CHAPTER 9

Symplectic Diffeology

The generalization of classical symplectic geometry to diffeology needs first an appropriate extension of the classical notion of moment map.[1] We know already that diffeology is suitable for describing, in a unique and satisfactory way, manifolds or infinite dimensional spaces, as well as singular quotients. But, if diffeology excels with covariant objects, like differential forms, it is more subtle when it is a question of contravariant objects like vector fields, Lie algebra,[2] kernels, etc. Thus, in order to build a good diffeological theory of the moment map and to avoid any unnecessary debate, we must absolutely avoid depending on contravariant geometrical objects.

Actually, the notion of moment map is not really an object of the symplectic world, but relates more generally to the category of spaces equipped with closed 2-forms. The nondegeneracy condition is secondary and can be first skipped from the data. This has been underlined explicitly by Souriau in his symplectic formulation of Noether's theorem, which involves presymplectic manifolds. On symplectic manifolds Noether's theorem is void.[3] The moment map is just an object of the world of differential closed forms,[4] and there is no reason *a priori* that it could not be extended to diffeology which offers a pretty well developed framework for Cartan-De Rham calculus.

In order to generalize the moment map in diffeology, we need to understand its meaning, and this meaning lies in the following simplest possible case. Let M be a manifold equipped with a closed 2-form ω. Let G be a Lie group acting smoothly on M and preserving ω, that is, $g_M^*(\omega) = \omega$ for all elements g of G, where g_M denotes the action of g on M. Let us assume that ω is exact, $\omega = d\lambda$, and moreover that λ also is invariant by the action of G. Then, for every point m of M, the pullback

[1] The notion of moment map in the framework of classical symplectic geometry was originally introduced in the early 1970s by Souriau; see [Sou70].

[2] Several authors, beginning with Souriau, proposed some generalizations of Lie algebra in diffeology. But it does not seem to exist a unique good choice. Such generalizations rely actually on the kind of problem treated. I have preferred, until now, not to choose one definition over another.

[3] Noether's theorem states that the moment map is constant on the characteristics of the 2-form; if the form is nondegenerate, then the characteristics are justs the points.

[4] I recently introduced the name *parasymplectic* to denote a closed 2-form; see [PIZ21a].

of λ by the orbit map $\hat{m} : g \mapsto g_M(m)$ is a left-invariant 1-form of G, that is, an element of the dual of the Lie algebra $\mathscr{G}^*$. The map $\mu : m \mapsto \hat{m}^*(\lambda)$ is exactly the moment map of the action of G on the pair (M, ω) — at least one of the moment maps, since they are defined up to constants. As we can see, this construction does not involve really the Lie algebra of G but the space $\mathscr{G}^*$ of left-invariant 1-forms on G. Since this space is well defined in diffeology, we just have to replace "manifold" by "diffeological space", and "Lie group" by "diffeological group", and everything works the same way. Thus, let us change the manifold M for a diffeological space[5] X, and let G be some diffeological group. Let us continue to denote the space of left-invariant 1-forms on G by $\mathscr{G}^*$, even if the star does not refer *a priori* to some duality, and let us simply call it the *space of momenta* of the group G. Note that the group G continues to act on $\mathscr{G}^*$ by pullback of its adjoint action $\mathrm{Ad} : (g, k) \mapsto gkg^{-1}$, so we do not lose the notions of coadjoint action and coadjoint orbits.

Next, if we got the good space of momenta, which is the space where the moment maps are assumed to take their values, the problem remains that not every G-invariant closed 2-form is exact. And moreover, even if such form is exact, there is no reason for some of its primitives to be G-invariant. We shall pass over this difficulty by introducing an intermediary, on which we can realize the simple case described above. This intermediary is the space Paths(X), of all the smooth paths in X, where the group G acts naturally by composition. And since Paths(X) carries a natural functional diffeology, it is legitimate to consider its differential forms, and this is what we do. By integrating ω along the paths, we get a differential 1-form defined on Paths(X), invariant by the action of G. The exact tool used here is the Chain-Homotopy operator $\mathscr{K}$. The 1-form $\Lambda = \mathscr{K}\omega$, defined on Paths(X), is a G-invariant primitive of the 2-form $\Omega = (\hat{1}^* - \hat{0}^*)(\omega)$, where $\hat{1}$ and $\hat{0}$ map every path in X to its ends. Thus, thanks to the construction described above, we get a moment map Ψ for the 2-form $\Omega = d\Lambda$ and the action of G on Paths(X). But this *paths moment map* Ψ is not the one we are waiting for. We need to push it down on X, or rather on $X \times X$. Now, if we get this way a 2-*points moment map* ψ well defined on $X \times X$, it no longer takes its value in $\mathscr{G}^*$, as Ψ does, but in the quotient $\mathscr{G}^*/\Gamma$, where Γ is the image by Ψ of all the loops in X. Fortunately, $\Gamma = \Psi(\mathrm{Loops}(X))$ is a subgroup of $(\mathscr{G}^*, +)$ and depends on the loops only through their free homotopy classes. In other words, Γ is a homomorphic image of the Abelianized fundamental group $\pi_1^{\mathrm{Ab}}(X)$ of X. Well, it is not a big deal to have the moment map taking its values in some quotient of the space of momenta, we can live with that, especially if the group Γ is invariant under the coadjoint action of G, which is actually the case.[6] But we are not completely done: the usual moment map is not a 2-points

[5]The space X will be assumed to be connected, as many results need this hypothesis.

[6]More precisely, the elements of Γ are not just elements of $\mathscr{G}^*$ but are moreover closed, and therefore invariant, each of them, by the coadjoint action of G.

function, but a 1-point function. Hence, we have to extract our usual moment maps from this 2-points function ψ. This is fairly easy, thanks to its very definition, the moment map Ψ satisfies an additive property for the concatenation of paths, and the moment map ψ inherits this property as a cocycle condition: for any three points x, x' and x'' of X we have $\psi(x,x') + \psi(x',x'') = \psi(x,x'')$. Therefore, for X connected, there exists always a map μ such that $\psi(x,x') = \mu(x') - \mu(x)$, and any two such maps differ just by a constant. We get finally our wanted *moment maps* μ, defined in the diffeological framework. The only difference, with the simplest case described above, is that a moment map takes its values in some quotient of the space of momenta, instead the space of momenta itself. But this is in fact already the case in the classical theory. It does not appear explicitly because people focus more on Hamiltonian actions than just on symplectic actions. Actually, the group Γ represents the very obstruction, for the action of G on (X,ω), to be *Hamiltonian*. We shall call Γ the *holonomy* of the action of G.

