t \mapsto T_g(tf) = tf + g]$, thus $\hat{1} \circ \hat{\gamma}_f(g) = f + g$, that is, $\hat{1} \circ \hat{\gamma}_f = T_f$. Then,

$$\begin{aligned}
T_f^*(\varepsilon)(r \mapsto g_r)_r(\delta r) &= \varepsilon(r \mapsto g_r + f)_r(\delta r) \\
&= \frac{1}{2i\pi} \int_0^1 \overline{(g_r(x) + f(x))} \frac{\partial g_r}{\partial r}(\delta r) dx \\
&= \varepsilon(r \mapsto g_r)_r(\delta r) \\
&+ d\left[g \mapsto \frac{1}{2i\pi} \int_0^1 \overline{f(x)} g(x) dx\right](r \mapsto g_r)_r(\delta r),
\end{aligned}$$

that is,

$$(\hat{1} \circ \hat{\gamma}_f)^*(\varepsilon) = \varepsilon + d\left[g \mapsto \frac{1}{2i\pi} \int_0^1 \overline{f(x)} g(x) dx\right].$$

On the other hand, $\hat{0} \circ \hat{\gamma}_f(g) = g$, that is, $\hat{0} \circ \hat{\gamma}_f = 1$ and $(\hat{0} \circ \hat{\gamma}_f)^*(\varepsilon) = \varepsilon$. Next, since ε is a 1-form, $\mathcal{H}\varepsilon$ is a function on $\text{Paths}(\mathcal{C}_{\text{per}}^\infty(\mathbf{R}, \mathbf{C}))$, precisely $\mathcal{H}\varepsilon(\gamma) = \int_\gamma \varepsilon$. Thus $\hat{\gamma}_f^*(\mathcal{H}\varepsilon) = (\mathcal{H}\varepsilon) \circ \hat{\gamma}_f$. But $(\mathcal{H}\varepsilon) \circ \hat{\gamma}_f(g) = \mathcal{H}\varepsilon(t \mapsto tf + g) = \int_0^1 \varepsilon(t \mapsto tf + g)_t(1) dt$, that is,

$$\begin{aligned}
(\mathcal{H}\varepsilon) \circ \hat{\gamma}_f(g) &= \int_0^1 \varepsilon(t \mapsto tf + g)_t(1) dt \\
&= \frac{1}{2i\pi} \int_0^1 \left\{ \int_0^1 (tf(x) + g(x)) f(x) dx \right\} dt \\
&= \frac{1}{2i\pi} \frac{1}{2} \int_0^1 |f(x)|^2 dx + \frac{1}{2i\pi} \int_0^1 \overline{g(x)} f(x) dx
\end{aligned}$$

Therefore,

$$d[(\mathcal{H}\varepsilon) \circ \hat{\gamma}_f] = d\left[g \mapsto \frac{1}{2i\pi} \int_0^1 \overline{g(x)} f(x) dx\right].$$

Combining everything into the expression of $\mu(f)$ we get what was announced. \square

5. Action of the Infinite Torus and Moment Maps — We consider the 2-form ω defined above pushed on the space of Fourier coefficients \mathcal{E} . Evaluated on a plot $P : r \mapsto (f_n(r))_{n \in \mathbf{Z}}$,

$$\omega(P)_r(\delta r, \delta' r) = \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} \frac{\partial \overline{f_n(r)}}{\partial r}(\delta r) \frac{\partial f_n(r)}{\partial r}(\delta' r) - \frac{\partial \overline{f_n(r)}}{\partial r}(\delta' r) \frac{\partial f_n(r)}{\partial r}(\delta r).$$

The ordinary action $(z_n)_{n \in \mathbf{Z}} \cdot (Z_n)_{n \in \mathbf{Z}} = (z_n Z_n)_{n \in \mathbf{Z}}$ of the infinite torus T^∞ on \mathcal{E} , is Hamiltonian and exact⁸. The Moment Map μ is given by

$$\mu(Z) = \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} |Z_n|^2 \pi_n^*(\theta) + \sigma,$$

where $\pi_n : T^\infty \rightarrow U(1)$ is the n -th projection $\pi_n(Z) = Z_n$, θ is the canonical invariant 1-form on $U(1)$, and σ is some constant momentum of T^∞ (a constant invariant 1-form).

