Example of Singular Reduction in Symplectic Diffeology

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Geometry and Topology Seminar
17 September 2020 — Shantou University, China

Ref. `http://math.huji.ac.il/~piz/documents/EOSRISD.diapos.pdf`

The Context

Symplectic reduction is a standard procedure in Symplectic geometry with application in Classical Mechanics:

- Reduction of Cartan-Souriau presymplectic manifold
- Reduction of an invariant level of moment map

These constructions are defined on symplectic or presymplectic manifolds (or submanifolds). They are well documented and applied when the reduction is regular, that is, does not involved singularities such that the reduced space is itself a manifold.

References

- Symplectic Mechanics
  "*Structure des Systèmes Dynamiques*" J.-M. Souriau, Dunod Editor (1969) Paris.
- Symplectic Reduction
  "*Reduction of Symplectic Manifolds with Symmetry*" J. Marsden and A. Weinstein, Rep. Math. Phys. 5 (1974), pp. 121130.

The Chalenge

Among unusual situations for symplectic reduction, two cases appears frequently in mathematics or in physics:

- The symplectic space is infinite dimensional, for example a sphere S^∞ in an infinite dimensional Hilbert space.
- The reduction has singularities, for example some orbits are non closed lines and other are circles.

We shall show how these questions are solved, in the framework of diffeology, in the particular example of the construction of an infinite dimensional quasiprojective space, that mixes the two situations.

References

- Original Paper: The Axiomatic
  "*Groupes différentiels*". Jean-Marie Souriau. Lecture notes in mathematics, vol. 836 (1980) pp.91128.
- The Recent Textbook
  "*Diffeology*". Patrick Iglesias-Zemmour. Mathematical Surveys and Monographs vol. 185 (2013), Am. Math. Soc.

What is a Diffeology

A diffeology is a smooth structure defined by means of parametrizations:

- A parametrization in a set X is any map $\mathrm{P} \colon \mathrm{U} \to \mathrm{X}$, defined on some open subset of some Euclidean space $\mathbf{R}^n$.

A diffeology on a set X is defined as a set $\mathcal{D}$ of parametrizations, called plots, that satisfies three axioms:

- Covering The set $\mathcal{D}$ contains the constant parametrizations.
- Locality A parametrization which belongs locally to $\mathcal{D}$ belongs globally to $\mathcal{D}$.
- Smooth Compatibility The composite of any element of $\mathcal{D}$ by a smooth parametrization of its domain belongs to $\mathcal{D}$.

A diffeological space is a set X equiped with a diffeology $\mathcal{D}$.

Category {Diffeology}

Diffeological spaces are the objects of the category {Diffeology}, whose morphisms are the smooth maps:

- A smooth map from a diffeological space X to another X′, is any map $f: X \to X'$ such that $f \circ P \in \mathcal{D}'$ for all $P \in \mathcal{D}$.

Smooth maps are denoted by $\mathcal{C}^\infty(X, X')$.

The isomorphisms are called diffeomorphisms, they are bijective smooth maps as well as their inverse.

Category {Diffeology} is stable by any set theoretic operation:

- Sum $X = \coprod_i X_i$.
- Subset $X \subset X'$.
- Product $X = \prod_i X_i$.
- Quotient $X = X'/\sim$.

Quotient Spaces

A striking and important construction is the quotient diffeology, for any kind of partition:

Let $\sim$ be any equivalence relation on a diffeological space X, that is, a partition of X. We can push forward the diffeology of X onto the quotient set $\mathrm{Q} = \mathrm{X}/\sim$, by the natural projection $\mathrm{class}\colon \mathrm{X} \to \mathrm{Q}$.

A plot of this quotient diffeology is any parametrization $\mathrm{P}\colon \mathrm{U} \to \mathrm{Q}$ such that everywhere,

$$\mathrm{P} \underset{\mathrm{loc}}{=} \mathrm{class} \circ \mathrm{R},$$

where R is some plot of X and the suffix loc means that R is required only locally.