Now, let us come back to some properties of the various moment maps introduced above. The paths moment map Ψ and its projection ψ are equivariant with respect to the action of G on X and the coadjoint action of G on $\mathscr{G}^*$, or the projection of the coadjoint action on $\mathscr{G}^*/\Gamma$. But this is no longer the case for the moment maps μ. The variance of the maps μ reveals a family of cocycles θ from G to $\mathscr{G}^*/\Gamma$ differing just by coboundaries, and generalizing the *Souriau cocycles* [Sou70]. Their common class σ belongs to the cohomology group $H^1(G,\mathscr{G}^*/\Gamma)$, and will be called the *Souriau class* of the action of G of (X,ω). The Souriau class σ is precisely the obstruction for the 2-points moment map ψ to be exact, that is, for some moment map μ to be equivariant. Actually, σ is just the pullback, at the group level, of the class of ψ regarded as a cocycle. In parallel with the classical situation, every Souriau cocycle θ defines a new action of G on $\mathscr{G}^*/\Gamma$, which we still call the *affine coadjoint action* (associated with θ). And the image of a moment map μ is a collection of coadjoint orbits for this action. We call these orbits the (Γ,θ)-coadjoint orbits of G. Two different cocycles give two families of orbits translated by the same constant.

Let us remark that the holonomy group Γ and the Souriau class σ appear clearly on a different level of meaning: the first one is responsible for the non-Hamiltonian character of the action of G and the second characterizes the lack of equivariance of the moment maps.

Well, until now we did not use all the facilities offered by the diffeological framework. Since we do not restrict ourselves to the category of Lie groups, nothing prevents us from considering the group of all the *automorphisms* of the pair (X,ω), that is, the group $\mathrm{Diff}(X,\omega)$ of all the diffeomorphisms of X preserving ω. This group is a natural diffeological group, acting smoothly on X. Thus, everything built above applies to $\mathrm{Diff}(X,\omega)$, and every other action preserving ω, of any diffeological group, passes through $\mathrm{Diff}(X, \omega)$, and through the associated object of the theory developed here. Therefore, considering the whole group of automorphisms of the closed 2-form ω of X, we get a natural notion of universal moment maps Ψ_ω, ψ_ω and μ_ω, universal holonomy Γ_ω, universal Souriau cocycles θ_ω, and universal Souriau class σ_ω. By the way, this universal construction suggests a simple and new characterization, for any diffeological space X equipped with a closed 2-form ω, of the group of *Hamiltonian diffeomorphisms* $\mathrm{Ham}(X, \omega)$, as the largest connected subgroup of $\mathrm{Diff}(X, \omega)$ whose holonomy vanishes.

It is interesting to note that, unlike the original constructions [Sou70] and most of their generalizations, the theory described above is essentially global, more or less algebraic, does not refer to any differential or partial differential, equation, and does not involve any notion of vector field or functional analysis techniques.

Considering the classical case of a closed 2-form ω defined on a manifold M, we show in particular that ω is nondegenerate if and only if the group $\mathrm{Diff}(M, \omega)$ is transitive on M, and if the universal moment map μ_ω is injective. In other words, symplectic manifolds are identified, by the universal moment maps, with some coadjoint orbits — in our general sense— of their group of symplectomorphisms. This idea that *every symplectic manifold is a coadjoint orbit* is not new, it is actually very natural and suggested by a well known classification theorem for symplectic homogeneous Lie group actions [Kir76], [Kos70], [Sou70]. This has been stated already in a different context, for example in [MW82] and [Omo86]. But the real question is: How can we make this statement rigorous at a good price, without involving the heavy functional analysis apparatus? This is what brings diffeology.

The examples and exercises at the end of this chapter show several situations involving diffeological groups which are not Lie groups, or involving diffeological spaces which are not manifolds. We can see, for example, the general theory applying meaningfully to the singular *symplectic irrational tori* for which topology is irrelevant. These general constructions of moment maps are also applied to a few examples in infinite dimension, and also when finite and infinite dimensions are mixed. Finally, two exercises on orbifolds exhibit a strong difference between classical symplectic geometry and what we expect from its diffeological counterpart. These examples show without any doubt the ability of this theory to treat correctly, in a unique framework, avoiding heuristic arguments, the large variety of situations we can find in the mathematical literature today. Infinite dimensional heuristic examples can be found for instance in [Dnl99]. The solutions of some of the exercises need tedious computations, which just shows diffeology at work in this particular field.

In conclusion, besides the point that the construction developed in this chapter is a first step in the elaboration of the *symplectic diffeology program*, I would emphasize the fact that, since {Manifolds} is a full subcategory of {Diffeology}, all the constructions developed here apply to manifolds and give a faithful description of the classical theory of moment maps. As we have seen, there is no mention, and no use, of Lie algebra or vector fields in this presentation. This reveals the fact that these objects also are useless in the traditional approach, and can be avoided. And, I would add, they should be avoided. Not just because they can then be extended to larger categories, but because the use of contravariant objects hides the deep fact that the theory of moment maps is a pure covariant theory. Let us take an example. We know that since coadjoint orbits of Lie groups are symplectic they are even dimensional. This is often regarded as a miracle, since it is not necessarily the case for adjoint orbits. But if we think that the Lie algebra has little to do with the space of momenta of a Lie group, there is no more miracle, just different behaviors for different objects, which is unsurprising. Moreover I would add, but this can appear as more or less subjective, that avoiding all this *va-et-vient* between Lie algebra and dual of Lie algebra, the diffeological approach of the moment maps is much simpler, and even deeper, than the classical approach. Compare for example the Souriau cocycle constructions in the original “Structure des systèmes dynamiques” [Sou70], and in diffeology. The only crucial property used here is connectedness, that is, the existence of enough smooth paths connecting points in spaces. Finally, it goes without saying that the diffeological approach respects scrupulously the principle of minimality required by mathematics.

The Paths Moment Map

We shall introduce the various flavors of moment map in diffeology step by step. The first step consists, in this section, in defining the paths moment map.