Let $(\alpha_n)_{n \in \mathbf{Z}}$ be a sequence of numbers independent over \mathbf{Q} . We denote by $\iota : \mathbf{R} \rightarrow T^\infty$ the induction of \mathbf{R} in T^∞ , $\iota(t) = (e^{2i\pi\alpha_n t})_{n \in \mathbf{Z}}$, and by $\underline{\iota}(Z_n)_{n \in \mathbf{Z}} = (e^{2i\pi\alpha_n t} Z_n)_{n \in \mathbf{Z}}$, the induced action of \mathbf{R} on \mathcal{E} . Its moment map is given by

$$\nu(Z) = h(Z) dt \quad \text{with} \quad h(Z) = \sum_{n \in \mathbf{Z}} \alpha_n |Z_n|^2 + c,$$

where c is some constant.

Proof. The primitive ε of ω , that is,

$$\varepsilon(P)_r(\delta r) = \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} \overline{Z_n(r)} \frac{\partial Z_n(r)}{\partial r}(\delta r),$$

where $r \mapsto (Z_n(r))_{n \in \mathbf{Z}}$ is a plot of \mathcal{E} , is obviously invariant under the action of T^∞ . Therefore, the action is Hamiltonian and equivariant [PIZ13, §9.11]. And, the moment map is, up to a constant, the pullback of ε by the orbit map $\hat{Z} : T^\infty \rightarrow \mathcal{E}$, $\hat{Z}(z) = zZ$, where $z = (z_n)_{n \in \mathbf{Z}}$ and $Z = (Z_n)_{n \in \mathbf{Z}}$. Therefore, $\mu(Z) = \hat{Z}^*(\varepsilon) + \sigma$. Now, let θ be the canonical 1-form on $U(1)$ defined by

$$\theta(\zeta)_r(\delta r) = \bar{\zeta}(r) \frac{\partial \zeta(r)}{\partial r}(\delta r),$$

for all plots ζ in $U(1)$. Let $\pi_n : T^\infty \rightarrow U(1)$ be the n -th projection $Z \mapsto Z_n$, for all $Z = (Z_n)_{n \in \mathbf{Z}} \in \mathcal{E}$. Let $\zeta : r \mapsto (\zeta_n(r))_{n \in \mathbf{Z}}$ be a plot in T^∞ , the moment map μ is then explicitly given, up to the constant σ , by:

$$\begin{aligned} \mu(Z)(\zeta)_r(\delta r) &= \hat{Z}^*(\varepsilon)(\zeta)_r(\delta r) \\ &= \varepsilon(\hat{Z} \circ \zeta)_r(\delta r) \\ &= \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} \overline{Z_n \zeta_n(r)} \frac{\partial Z_n \zeta_n(r)}{\partial r}(\delta r) \\ &= \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} |Z_n|^2 \bar{\zeta}_n(r) \frac{\partial \zeta_n(r)}{\partial r}(\delta r) \\ &= \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} |Z_n|^2 \pi_n^*(\theta)(\zeta)_r(\delta r). \end{aligned}$$

Therefore, the general moment map is written:

$$\mu(Z) = \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} |Z_n|^2 \pi_n^*(\theta) + \sigma.$$

⁸Being exact means having an equivariant solution to the equation in μ , $\phi(x, x') = \mu(x') - \mu(x)$ [PIZ13, Chapter 9]

Remark that, for all $n \in \mathbf{Z}$, for all ball $\mathcal{B} \subset \text{dom}(\zeta)$, there exists a smooth function τ_n , defined on \mathcal{B} , such that $\zeta_n(r) = \exp(2i\pi\tau_n(r))$. Then, modulo a constant, the moment map is also given by

$$\mu(\mathbf{Z})(\zeta)_r(\delta r) = \sum_{n \in \mathbf{Z}} |Z_n|^2 \frac{\partial \tau_n(r)}{\partial r}(\delta r).$$

Note also that, since ζ is a plot in T^∞ for the tempered diffeology, the norm of the derivatives $\partial \zeta_n(r)/\partial r$, that is, the norm of $\partial \tau_n(r)/\partial r$, are upper bounded by a polynomial in n what insures the convergence of the series defining the moment map just above.