Differential Forms

A differential k-form α on a diffeological space X is defined by its pullback on the plots. Precisely, For every plot $P\colon U \to X$,

$$\alpha(P) \in \Omega^k(U)$$

and for all smooth parametrization $F\colon V \to U$,

$$\alpha(P \circ F) = F^*(\alpha(P)).$$

The form α is just the mapping

$$\alpha\colon P \mapsto \alpha(P).$$

Exterior Derivative

The exterior derivative of a k-form is given by:

$$d\alpha(P) = d[\alpha(P)],$$

With

$$d\colon \Omega^k(X) \to \Omega^{k+1}(X) \quad \text{and} \quad d \circ d = 0.$$

That defines a De Rham Complex for every diffeological space.

$$H^k_{DR}(X) = \ker[d\colon \Omega^k(X) \to \Omega^{k+1}(X)]/d[\Omega^{k-1}(X)].$$

Pullbacks

Let X and X′ be two diffeological spaces. Let

$$f\colon X \to X'$$

be a smooth map and $\alpha' \in \Omega^k(X')$.

The pullback $\alpha = f^*(\alpha')$ is defined by

$$\alpha(P) = \alpha'(f \circ P), \quad \alpha \in \Omega^k(X)$$

Quotients

Let X be a diffeological space and

$$\pi\colon X \to Q$$

be a projection onto a quotient, and let $\alpha \in \Omega^k(X)$.

Criterion There exists $\beta \in \Omega^k(Q)$ such that $\alpha = \pi^*(\beta)$ if and only if, for all plots P and P' in X,

$$\pi \circ P = \pi \circ P' \quad \Rightarrow \quad \alpha(P) = \alpha(P').$$

Functional Diffeology on Complex Periodic Functions

Let X and X' be two diffeological spaces, $\mathcal{C}^\infty(X, X')$ carries a natural diffeology called the functionnal diffeology. The plots are the parametrizations $r \mapsto f_r$, defined on some Euclidean domain U such that

$$[(r, x) \mapsto f_r(x)] \in C^\infty(U \times X, X').$$

That diffeology makes the category Cartesian closed.

The space we will consider in the following is the space od complex periodic functions

$$C^\infty_{\text{per}}(\mathbf{R}, \mathbf{C}) = \{f \in C^\infty(\mathbf{R}, \mathbf{C}) \mid f(x+1) = f(x)\},$$

equipped with the functional diffeology.

First, Fourier Transform

For all f in $C^\infty_{\text{per}}(\mathbf{R}, \mathbf{C})$, we associate the sequence of its Fourier coefficients $(f_n)_{n\in\mathbf{Z}}$

$$f_n = \int_0^1 f(x) e^{-2i\pi n x}\, dx,\ \forall n \in \mathbf{Z}.$$

The image of $f \mapsto (f_n)_{n\in\mathbf{Z}}$ is the vector space $\mathcal{E}$ of rapidly decreasing infinite complex series

$$\mathcal{E} = \left\{ (f_n)_{n\in\mathbf{Z}} \;\middle|\; f_n \in \mathbf{C} \;\&\; \forall p \in \mathbf{N},\ n^p f_n \xrightarrow[|n|\to\infty]{} 0 \right\}.$$

We push the functional diffeology on $C^\infty_{\text{per}}(\mathbf{R}, \mathbf{C})$ to $\mathcal{E}$. A plot $r \mapsto f_r$ will give a plot of $\mathcal{E}$

$$r \mapsto (f_n(r))_{n\in\mathbf{Z}} \quad \text{with} \quad f_n(r) = \int_0^1 f_r(x) e^{-2i\pi n x}\, dx,\ \forall n \in \mathbf{Z}.$$

Functional Diffeology on Fourier Coefficients – I

How to recognize a family $(f_n(r))_{n \in \mathbf{Z}}$ of smooth parametrizations in $\mathbf{C}$ coming from $C^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$?