9.1. Definition of the paths moment map. Let X be a diffeological space, and let ω be a closed 2-form defined on X. Let G be a diffeological group, and let $\rho : \mathrm{G} \to \mathrm{Diff}(\mathrm{X})$ be a smooth action. Let us denote by the same letter the natural action of G on Paths(X), induced by the action ρ of G on X, that is, for all $g \in \mathrm{G}$, for all $p \in \mathrm{Paths}(\mathrm{X})$,

$$\rho(g)(p) = \rho(g) \circ p = [t \mapsto \rho(g)(p(t))].$$

Let us assume now that the action ρ of G on X preserves ω, that is, for all $g \in \mathrm{G}$,

$$\rho(g)^*(\omega) = \omega, \text{ or } \rho \in \mathrm{Hom}^\infty(\mathrm{G}, \mathrm{Diff}(\mathrm{X}, \omega)).$$

Let $\mathscr{K}$ be the Chain-Homotopy operator (art. 6.83), so $\mathscr{K}\omega$ is a 1-form on Paths(X), and the action of G on Paths(X) preserves $\mathscr{K}\omega$. This is a consequence of the variance of the Chain-Homotopy operator (art. 6.84). Thus, for all $g \in \mathrm{G}$,

$$\rho(g)^*(\mathscr{K}\omega) = \mathscr{K}\omega.$$

Now, let p be a path in X, and let $\hat{p} : \mathrm{G} \to \mathrm{Paths}(\mathrm{X})$ be the orbit map, $\hat{p}(g) = \rho(g) \circ p$. Then, the pullback $\hat{p}^*(\mathscr{K}\omega)$ is a left-invariant 1-form of G, that is, an element of $\mathscr{G}^*$. The map

$$\Psi : \mathrm{Paths}(\mathrm{X}) \to \mathscr{G}^*, \text{ defined by } \Psi(p) = \hat{p}^*(\mathscr{K}\omega),$$

is smooth with respect to the functional diffeology, $\Psi \in \mathscr{C}^\infty(\mathrm{Paths}(\mathrm{X}), \mathscr{G}^*)$. The map Ψ will be called the *paths moment map*.

9.2. Evaluation of the paths moment map. Let X be a diffeological space and ω be a closed 2-form defined on X. Let G be a diffeological group, and let ρ be a smooth action of G on X, preserving ω. Let p be a path in X. Thanks to the explicit expression of the Chain-Homotopy operator (art. 6.83), we get the evaluation of the momentum $\Psi(p)$ on any n-plot P of G,

$$\Psi(p)(\mathrm{P})_r(\delta r) = \int_0^1 \omega\left[\begin{pmatrix} s \\ u \end{pmatrix} \mapsto (\rho \circ \mathrm{P})(u)(p(s+t))\right]_{\binom{s=0}{u=r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \delta r \end{pmatrix} dt, \qquad (\heartsuit)$$

for all r in def(P) and all δr in $\mathbf{R}^n$. Now, as a differential 1-form, $\Psi(p)$ is characterized by its values on the 1-plots (art. 6.37). Then, let $f : t \mapsto f_t$ be a 1-plot of G centered at the identity $\mathbf{1}_\mathrm{G}$, that is, $f \in \mathrm{Paths}(\mathrm{G})$ and $f(0) = \mathbf{1}_\mathrm{G}$. For every $t \in \mathbf{R}$, let F_t be the path in Diff(X, ω) — centered at the identity $\mathbf{1}_\mathrm{X}$ — defined by

$$\mathrm{F}_t : s \mapsto \rho(f_t^{-1} \circ f_{t+s}).$$

We have then

$$\Psi(p)(f)_t(1) = -\int_p i_{\mathrm{F}_t}(\omega) = -\int_0^1 i_{\mathrm{F}_t}(\omega)(p)_s(1)ds, \qquad (\clubsuit)$$

where $i_{\mathrm{F}_t}(\omega)$ is the contraction of ω by F_t (art. 6.56). But, as an invariant 1-form on G, the moment $\Psi(p)$ is characterized by its *value at the identity*, for $t = 0$,

$$\Psi(p)(f)_0(1) = -\int_p i_\mathrm{F}(\omega) = -\int_0^1 i_\mathrm{F}(\omega)(p)_t(1)\,dt \text{ with } \mathrm{F} = \rho \circ f. \qquad (\diamondsuit)$$

NOTE. Let $f \in \mathrm{Hom}^\infty(\mathbf{R}, \mathrm{G})$, then $\Psi(p)(f)$ is an invariant 1-form on $\mathbf{R}$ whose coefficient is just $\int_p i_\mathrm{F}(\omega)$, that is,

$$\Psi(p)(f) = h_f(p) \times dt, \text{ where } h_f(p) = -\int_p i_\mathrm{F}(\omega).$$

The smooth map $h_f : \mathrm{Paths}(\mathrm{X}) \to \mathbf{R}$ is the *Hamiltonian* of f, or the Hamiltonian of the 1-parameter group $f(\mathbf{R})$. Also note that the map $h : \mathrm{Hom}^\infty(\mathbf{R}, \mathrm{G}) \to \mathscr{C}^\infty(\mathrm{Paths}(\mathrm{X}), \mathbf{R})$, defined above, is smooth.

PROOF. Let us prove (♡). Let us recall that for every $p \in \mathrm{Paths}(X)$ and every $g \in G$, $\hat{p}(g) = \rho(g) \circ p = [t \mapsto \rho(g)(p(t))]$. Thus, by definition,

$$\begin{aligned}
\Psi(p)(P)_r(\delta r) &= \hat{p}^*(\mathcal{K}\omega)(P)_r(\delta r) \\
&= \mathcal{K}\omega(\hat{p} \circ P)_r(\delta r) \\
&= \int_0^1 \omega\left[\begin{pmatrix} s \\ r \end{pmatrix} \mapsto \hat{p} \circ P(r)(s+t)\right]_{\binom{0}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ \delta r \end{pmatrix} dt \\
&= \int_0^1 \omega\left[\begin{pmatrix} s \\ r \end{pmatrix} \mapsto (\rho \circ P)(r)(p(s+t))\right]_{\binom{0}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ \delta r \end{pmatrix} dt.
\end{aligned}$$