Now, consider the induction ι . It induces a projection $\iota^* : \mathcal{T}^{\infty*} \rightarrow \mathbf{R}^*$, where $\mathcal{T}^{\infty*}$ is the space of momenta of T^∞ . The moment maps with respect to the group \mathbf{R} are then the composites $\nu = \iota^* \circ \mu$, that is,

$$\nu = \iota^* \left\{ \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} |Z_n|^2 \pi_n^*(\theta) + \sigma \right\} = \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} |Z_n|^2 (\pi_n \circ \iota)^*(\theta) + \iota^*(\sigma).$$

But $\pi_n \circ \iota : t \mapsto \exp(2i\pi\alpha_n t)$ then $(\pi_n \circ \iota)^*(\theta) = 2i\pi\alpha_n dt$. Thus, $\nu(\mathbf{Z}) = b(\mathbf{Z}) dt$ with $b(\mathbf{Z}) = \sum_{n \in \mathbf{Z}} \alpha_n |Z_n|^2 + c$, where $\iota^*(\sigma) = c dt$, $c \in \mathbf{R}$. \square

QUASIPROJECTIVE ACTION OF THE REAL LINE AND REDUCTION

Let $\alpha = (\alpha_n)_{n \in \mathbf{Z}}$ be a sequence of positive numbers, independent over \mathbf{Q} . Let S_α^∞ be the unit level⁹ of the moment map ν of the action of \mathbf{R} ,

$$S_\alpha^\infty = \left\{ Z = (Z_n)_{n \in \mathbf{Z}} \in \mathcal{E} \mid \sum_{n \in \mathbf{Z}} \alpha_n |Z_n|^2 = 1 \right\}.$$

Let QP_α^∞ be the quotient of S_α^∞ under the action of \mathbf{R} , and pr be the projection,

$$\text{pr} : S_\alpha^\infty \rightarrow \text{QP}_\alpha^\infty \quad \text{and} \quad \text{QP}_\alpha^\infty = S_\alpha^\infty / \mathbf{R}.$$

We say that the quotient space is an *Infinite Quasiprojective Space*.

NOTE. Consider a general level $\sum_{n \in \mathbf{Z}} \alpha_n |Z_n|^2 - c$, with $c > 0$. By the change $Z_n \mapsto \sqrt{\alpha_n} Z_n / \sqrt{c}$, the subspace S_α^∞ is mapped into $S^\infty \subset \mathcal{E}$, the unit sphere in \mathcal{E} . Now, if all α_n would be equal to 1, then the action of \mathbf{R} would give the action of S^1 and QP_α^∞ would be diffeomorphic to the infinite projective space $\mathbf{CP}^\infty = S^\infty / S^1$, and the projection $\text{pr} : S_\alpha^\infty \rightarrow \text{QP}_\alpha^\infty$ would be the infinite Hopf fibration¹⁰. And that justifies our choice of vocabulary.

6. The Orbits of the Quasiprojective Action of \mathbf{R} — Considering the action of \mathbf{R} on S_α^∞ , the analysis of the action of \mathbf{R} in terms of orbits is given by the following properties.

- (1) Let $Z \in S_\alpha^\infty$. If there exist $Z_n \neq 0$ and $Z_m \neq 0$, then the stabilizer of Z is $\{0\}$ and the orbit of Z by \mathbf{R} , equipped with the subset diffeology, is diffeomorphic to \mathbf{R} . They are the *principal orbits*.

⁹The choice of the constant c in the moment map ν being irrelevant for the following, we choose $c = 0$.

¹⁰Same set but with another diffeology as in [PIZ13, §4.11].

(2) The *singular orbits*, that is, the non principal orbits, are the subspaces

$$S_n^1 = \{Z \in S_\alpha^\infty \mid Z_m = 0 \text{ if } m \neq n\}, \quad \text{with } n \in \mathbf{Z}.$$

Each singular orbit, equipped with the subset diffeology, is diffeomorphic to the circle S^1 .

(3) The singular locus, that is, the union $\mathcal{S} = \bigcup_{n \in \mathbf{Z}} S_n^1 \subset S_\alpha^\infty$, equipped with the subset diffeology, is actually the diffeological sum¹¹

$$\mathcal{S} = \coprod_{n \in \mathbf{Z}} S_n^1, \quad \text{and } \dim(\mathcal{S}) = 1.$$

(4) The regular locus, that is, the subset $S_\alpha^\infty - \mathcal{S}$, is D-open¹².