Theorem. A parametrizations $P : r \mapsto (f_n(r))_{n \in \mathbf{Z}}$ in $\mathcal{E}$ is a plot, for the pushforward of the functional diffeology on $C^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$, if and only if:

1. The functions $f_n : \mathrm{dom}(P) \to \mathbf{C}$ are smooth.
2. For all closed ball $\overline{\mathcal{B}} \subset \mathrm{dom}(P)$, for every $k \in \mathbf{N}$, for all $p \in \mathbf{N}$, there exists a positive number $M_{k,p}$ such that, for all integer $n \neq 0$,

$$\left| \frac{\partial^k f_n(r)}{\partial r^k} \right| \leq \frac{M_{k,p}}{|n|^p} \quad \text{for all } r \in \mathcal{B}. \qquad (\diamondsuit)$$

Functional Diffeology on Fourier Coefficients – II

Remark 1. In other words, the parametrization $r \mapsto (f_n(r))_{n\in\mathbf{Z}}$ is a plot of this diffeology if the functions f_n are smooth and their derivatives are uniformly rapidly decreasing:

$$n^p \frac{\partial^k f_n(r)}{\partial r^k} \xrightarrow[|n|\to\infty]{} 0, \quad \text{for all } p \in \mathbf{N}.$$

Remark 2. By compactness, it is enough that, for every point $r_0 \in \mathrm{dom}(P)$, there exists a ball $\mathcal{B}'$ centered at r_0 such that $(\diamondsuit)$ holds to ensure that $(\diamondsuit)$ holds on every closed ball $\overline{\mathcal{B}} \subset \mathrm{dom}(P)$.

Remark 3. This is a nice example of a non conventional diffeology when we forget where it comes from.

The Infinite Torus – I

Let T^∞ be the group of infinite sequences of unit number:

$$T^\infty = \prod_{n \in \mathbf{Z}} U(1),$$

acting $\mathbf{C}$-linearly on $\mathcal{E}$ by

$$(z_n)_{n\in\mathbf{Z}} \cdot (Z_n)_{n\in\mathbf{Z}} = (z_n Z_n)_{n\in\mathbf{Z}}.$$

A rapidly decreasing complex sequence is obviously transformed into another.

Every element $z = (z_n)_{n\in\mathbf{Z}} \in T^\infty$ is invertible,

$$(z_n)^{-1}_{n\in\mathbf{Z}} = (\bar{z}_n)_{n\in\mathbf{Z}}, \quad \bar{z} = z^*.$$

Action of The Infinite Torus

For every plot $r \mapsto (Z_n(r))_{n\in\mathbf{Z}}$ in $\mathcal{E}$, for all $p \in \mathbf{N}$,

$$\left|\frac{\partial^k z_n Z_n(r)}{\partial r^k}\right| = \left|z_n \frac{\partial^k Z_n(r)}{\partial r^k}\right| = \left|\frac{\partial^k Z_n(r)}{\partial r^k}\right|.$$

Then:

Proposition. The action of $(z_n)_{n\in\mathbf{Z}}$ on $\mathcal{E}$ is smooth as well as its inverse, $(z_n)_{n\in\mathbf{Z}}$ acts on $\mathcal{E}$ by diffeomorphism. We got a monomorphism

$$\eta : \mathrm{T}^\infty \to \mathrm{GL}^\infty(\mathcal{E}) = \mathrm{GL}(\mathcal{E}) \cap \mathrm{Diff}(\mathcal{E}).$$

Diffeology of The Infinite Torus

DEFINITION. A tempered parametrization in T^∞ is a parametrization

$$\zeta : r \mapsto (z_n(r))_{n\in\mathbf{Z}}$$

that satisfies:

- The z_n are smooth and if for every $k \in \mathbf{N}$.
- For every r_0 in $\mathrm{dom}(\zeta)$, there exist a closed ball $\overline{\mathcal{B}} \subset \mathrm{dom}(\zeta)$ centered at r_0, a polynomial P_k and an integer N such that:

$$\forall r \in \mathcal{B}, \forall n > N, \quad \left|\frac{\partial^k z_n(r)}{\partial r^k}\right| \leq P_k(n).$$

PROPOSITION. The tempered parametrizations form a group diffeology on T^∞.

PROPOSITION. Equipped with the tempered diffeology, the action of the group T^∞ on $\mathcal{E}$ is smooth. That is, the monomorphism $\eta : T^\infty \to GL^\infty(\mathcal{E})$ is smooth.

Next, for all $N \in \mathbf{N}$, let $\iota_N : T^N \to T^\infty$ be defined as follows:

$$\iota_N(z_n)_{n=1}^N = Z \quad \text{with} \quad \begin{cases} Z_n = z_n & \text{if } n \in \{1, \ldots, N\}, \\ Z_n = 1 & \text{otherwise.} \end{cases}$$

PROPOSITION. The smooth injection ι_N is a diffeomorphism from T^N onto its image equipped with the subset diffeology. That is, an induction.