Let us prove (♣). Let us apply the general formula (♡) for $P = f$. Introducing $u' = u - t$ and $s'' = s + s'$, using the compatibility property of $\omega(P \circ Q) = Q^*(\omega(P))$ and the $\rho(f_t)$ invariance of ω, we get

$$\begin{aligned}
\Psi(p)(f)_t(1) &= \int_0^1 \omega\left[\begin{pmatrix} s \\ u \end{pmatrix} \mapsto \rho(f_u)(p(s+s'))\right]_{\binom{s=0}{u=t}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} ds' \\
&= \int_0^1 \omega\left[\begin{pmatrix} s'' \\ u' \end{pmatrix} \mapsto \rho(f_{t+u'})(p(s''))\right]_{\binom{s''=s'}{u'=0}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} ds' \\
&= \int_0^1 \omega\left[\begin{pmatrix} s'' \\ u' \end{pmatrix} \mapsto \rho(f_t \circ f_t^{-1} \circ f_{t+u'})(p(s''))\right]_{\binom{s''=s'}{u'=0}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} ds' \\
&= \int_0^1 \omega\left[\begin{pmatrix} s'' \\ u' \end{pmatrix} \mapsto \rho(f_t)\Big(F_t(u')(p(s''))\Big)\right]_{\binom{s''=s'}{u'=0}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} ds' \\
&= \int_0^1 \omega\left[\begin{pmatrix} s'' \\ u' \end{pmatrix} \mapsto F_t(u')(p(s''))\right]_{\binom{s''=s'}{u'=0}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} ds' \\
&= \int_0^1 \omega\left[\begin{pmatrix} u' \\ s'' \end{pmatrix} \mapsto F_t(u')(p(s''))\right]_{\binom{u'=0}{s''=s'}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} ds' \\
&= -\int_0^1 \omega\left[\begin{pmatrix} u' \\ s'' \end{pmatrix} \mapsto F_t(u')(p(s''))\right]_{\binom{u'=0}{s''=s'}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} ds' \\
&= -\int_0^1 i_{F_t}(\omega)(p)_{s'}(1) ds' \\
&= -\int_p i_{F_t}(\omega).
\end{aligned}$$

Let us prove the Note. Let $f \in \mathrm{Hom}^\infty(\mathbf{R}, G)$. By definition of differential forms and pullbacks, $\Psi(p)(f) = f^*(\Psi(p))$, but since f is a homomorphism from $\mathbf{R}$ to $\mathrm{Diff}(X, \omega)$ and $\Psi(p)$ is a left-invariant 1-form on $\mathrm{Diff}(X, \omega)$, $f^*(\Psi(p))$ is an invariant 1-form of $\mathbf{R}$, then $\Psi(p)(f) = f^*(\Psi(p)) = a \times dt$, for some real a. Thus,

$\Psi(p)(f)_r = \Psi(p)(f)_0(1) \times dt = h_f(p) \times dt$, with $h_f(p) = \Psi(p)(f)_0(1) = -\int_p i_F(\omega)$, where dt is the canonical 1-form on $\mathbf{R}$. ☐

9.3. Variance of the paths moment map. Let X be a diffeological space, and let ω be a closed 2-form defined on X. Let G be a diffeological group and ρ be a smooth action of G on X, preserving ω. The paths moment map Ψ, defined in (art. 9.1), is equivariant under the action of G, that is, for all $g \in G$,

$$\Psi \circ \rho(g) = \mathrm{Ad}(g)_* \circ \Psi.$$

PROOF. Let us denote here the orbit map $\hat{p}$, of $p \in \mathrm{Paths}(X)$, by $R(p)$, that is, $R(p)(g) = \rho(g) \circ p$ and $\Psi(p) = R(p)^*(\mathcal{K}\omega)$. Then, $\Psi(\rho(g)(p)) = \Psi(\rho(g) \circ p) = R(\rho(g) \circ p)^*(\mathcal{K}\omega)$. But $R(\rho(g) \circ p)(g') = \rho(g')(\rho(g) \circ p) = \rho(g') \circ \rho(g) \circ p = \rho(g'g) \circ p = R(p)(g'g) = R(p) \circ R(g)(g')$, thus $R(\rho(g) \circ p) = R(p) \circ R(g)$, and $\Psi(\rho(g)(p)) = (R(p) \circ R(g))^*(\mathcal{K}\omega) = R(g)^*(R(p)^*(\mathcal{K}\omega)) = R(g)^*(\Psi(p))$. But since $\Psi(p)$ is left-invariant, $R(g)^*(\Psi(p)) = \mathrm{Ad}(g)_*(\Psi(p))$. ☐

9.4. Additivity of the paths moment map. Let X be a diffeological space and ω be a closed 2-form defined on X. Let G be a diffeological group and ρ be a smooth action of G on X, preserving ω. The paths moment map Ψ, defined in (art. 9.1), satisfies the following additive property,

$$\Psi(p \vee p') = \Psi(p) + \Psi(p') \text{ and } \Psi(\bar{p}) = -\Psi(p), \text{ with } \bar{p}(t) = p(1-t),$$

for any two juxtaposable paths p and p' in X.

PROOF. This is a direct application of the expression given in (art. 9.2, (♢)), and of the additivity of the integral of differential forms on paths. ☐

9.5. Differential of the paths moment map. Let X be a diffeological space, and let ω be a closed 2-form defined on X. Let G be a diffeological group, and let ρ be a smooth action of G on X, preserving ω. Let p be a path in X. Then, the exterior derivative of the paths momentum $\Psi(p)$ is given by

$$d(\Psi(p)) = \hat{x}_1^*(\omega) - \hat{x}_0^*(\omega),$$

where $x_0 = p(0)$ and $x_1 = p(1)$, and the $\hat{x}_i$ denote the orbit maps.

PROOF. This is a direct application of the main property of the Chain-Homotopy operator, $d \circ \mathcal{K} + \mathcal{K} \circ d = \hat{1}^* - \hat{0}^*$. Since $d\omega = 0$, $d(\mathcal{K}\omega) = \hat{1}^*(\omega) - \hat{0}^*(\omega)$, composed with $\hat{p}^*$ we get $\hat{p}^* \circ d(\mathcal{K}\omega) = \hat{p}^* \circ \hat{1}^*(\omega) - \hat{p}^* \circ \hat{0}^*(\omega)$, that is, $d(\hat{p}^*(\mathcal{K}\omega)) = (\hat{1} \circ \hat{p})^*(\omega) - (\hat{0} \circ \hat{p})^*(\omega)$. Thus, $d(\Psi(p)) = \hat{x}_1^*(\omega) - \hat{x}_0^*(\omega)$. ☐

9.6. Homotopic invariance of the paths moment map. Let X be a diffeological space and ω be a closed 2-form defined on X. Let G be a diffeological group, and let ρ be a smooth action of G on X, preserving ω. Let p_0 and p_1 be any two paths in

X. If p_0 and p_1 are fixed-ends homotopic, then $\Psi(p_0) = \Psi(p_1)$. In other words, Ψ passes on the Poincaré groupoid of X (art. 5.15).