Proof. For the first item, let $Z \in S_\alpha^\infty$ with $Z_n \neq 0$ and $Z_m \neq 0$, the map

$$t \mapsto (e^{2i\pi\alpha_n t} z_n, e^{2i\pi\alpha_m t} z_m) \quad \text{with } z_n = \frac{Z_n}{|Z_n|} \quad \text{and } z_m = \frac{Z_m}{|Z_m|}$$

is an induction from \mathbf{R} into \mathbf{T}^2 , because α_n and α_m are independent over \mathbf{Q} , see [PIZ13, Exercise 31]. It follows from there that the orbit map $t \mapsto \underline{t}(Z)$ is an induction.

For the second item, there exists n such that for all $m \neq n$, $Z_m = 0$ but $Z_n \neq 0$, since $\sum_{m \in \mathbf{Z}} \alpha_n |Z_n|^2 > 0$. The orbit map is a covering onto the circle S_n^1 induced in S_α^∞ .

For the third item, let $P : U \rightarrow \mathcal{S}$ be a plot. For every $n \in \mathbf{Z}$ let $\mathcal{O}_n = (\pi_n \circ P)^{-1}(\mathbf{C} - \{0\})$, where $\pi_n : S_\alpha^\infty \rightarrow \mathbf{C}$ is the projection $\pi_n((Z_m)_{m \in \mathbf{Z}}) = Z_n$. Since $\pi_n \circ P$ is smooth, thus continuous, every $\mathcal{O}_n \subset U$ is open. Moreover, let $n \neq m$, assume $r \in \mathcal{O}_n \cap \mathcal{O}_m$, that is, $Z_n(r) \neq 0$ and $Z_m(r) \neq 0$, but P takes its values in the union of the S_m^1 , $m \in \mathbf{Z}$, hence $Z_n(r) \neq 0$ implies $Z_m(r) = 0$ for all $m \neq n$ thus $\mathcal{O}_n \cap \mathcal{O}_m = \emptyset$. Therefore,

$$U = \bigcup_{n \in \mathbf{Z}} \mathcal{O}_n \quad \text{and } \mathcal{O}_n \cap \mathcal{O}_m = \emptyset \text{ for all } n \neq m.$$

That means that the \mathcal{O}_n make an open partition of U . Then, each \mathcal{O}_n is an union of connected components of U . Thus, P takes locally its values in the S_n^1 , $n \in \mathbf{Z}$, that means that \mathcal{S} is the diffeological sum of the circles S_n^1 , $n \in \mathbf{Z}$ (*op. cit.*).

For the fourth item, let $P : U \rightarrow S_\alpha^\infty$ be a plot, $P(r) = (Z_n(r))_{n \in \mathbf{Z}}$. For all $r_0 \in P^{-1}(S_\alpha^\infty - \mathcal{S})$ there exist at least two different indices n and m such that $Z_n(r_0) \neq 0$ and $Z_m(r_0) \neq 0$. Since Z_n and Z_m are smooth there exists an open neighborhood V of r_0 such that $Z_n(r) \neq 0$ and $Z_m(r) \neq 0$, for all $r \in V$, that is, $V \subset P^{-1}(S_\alpha^\infty - \mathcal{S})$. Thus, $P^{-1}(S_\alpha^\infty - \mathcal{S})$ is a union of open domains, it is then an open domain, and consequently $S_\alpha^\infty - \mathcal{S}$ is D-open. \square

7. Symplectic Reduction on the Quasiprojective Space — Independent of the various types of fibers of the projection $\text{pr} : S_\alpha^\infty \rightarrow \text{QP}_\alpha^\infty$, and the singularities induced on the quotient space QP_α^∞ , there exists a differential closed 2-form ϖ on QP_α^∞ such that

$$\omega \upharpoonright S_\alpha^\infty = \text{pr}^*(\varpi).$$

NOTE. The group \mathbf{T}^∞ being abelian, it still acts on the reduced space QP_α^∞ , its moment on this space is the projection of the moment μ of its action on S_α^∞ [PIZ13, §9.13].

¹¹See [PIZ13, 1.39].

¹²That is, open for the D-topology [PIZ13, 2.8].