Consider a sequence $\alpha = (\alpha_n)_{n\in\mathbf{Z}}$ of positive numbers, independent over $\mathbf{Q}$. That is,

$$\sum_{n\in\mathbf{Z}} q_n \alpha_n = 0 \quad \Rightarrow \quad q_n = 0,$$

for all finitely supported sequence of rational numbers $q_n \in \mathbf{Q}$.

In the following we will consider such sequences with $|\alpha_n| \leq 1$. Then, the map

$$\iota : \mathbf{R} \mapsto \mathrm{T}^\infty, \quad \text{defined by} \quad \iota(t) = \left(e^{2i\pi\alpha_n t}\right)_{n\in\mathbf{Z}},$$

which is obviously injective, is an induction, that is, a diffeomorphism onto its image equipped with the subset diffeology. We call the image $\iota(\mathbf{R}) \subset \mathrm{T}^\infty$, an irrational solenoid.

Symplectic Structure on $C^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$

Let Surf be the standard symplectic form on $\mathbf{C}$:

$$\mathrm{Surf}_z(\delta z, \delta' z) = \frac{1}{2i}\left[\delta\bar{z}\,\delta' z - \delta'\bar{z}\,\delta z\right] \quad \forall z, \delta z, \delta' z \in \mathbf{C}.$$

The evaluation map: for all $x \in \mathbf{R}$, let

$$\hat{x} : C^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C}) \to \mathbf{C} \quad \text{with} \quad \hat{x}(f) = f(x), \quad \forall x \in \mathbf{R}.$$

Because $\hat{x}$ is smooth, the mean value of the pullback $\hat{x}^*(\mathrm{Surf})$ is a 2-form on $C^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$, and closed.

$$\omega = \frac{1}{\pi}\int_0^1 \hat{x}^*(\mathrm{Surf})\, dx \quad \text{with} \quad \begin{cases} \omega \in \Omega^2(C^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})) \\ d\omega = 0. \end{cases}$$

The Form ω by Plots

Let $P : r \mapsto f_r$ be a plot in $C^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$,

$$\begin{aligned}\omega(P)_r(\delta r, \delta' r) &= \frac{1}{2i\pi}\int_0^1 \left\{ \frac{\partial \overline{f_r(x)}}{\partial r}(\delta r)\frac{\partial f_r(x)}{\partial r}(\delta' r) \right. \\ &- \left. \frac{\partial \overline{f_r(x)}}{\partial r}(\delta' r)\frac{\partial f_r(x)}{\partial r}(\delta r)\right\} dx.\end{aligned}$$

The Form ω is Indeed Symplectic

The closed 2-form ω is invariant by translation $\mathrm{Tr}_g : f \mapsto f + g$, and $C^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$ is an homogeneous space of itself. The moment map of this action is given, up to a constant, by:

$$\mu(f) = \frac{1}{2i\pi} d\left[g \mapsto \int_0^1 \bar{f}g - \bar{g}f\right].$$

PROPOSITION. $C^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$ acts transitively on itself by translation, preserving ω, and the moment map μ is injective. Thus it is a diffeological symplectic space.

NOTE This is the definition given in the Introduction of "*Example of singular reduction in symplectic diffeology*" (P.I-Z). Proc. Amer. Math. Soc., 144(2):1309–1324 (2016).

A Word on Moment Map

When we have a closed 2-form ω on a diffeological space X, invariant by a group G, there is a moment map

$$\mu\colon X \to \mathcal{G}^*/\Gamma,$$

where $\mathcal{G}^*$ is the space of momenta, that is, the spaces of left-invariant 1-forms on G, and Γ some representation of $\pi_1(X)$.

In the simplest case where $\omega = d\alpha$ and α is itself invariant:

$$\mu(x) = \hat{x}^*(\alpha) \in \mathcal{G}^*.$$

The case non invariant-exact is treated involving some diffeological constructions on the space Paths(X).

- "*The Moment Map in Diffeology*", P.I-Z. Memoir of the American Mathematical Society (2010) no.970, USA.