PROOF. Let $s \mapsto p_s$ be a fixed-ends homotopy connecting p_0 to p_1, for example let $p_s(0) = x_0$ and $p_s(1) = x_1$, for all s. Let f be a 1-plot of G centered at the identity $\mathbf{1}_G$, that is, $f(0) = \mathbf{1}_G$, and let $F = \rho \circ f$. We use the fact that the moment of paths is characterized by its value at the identity, $\Psi(p_s)(f)_0(1) = -\int_{p_s} i_F(\omega)$; see (art. 9.2, ($\diamondsuit$)). Let us differentiate this equality with respect to s,

$$\frac{\partial}{\partial s}\Big(\Psi(p_s)(f)_0(1)\Big) = -\delta \int_{p_s} i_F(\omega), \text{ with } \delta = \frac{\partial}{\partial s}.$$

The variation of the integral of differential forms on cubes (art. 6.70) gives

$$\begin{aligned} \delta \int_{p_s} i_F(\omega) &= \int_0^1 d\,[i_F(\omega)](\delta p_s) + \int_0^1 d[i_F(\omega)(\delta p_s)] \\ &= \int_0^1 d\,[i_F(\omega)](\delta p_s) + \Big[i_F(\omega)(\delta p_s)\Big]_0^1. \end{aligned}$$

The second summand of the right term vanishes because δp_s vanishes at the ends: $p_s(0) = \text{cst}$ and $p_s(1) = \text{cst}$. Now, thanks to the Cartan formula (art. 6.72), $d[i_F(\omega)] = £_F(\omega) - i_F(d\omega)$. But ω is invariant under the action of G, thus $£_F(\omega) = 0$, and $d\omega = 0$, so $d[i_F(\omega)] = 0$. Thus, $\delta \int_{p_s} i_F(\omega) = 0$ and therefore $\Psi(p_0) = \Psi(p_s) = \Psi(p_1)$, for all s. ☐

9.7. The holonomy group. Let X be a connected diffeological space, and let ω be a closed 2-form defined on X. Let G be a diffeological group, and let ρ be a smooth action of G on X, preserving ω. Let Ψ be the paths moment map (art. 9.1). We define the *holonomy* Γ of the action ρ by

$$\Gamma = \{\Psi(\ell) \mid \ell \in \text{Loops}(X)\}.$$

1. The holonomy Γ is an additive subgroup of the subspace of closed momenta, $\Gamma \subset Z$ (art. 7.17), that is,

   $$d\gamma = 0 \text{ and } \gamma - \gamma' \in \Gamma,$$

   for any two elements γ and γ' in Γ.
2. The paths moment map Ψ, restricted to Loops(X), factorizes through a homomorphism from $\pi_1(X)$ to $\mathscr{G}^*$. Thus, Γ is a homomorphic image of the Abelianized fundamental group $\pi_1^{\text{Ab}}(X)$.
3. In particular, every element γ of Γ is invariant by the coadjoint action of G on $\mathscr{G}^*$. For all g in G,

   $$\text{Ad}_*(g)(\gamma) = \gamma.$$

The holonomy Γ is the obstruction for the action ρ to be *Hamiltonian*. Precisely,

DEFINITION. *The action of* G *on* X *is said to be Hamiltonian if* $\Gamma = \{0\}$.

Note that if $\pi_1^{\rm Ab}(X) = \{0\}$, or if the group G has no Ad_*-invariant 1-form except 0, the action ρ is necessarily Hamiltonian.

PROOF. We get immediately that $\gamma \in \Gamma$ is closed, by application of (art. 9.5). Indeed, for all paths $p \in \mathrm{Paths}(X)$, $d(\Psi(p)) = \hat{x}_1^*(\omega) - \hat{x}_0^*(\omega)$, where $x_0 = p(0)$ and $x_1 = p(1)$. Thus, for a loop ℓ, since $\ell(0) = \ell(1)$, $d(\Psi(\ell)) = 0$. Now, let us choose a basepoint $x_0 \in X$. For every loop $\ell \in \mathrm{Loops}(X, x_0)$, the momentum $\Psi(\ell)$ depends on ℓ only through its homotopy class (art. 9.6), so Γ is the image of $\pi_1(X, x_0)$. And, thanks to the additive property of Ψ (art. 9.4), the map $\mathrm{class}(\ell) \mapsto \Psi(\ell)$ is a homomorphism. Now, since X is connected, for every other point x_1 of X, there exists a path c connecting x_0 to x_1, $\bar{c} = t \mapsto c(1-t)$ connects x_1 to x_0. Thanks again to the additive property of Ψ, $\Psi(\bar{c} \vee \ell \vee c) = \Psi(\bar{c}) + \Psi(\ell) + \Psi(c) = -\Psi(c) + \Psi(\ell) + \Psi(c) = \Psi(\ell)$. Then, since the map $\mathrm{class}(\ell) \mapsto \mathrm{class}(\bar{c} \vee \ell \vee c)$ is a conjugation from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$, Γ is the same homomorphic image of $\pi_1(X, x)$, for every point $x \in X$. Hence, we proved points 1 and 2. Point 3 is a direct consequence of (art. 7.17). □

Exercise

✎ EXERCISE 142 (Compact supported real functions I). Let us denote by X the space of compact supported real functions defined on **R**, equipped with the diffeology defined in Exercise 25, p. 29. Precisely, $P : U \to X$ is a plot if and only if (a) $(r, t) \mapsto P(r)(t)$ is a real smooth map defined on $U \times \mathbf{R}$, and (b) for every $r_0 \in U$ there exist an open neighborhood $V \subset U$ of r_0 and a compact $K \subset \mathbf{R}$ such that for every $r \in V$, $P(r)$ and $P(r_0)$ coincide out of K. We want to consider the following bilinear form $\bar{\omega}$ as a differential 2-form on X,

$$\bar{\omega}(f, g) = \int_{-\infty}^{+\infty} \dot{f}(t) g(t)\, dt,$$

where the dot denotes the derivative with respect to the parameter t of f. For all $n \in \mathbf{N}$, for every n-plot $P : U \to X$, for all $r \in U$ and $\delta r, \delta' r \in \mathbf{R}^n$, we define

$$\omega(P)_r(\delta r, \delta' r) = \int_{-\infty}^{+\infty} \frac{\partial}{\partial r}\left(\frac{\partial P(r)(t)}{\partial t}\right)(\delta r)\, \frac{\partial P(r)(t)}{\partial r}(\delta' r)\, dt\,.$$

1) Show that ω is a 2-form on X.