Proof. We shall apply the general criterion for a differential form to be the pullback of another one [PIZ13, §6.38]. Let $P : U \rightarrow S_\alpha^\infty$ and $P' : U \rightarrow S_\alpha^\infty$ be two plots

$$\begin{array}{ccc} U & \begin{array}{c} \xrightarrow{P} \\ \xrightarrow{P'} \end{array} & S_\alpha^\infty \\ & & \downarrow \text{pr} \\ & & QP_\alpha^\infty \end{array} \quad \text{such that} \quad \text{pr} \circ P = \text{pr} \circ P'.$$

We want to check if, in these conditions, $\omega(P) = \omega(P')$. That would insure the existence of ϖ , a (necessarily closed) 2-form on QP_α^∞ such that $\omega = \text{pr}^*(\varpi)$ [PIZ13, §6.38]. We consider first of all what happens on the open subset

$$U_0 = P^{-1}(S_\alpha^\infty - \mathcal{S}).$$

Since $\text{pr} \circ P = \text{pr} \circ P'$, $P^{-1}(S_\alpha^\infty - \mathcal{S}) = P'^{-1}(S_\alpha^\infty - \mathcal{S}) = U_0$. Now, the restrictions of P and P' on U_0 take their values in the subset of S_α^∞ made of principal orbits of \mathbf{R} , for which the stabilizer of the action of \mathbf{R} is $\{0\}$. Thus, for each $r \in U_0$ there is a unique $\tau(r) \in \mathbf{R}$ such that, for all n , $Z'_n(r) = e^{2i\pi\alpha_n\tau(r)}Z_n(r)$. The function τ is smooth. Indeed, for all $r_0 \in U_0$, there exists $n \in \mathbf{Z}$ such that $Z_n(r_0) \neq 0$. Then there exists a neighborhood of r_0 where $Z_n(r) \neq 0$. On this neighborhood $Z'_n(r) \neq 0$, and $e^{2i\pi\alpha_n\tau(r)} = Z'_n(r)/Z_n(r)$. But $r \mapsto Z'_n(r)$ and $r \mapsto Z_n(r)$ are smooth, thus $r \mapsto e^{2i\pi\alpha_n\tau(r)}$ is smooth, and therefore so is τ .

Now, $\omega = d\varepsilon$, and

$$\begin{aligned} \varepsilon(P')_r(\delta r) &= \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} \overline{Z'_n(r)} \frac{\partial Z'_n(r)}{\partial r}(\delta r) \\ &= \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} \overline{Z_n(r)} \frac{\partial Z_n(r)}{\partial r}(\delta r) \\ &\quad + \left(\sum_{n \in \mathbf{Z}} \alpha_n \overline{Z_n(r)} Z_n(r) \right) \frac{\partial \tau(r)}{\partial r}(\delta r) \\ &= \varepsilon(P)_r(\delta r) + \tau^*(dt)_r(\delta r). \end{aligned}$$

Therefore, $[\omega(P') - \omega(P)] \upharpoonright U_0 = 0$. Thus, by continuity, $[\omega(P') - \omega(P)] \upharpoonright \bar{U}_0 = 0$, where \bar{U}_0 is the closure of U_0 . It remains to check what happens on the complementary $V = U - \bar{U}_0$. The subset V is open, thus $P \upharpoonright V$ and $P' \upharpoonright V$ are two plots of S_α^∞ but with values in the subset of singular orbits \mathcal{S} . Since \mathcal{S} has dimension 1 and ω is a 2-form, $\omega(P \upharpoonright V) = \omega(P' \upharpoonright V) = 0$, see [PIZ13, §6.39 Note], In conclusion $\omega(P') = \omega(P)$ everywhere on U . That proves that there exists a 2-form ϖ on $QP_\alpha^\infty = S_\alpha^\infty/\mathbf{R}$ such that $\text{pr}^*(\varpi) = \omega$. \square

REFERENCES

- [Don00] Paul Donato, *Calcul différentiel pour la licence*. Dunod, Paris (2000).
- [PIZ10] Patrick Iglesias-Zemmour, *Moment Maps in Diffeology*. Memoir of the AMS, vol. 192. Am. Math. Soc., Providence RI, (2010).
- [PIZ13] Patrick Iglesias-Zemmour, *Diffeology*. Mathematical Surveys and Monographs, vol. 185. Am. Math. Soc., Providence RI, (2013).

- [EP01] Elisa Prato, *Simple Non-Rational Convex Polytopes via Symplectic Geometry*. *Topology*, **40** (2001), pp. 961–975.
- [Sou70] Jean-Marie Souriau. *Structure des systèmes dynamiques*, Dunod, Paris 1970.
- [Vil68] Naum Vilenkin, *Special Functions and the Theory of Group Representations*. Translations of Mathematical Monographs. Am. Math. Soc., Providence RI (1968).

E-mail address: `piz@math.huji.ac.il`

URL: `http://math.huji.ac.il/~piz/`