The Moment Map μ of the (Hamiltonian and exact) action of T^∞ on $\mathcal{E}$:

$$\mu(Z) = \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} |Z_n|^2 \, \pi_n^*(\theta) + \sigma,$$

- $\pi_n : T^\infty \to U(1)$ is the n-th projection $\pi_n(Z) = Z_n$,
- θ is the canonical invariant 1-form on $U(1)$,
- σ is some constant momentum of T^∞ (a constant invariant 1-form).

The Moment Map of the Solenoid

Let $(\alpha_n)_{n\in\mathbf{Z}}$ be a sequence of positive numbers, independent over $\mathbf{Q}$. Let

$$\underline{t}(Z_n)_{n\in\mathbf{Z}} = (e^{2i\pi\alpha_n t} Z_n)_{n\in\mathbf{Z}}$$

be the induced action of $\mathbf{R}$ on $\mathcal{E}$. We call α-solenoid the subgroup

$$\mathcal{S}_\alpha = \left\{(e^{2i\pi\alpha_n t})_{n\in\mathbf{Z}}\right\}_{t\in\mathbf{R}} \subset \mathrm{T}^\infty.$$

Its moment map is given by reduction of μ:

$$\nu(Z) = h(Z)\,dt \quad \text{with} \quad h(Z) = \sum_{n\in\mathbf{Z}} \alpha_n |Z_n|^2 + c,$$

where c is some constant. The function h is called Hamiltonian.

The Infinite Sphere and the Solenoid

Let S_α^∞ be the unit level of the Hamiltonian h, for $c = 0$.

$$S_\alpha^\infty = \left\{ Z = (Z_n)_{n\in\mathbf{Z}} \in \mathcal{E} \;\middle|\; \sum_{n\in\mathbf{Z}} \alpha_n |Z_n|^2 = 1 \right\}.$$

Let QP_α^∞ be the quotient of S_α^∞ by the action of the solenoid, equipped with the quotient diffeology, and pr be the projection,

$$\mathrm{pr} : S_\alpha^\infty \to QP_\alpha^\infty \quad \text{and} \quad QP_\alpha^\infty = S_\alpha^\infty / \underline{\mathbf{R}}.$$

We call the quotient space an: Infinite Quasiprojective Space, since it is a generalization of Prato's quasisphere [EP01].

The orbit of $Z = (Z_n)_{n \in \mathbf{N}} \in S^\infty_\alpha$ by the solenoid $\mathcal{S}_\alpha$:

1. If there exist $Z_n \neq 0$ and $Z_m \neq 0$, then the stabilizer of Z is $\{0\}$ and the orbit is equivalent to the line $\mathbf{R}$. These are the principal orbits.
2. The singular orbits, aka non principal orbits, are the subspaces

$$S^1_n = \{Z \in S^\infty_\alpha \mid Z_m = 0 \text{ if } m \neq n\}, \quad \text{with} \quad n \in \mathbf{Z}.$$

Each singular orbit, equipped with the subset diffeology, is equivalent to the circle S^1.

The Principal and Singular Loci

The singular locus of the action of the solenoid is

$$\mathcal{S}\mathrm{ing} = \bigcup_{n \in \mathbf{Z}} S^1_n \subset S^\infty_\alpha.$$

Equipped with the subset diffeology, it is the diffeological sum

$$\mathcal{S}\mathrm{ing} = \coprod_{n \in \mathbf{Z}} S^1_n, \quad \text{and} \quad \dim(\mathcal{S}) = 1.$$

It is a closed subset for the D-topolgy.

The regular or principal locus, that is,

$$\mathcal{R}\mathrm{eg} = S^\infty_\alpha - \bigcup_{n \in \mathbf{Z}} S^1_n,$$

is an open dense subset for the D-topology.

THEOREM. There exists a closed 2-form ϖ on $\mathrm{QP}^\infty_\alpha$ such that:

$$\mathrm{pr}^*(\varpi) = \omega \restriction \mathrm{S}^\infty_\alpha.$$

NOTE 1. Because of the singular orbits, the quasi projective space is not transitive under the pseudogroup of automorphisms, and therefore ϖ is not symplectic. I didn't check if the universal moment map is injective, it would not be surprising. But this is could be the subject of a separate work.