2) Check that ω realizes $\bar{\omega}$, that is, for any $f, g \in X$,

$$\bar{\omega}(f, g) = \omega\left(\begin{pmatrix} s \\ s' \end{pmatrix} \mapsto sf + s'g\right)_{\binom{s}{s'}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let us consider now the group $G = (X, +)$ acting on itself by translation, that is, for all $u \in X$,

$$T_u(f) = f + u \text{ for all } f \in X.$$

3) Show that ω is invariant by G.

4) Say why the holonomy group Γ, associated with the action of G on (X, ω), must vanish.

5) Show that, for any path p connecting $f = p(0)$ to $g = p(1)$, the paths moment map is given, for any plot F of G, that is, a plot of X, by

$$\Psi(p)(F)_r(\delta r) = \int_{-\infty}^{+\infty} \Big(\dot g(t) - \dot f(t)\Big) \frac{\partial F(r)(t)}{\partial r} \delta r \, dt\,.$$

The 2-points Moment Map

The definition of the paths moment map leads immediately to the 2-points moment map. *The* 2*-points moment map satisfies a cocycle condition inherited from the additive property of the paths moment map. This is the second step in the general construction.*

9.8. Definition of the 2-points moment map. Let X be a connected diffeological space, and let ω be a closed 2-form defined on X. Let G be a diffeological group and ρ be a smooth action of G on X, preserving ω. Let Ψ be the paths moment map (art. 9.1), and let Γ be the holonomy of the action ρ (art. 9.7). Then, there exists a smooth map $\psi : X \times X \to \mathscr{G}^*/\Gamma$ such that the following diagram commutes,

$$\begin{array}{ccc}
\mathrm{Paths}(X) & \xrightarrow{\ \Psi\ } & \mathscr{G}^* \\
{\scriptstyle \mathrm{ends}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}} \\
X \times X & \xrightarrow[\ \psi\ ]{} & \mathscr{G}^*/\Gamma
\end{array}$$

where pr is the canonical projection from $\mathscr{G}^*$ onto its quotient, and ends maps p to $(p(0), p(1))$. The map $\psi \in \mathscr{C}^\infty(X \times X, \mathscr{G}^*/\Gamma)$ will be called the 2*-points moment map*.

1. The 2-points moment map ψ satisfies the Chasles cocycle relation, for any three points x, x', x'' of X,
$$\psi(x, x') + \psi(x', x'') = \psi(x, x''). \qquad (\heartsuit)$$
2. The 2-points moment map ψ is equivariant under the action of G. Precisely, for any $g \in G$, and any pair of points x and x' of X,
$$\psi(\rho(g)(x), \rho(g)(x')) = \mathrm{Ad}_*^\Gamma(g)(\psi(x, x')).$$

PROOF. By construction ψ is defined by $\psi(x,x') = \text{class}_\Gamma(\Psi(p))$, where $p \in \text{Paths}(X)$, $x = p(0)$, $x' = p(1)$, and $\text{class}_\Gamma(\alpha)$ denotes the class of $\alpha \in \mathscr{G}^*$ in $\mathscr{G}^*/\Gamma$. The map ψ is smooth simply by the general properties of subductions in diffeology. Next, the first point is a consequence of the additivity of the paths moment map (art. 9.4). The second point is a consequence of the equivariance of the paths moment map of the Ad_* invariance of Γ (art. 9.3), and of the definition of the Ad_*^Γ action (art. 7.16). ☐

The Moment Maps

From the construction of the paths moment map given in (art. 9.1)*, and the* 2*-points moment map given in* (art. 9.8)*, we get the notion of* 1*-point moment maps, or simply moment maps. This is the third step of the general construction, and the generalization of the classical notion of moment maps.*

9.9. Definition of the moment maps. Let X be a connected diffeological space, and let ω be a closed 2-form defined on X. Let G be a diffeological group and ρ be a smooth action of G on X, preserving ω. Let ψ be the 2-points moment map defined in (art. 9.8). There exists always a smooth map $\mu : X \to \mathscr{G}^*/\Gamma$, called a *primitive* of ψ, such that, for any two points x and x' of X,

$$\psi(x,x') = \mu(x') - \mu(x).$$

For every point $x_0 \in X$, for every constant $c \in \mathscr{G}^*/\Gamma$, the map μ defined by

$$\mu(x) = \psi(x_0, x) + c$$

is a primitive of ψ. Every primitive μ of ψ is of this kind, and any two primitives μ and μ' of ψ differ only by a constant. The 2-points moment map ψ will be said to be *exact* if there exists a primitive μ, *equivariant* by the action of G, that is, if there exists a primitive μ such that, for all $g \in G$,

$$\mu \circ \rho(g) = \text{Ad}_*^\Gamma(g) \circ \mu.$$

The primitives μ of ψ, equivariant or not, will be called the *moment maps*.[7] Once one μ will be chosen, we shall call it "the moment map" since it is essentially unique, means unique up to a constant.

NOTE. By the identity (♡) of (art. 9.8), ψ is a 1-cocycle of the G-equivariant cohomology of X with coefficients in $\mathscr{G}^*/\Gamma$, twisted by the coadjoint action. Two cocycles ψ and ψ' are cohomologous if and only if there exists a smooth equivariant map $\mu : X \to \mathscr{G}^*/\Gamma$ such that $\psi'(x,x') = \psi(x,x') + \Delta\mu(x,x')$, where $\Delta\mu(x,x') = \mu(x') - \mu(x)$, and $\Delta\mu$ is a coboundary. So, the 2-points moment map ψ

[7]These maps should have been called "1-point moment maps", but to conform to the usual denomination, we chose to call them simply "moment maps".

defines a class belonging to $\mathrm{H}^1_\mathrm{G}(\mathrm{X}, \mathscr{G}^*/\Gamma)$ which depends only on the form ω and on the action ρ of G on X. If the moment map ψ is exact, that is, if $\mathrm{class}(\psi) = 0$, we shall say that the action ρ of G on X is *exact*, with respect to ω. In this case, there exists a point x_0 of X and a constant c such that $\mu : x \mapsto \psi(x_0, x) + c$ is an equivariant primitive for ψ.

PROOF. Let us choose a basepoint $x_0 \in \mathrm{X}$. Since X is connected, for any $x \in \mathrm{X}$ there exists always a path $p \in \mathrm{X}$ such that $p(0) = x_0$ and $p(1) = x$. Thus, defining $\mu(x) = \psi(x_0, x) = \mathrm{class}(\Psi(p))$, and thanks to the cocycle properties of ψ, we have $\psi(x, x') = \psi(x, x_0) + \psi(x_0, x') = \psi(x_0, x') - \psi(x_0, x) = \mu(x') - \mu(x)$. Now, since ψ is smooth, μ is smooth. Therefore, the equation $\psi(x', x) = \mu(x') - \mu(x)$ always has a solution in μ.

Now, let μ and μ' be two primitives of ψ. For each pair x, x' of points of X we have $\mu'(x') - \mu'(x) = \mu(x') - \mu(x)$, that is, $\mu'(x') - \mu(x') = \mu'(x) - \mu(x)$. Then, the map $x \mapsto \mu'(x) - \mu(x)$ is constant. There exists $c \in \mathscr{G}^*/\Gamma$ such that $\mu'(x) - \mu(x) = c$, that is, $\mu'(x) = \mu(x) + c$. Since the map $x \mapsto \psi(x_0, x)$ is a special solution of the equation in μ: $\psi(x', x) = \mu(x') - \mu(x)$, any solution writes $\mu(x) = \psi(x_0, x) + c$ for some point $x_0 \in \mathrm{X}$ and some constant $c \in \mathscr{G}^*/\Gamma$. ☐

9.10. The Souriau cocycle. Let X be a connected diffeological space, and let ω be a closed 2-form defined on X. Let G be a diffeological group and ρ be a smooth action of G on X, preserving ω. Let ψ be the 2-points moment map defined in (art. 9.8), and let μ be a primitive of ψ as defined in (art. 9.9). Then, there exists a map $\theta \in \mathscr{C}^\infty(\mathrm{G}, \mathscr{G}^*/\Gamma)$ such that

$$\mu(\rho(g)(x)) = \mathrm{Ad}^\Gamma_*(g)(\mu(x)) + \theta(g).$$

This map θ is a $(\mathscr{G}^*/\Gamma)$-cocycle, as defined in (art. 7.16). For all $g, g' \in \mathrm{G}$,

$$\theta(gg') = \mathrm{Ad}^\Gamma_*(g)(\theta(g')) + \theta(g).$$

We shall call the cocycle θ the *Souriau cocycle* of the moment μ.

1. Two Souriau cocycles θ and θ', associated with two moment maps μ and μ' are *cohomologous*, they differ by a *coboundary*
$$\Delta c : g \mapsto \mathrm{Ad}^\Gamma_*(g)(c) - c, \quad \text{where } c \in \mathscr{G}^*/\Gamma.$$
2. For the affine coadjoint action of G on $\mathscr{G}^*/\Gamma$ defined by θ (art. 7.16), the moment map μ is equivariant. For all $g \in \mathrm{G}$,
$$\mu \circ \rho(g) = \mathrm{Ad}^{\Gamma,\theta}_*(g) \circ \mu.$$
3. For every cocycle θ, associated with some moment map μ, there always exist a point $x_0 \in \mathrm{X}$ and a constant $c \in \mathscr{G}^*/\Gamma$ such that,
$$\theta(g) = \psi(x_0, \rho(g)(x_0)) + \Delta c(g), \text{ for all } g \in \mathrm{G}.$$

4. The cohomology class σ of θ belongs to a cohomology group denoted by $\mathrm{H}^1(\mathrm{G}, \mathscr{G}^*/\Gamma)$. It depends only on the cohomology class of the 2-points moment map ψ. This class σ will be called the *Souriau cohomology class*.

NOTE 1. Let x_0 be some point of X. The 2-points moment map ψ (which can also be regarded as a 1-cocycle) defines a 1-cocycle f from G to $\mathscr{G}^*/\Gamma$ by $f(g, g') = \psi(\rho(g)(x_0), \rho(g')(x_0))$. The cocycle f' associated with another point x'_0 will differ only by a coboundary. So, the Souriau cocycle σ represents just the class of the pullback $f = \hat{x}_0^*(\psi)$ by the orbit map $\hat{x}_0$, where $\hat{x}_0^* : \mathrm{H}^1_\rho(\mathrm{X}, \mathscr{G}^*/\Gamma) \to \mathrm{H}^1(\mathrm{G}, \mathscr{G}^*/\Gamma)$. And, by the way, it depends only on the restriction of ω to any one orbit of G on X. Hence, a good choice of the point x_0 can simplify the computation of σ.

NOTE 2. The nature of the action ρ has strong consequences on the Souriau class. For example, thanks to the third item, if the group G has a fixed point x_0, that is, $\rho(g)(x_0) = x_0$ for all g in G, then the Souriau class is zero and the cocycle ψ is exact, *i.e.*, there exists an equivariant primitive μ of ψ.

PROOF. Thanks to (art. 9.9), every moment map μ writes $\mu(x) = \psi(x_0, x) + c$, where x_0 is some fixed point of X and $c \in \mathscr{G}^*/\Gamma$. Thus, $\mu(\rho(g)(x)) - \mathrm{Ad}_*^\Gamma(g)(\mu(x)) = \psi(x_0, \rho(g)(x)) + c - \mathrm{Ad}_*^\Gamma(g)(\psi(x_0, x) + c) = \psi(x_0, \rho(g)(x)) + c - \mathrm{Ad}_*^\Gamma(g)(\psi(x_0, x)) - \mathrm{Ad}_*^\Gamma(g)(c) = \psi(x_0, \rho(g)(x)) - \psi(\rho(g)(x_0), \rho(g)(x)) - \Delta c(g) = \psi(x_0, \rho(g)(x)) + \psi(\rho(g)(x), \rho(g)(x_0)) - \Delta c(g) = \psi(x_0, \rho(g)(x_0)) - \Delta c(g)$. Therefore, $\mu(\rho(g)(x)) - \mathrm{Ad}_*^\Gamma(g)(\mu(x))$ is constant with respect to x. That proves points 1 and 4.

Now, the variance of θ, with respect to the multiplication of G, is a classical result of cohomology (see for example [Kir76]). It is then obvious that, since two moment maps μ and μ' differ only by a constant, the associated cocycles θ and θ' differ by a coboundary. The remaining items of the proposition are just the results of elementary algebraic computations. □

Exercise

✎ EXERCISE 143 (Compact supported real functions, II). Let us consider the data and notations of Exercise 142, p. 368.

1) Show that the moment map of the action of G on (X, ω) is given, for any plot F of G, that is, $(\mathrm{X}, +)$, by:

$$\mu(f)(\mathrm{F})_r(\delta r) = \int_{-\infty}^{+\infty} \dot{f}(t) \frac{\partial \mathrm{F}(r)(t)}{\partial r} \delta r \, dt + \mathrm{cst}.$$

2) Compute the associated Souriau cocycle θ. Is this cocycle trivial?