NOTE 2. Considering the mechanism of the proof, it is clear that this situation is a special case of a more general theorem on reduction by $\mathbf{R}$ or S^1 actions.

Proof We shall apply the general criterion for a differential form to be a pullback by a subduction. Let $\mathrm{P} : \mathrm{U} \to \mathrm{S}^\infty_\alpha$ and $\mathrm{P}' : \mathrm{U} \to \mathrm{S}^\infty_\alpha$ be two plots

$$\begin{array}{ccc} \mathrm{U} \underset{\mathrm{P}'}{\overset{\mathrm{P}}{\rightrightarrows}} \mathrm{S}^\infty_\alpha & & \\ \downarrow \mathrm{pr} & \quad \text{such that} \quad & \mathrm{pr} \circ \mathrm{P} = \mathrm{pr} \circ \mathrm{P}'. \\ \mathrm{QP}^\infty_\alpha & & \end{array}$$

We want to check if, in these conditions, $\omega(\mathrm{P}) = \omega(\mathrm{P}')$. That would insure the existence of ϖ, a (necessarily closed) 2-form on QP^∞_α such that $\omega = \mathrm{pr}^*(\varpi)$. We consider first of all what happens on the open subset

$$U_0 = P^{-1}(S^\infty_\alpha - \mathcal{S}\mathrm{ing}).$$

Since $\mathrm{pr} \circ P = \mathrm{pr} \circ P'$, $P^{-1}(S^\infty_\alpha - \mathcal{S}\mathrm{ing}) = P'^{-1}(S^\infty_\alpha - \mathcal{S}\mathrm{ing}) = U_0$. Now, the restrictions of P and P' on U_0 take their values in the subset of S^∞_α made of principal orbits of $\mathbf{R}$, for which the stabilizer of the action of $\mathbf{R}$ is $\{0\}$. Thus, for each $r \in U_0$ there is a unique $\tau(r) \in \mathbf{R}$ such that, for all n, $Z'_n(r) = e^{2i\pi\alpha_n\tau(r)}Z_n(r)$. The function τ is smooth. Indeed, for all $r_0 \in U_0$, there exists $n \in \mathbf{Z}$ such that $Z_n(r_0) \neq 0$. Then there exists a neighborhood of r_0 where $Z_n(r) \neq 0$. On this neighborhood $Z'_n(r) \neq 0$, and

$e^{2i\pi\alpha_n\tau(r)} = Z'_n(r)/Z_n(r)$. But $r \mapsto Z'_n(r)$ and $r \mapsto Z_n(r)$ are smooth, thus $r \mapsto e^{2i\pi\alpha_n\tau(r)}$ is smooth, and therefore so is τ.

Now, $\omega = d\varepsilon$, and

$$\begin{aligned}
\varepsilon(P')_r(\delta r) &= \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} \overline{Z'_n(r)}\frac{\partial Z'_n(r)}{\partial r}(\delta r) \\
&= \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} \overline{Z_n(r)}\frac{\partial Z_n(r)}{\partial r}(\delta r) \\
&+ \left(\sum_{n\in\mathbf{Z}} \alpha_n \overline{Z_n(r)}\, Z_n(r)\right)\frac{\partial \tau(r)}{\partial r}(\delta r) \\
&= \varepsilon(P)_r(\delta r) + \tau^*(dt)_r(\delta r).
\end{aligned}$$

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Therefore, $[\omega(P') - \omega(P)] \restriction U_0 = 0$. Thus, by continuity, $[\omega(P') - \omega(P)] \restriction \overline{U}_0 = 0$, where $\overline{U}_0$ is the closure of U_0. It remains to check what happens on the complementary $V = U - \overline{U}_0$. The subset V is open, thus $P \restriction V$ and $P' \restriction V$ are two plots of S^∞_α but with values in the subset of singular orbits $\mathcal{S}$ing. Since $\mathcal{S}$ing has dimension 1 and ω is a 2-form, $\omega(P \restriction V) = \omega(P' \restriction V) = 0$. This is a general result also in diffeology. In conclusion $\omega(P') = \omega(P)$ everywhere on U. That proves that there exists a 2-form ϖ on $QP^\infty_\alpha = S^\infty_\alpha/\mathbf{R}$ such that $pr^*(\varpi) = \omega$. □

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