The Moment Maps for Exact 2-Forms

The special case where a closed 2*-form is the exterior derivative of an invariant* 1*-form deserves special care, since it justifies* a posteriori *the constructions above, by analogy with the classical moment maps* [Sou70].

9.11. The exact case. Let X be a connected diffeological space, and let ω be a closed 2-form defined on X. Let G be a diffeological group, and let ρ be a smooth action of G on X, preserving ω. Let us assume that $\omega = d\alpha$ and that α also is invariant under the action of G, that is, $\rho(g)^*(\alpha) = \alpha$ for all g in G. Let Ψ be the paths moment map defined in (art. 9.1), and ψ be the 2-points moment map defined in (art. 9.8). Then, for every $p \in \mathrm{Paths}(\mathrm{X})$

$$\Psi(p) = \psi(x, x') = \hat{x}_1^*(\alpha) - \hat{x}_0^*(\alpha),$$

where $x_1 = p(1)$ and $x_0 = p(0)$. Moreover, the 2-points moment map ψ is exact, and every equivariant moment map is cohomologous to

$$\mu : x \mapsto \hat{x}^*(\alpha).$$

The action of G is Hamiltonian, $\Gamma = \{0\}$ and exact $\sigma = 0$; see (art. 9.7) and (art. 9.10). This shows, in particular, the coherence of the general constructions developed until now.

Proof. By definition of the paths moment map $\Psi(p) = \hat{p}^*(\mathscr{K}\omega)$, that is, $\Psi(p) = \hat{p}^*(\mathscr{K}(d\alpha))$. But $\mathscr{K}(d\alpha) + d(\mathscr{K}\alpha) = \hat{1}^*(\alpha) - \hat{0}^*(\alpha)$, hence $\hat{p}^*(\mathscr{K}(d\alpha)) = \hat{p}^*[\hat{1}^*(\alpha) - \hat{0}^*(\alpha) - d(\mathscr{K}\alpha)]$, and $\Psi(p) = (\hat{1} \circ \hat{p})^*(\alpha) - (\hat{0} \circ \hat{p})^*(\alpha) - d[\hat{p}^*(\mathscr{K}(\alpha))]$. But $\hat{1} \circ \hat{p} = \hat{x}_1$ and $\hat{0} \circ \hat{p} = \hat{x}_0$, then $\Psi(p) = \hat{x}_1^*(\alpha) - \hat{x}_0^*(\alpha) - d[\hat{p}^*(\mathscr{K}\alpha)]$. Now, $\mathscr{K}\alpha$ is the real function

$$\mathscr{K}\alpha : p \mapsto \int_p \alpha,$$

since $\hat{p}^*(\mathscr{K}\alpha) = \mathscr{K}\alpha \circ \hat{p}$, for all $g \in \mathrm{G}$,

$$\mathscr{K}\alpha(\hat{p}(g)) = \int_{\rho(g)\circ p} \alpha = \int_p \rho(g)^*(\alpha) = \int_p \alpha.$$

Thus, the function $\hat{p}^*(\mathscr{K}\alpha) : \mathrm{G} \to \mathbf{R}$ is constant and equal to $\int_p \alpha$. Then, $d[\hat{p}^*(\mathscr{K}\alpha)] = 0$, and $\Psi(p) = \hat{x}_1^*(\alpha) - \hat{x}_0^*(\alpha)$. Hence, $\Psi(p) = \psi(x_0, x_1)$ and $\Gamma = \{0\}$.

Next, the function $\mu : x \mapsto \hat{x}^*(\alpha)$ is clearly a primitive of ψ, that is, $\psi(x_0, x_1) = \mu(x_1) - \mu(x_0)$. But $\mathrm{R}(\rho(g)(x)) = \hat{x} \circ \mathrm{R}(g)$, where $\mathrm{R}(\rho(g)(x))$ is the orbit map of $\rho(g)(x)$, $g \in \mathrm{G}$. Thus, $\mu(\rho(g)(x)) = (\hat{x} \circ \mathrm{R}(g))^*(\alpha) = \mathrm{R}(g)^*(\hat{x}^*(\alpha)) = \mathrm{R}(g)^*(\mu(x)) = \mathrm{Ad}_*(g)(\mu(x))$. Hence, μ is an equivariant primitive of ψ and $\sigma = 0$. ☐

Functoriality of the Moment Maps

In this section we focus on the behavior of the moment maps, and the various associated objects, under natural transformations.

9.12. Images of the moment maps by morphisms. Let X be a connected diffeological space, and let ω be a closed 2-form defined on X. Let G be a diffeological group, and let ρ be a smooth action of G on X, preserving ω. Let G' be another diffeological group, and let $h : \mathrm{G}' \to \mathrm{G}$ be a smooth homomorphism. Let $\rho' = \rho \circ h$ be the induced action of G' on X. Let us recall that the pullback $h^* : \mathscr{G}^* \to \mathscr{G}'^*$ is a linear smooth map.

1. Let $\Psi : \mathrm{Paths}(\mathrm{X}) \to \mathscr{G}$, and $\Psi' : \mathrm{Paths}(\mathrm{X}) \to \mathscr{G}'$ be the paths moment maps with respect to the actions of G and G' on X. Then, $\Psi' = h^* \circ \Psi$.
2. Let Γ and Γ' be the holonomy groups with respect to the actions of G and G' on X. Then, $\Gamma' = h^*(\Gamma)$.
3. The linear map h^* projects on a smooth homomorphism $h^*_\Gamma : \mathscr{G}^*/\Gamma \to \mathscr{G}'^*/\Gamma'$, such that the following diagram commutes.

$$\begin{array}{ccc} \mathscr{G}^* & \xrightarrow{\;h^*\;} & \mathscr{G}'^* \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}'} \\ \mathscr{G}^*/\Gamma & \xrightarrow[h^*_\Gamma]{} & \mathscr{G}'^*/\Gamma' \end{array}$$

4. Let ψ and ψ' be the 2-points moment maps with respect to the actions of G and G'. Then, $\psi' = h^*_\Gamma \circ \psi$.
5. Let μ be a moment map relative to the action ρ of G. Then, $\mu' = h^*_\Gamma \circ \mu$ is a moment map relative to the action ρ' of G'.
6. Let μ be a moment map relative to the action ρ of G, and let $\mu' = \mu \circ h^*_\Gamma$ be the associated moment map relative to the action ρ' of G'. Then, the associated Souriau cocycles satisfy $\theta' = h^*_\Gamma \circ \theta \circ h$, which is summarized by the following commutative diagram.

$$\begin{array}{ccc} \mathrm{G} & \xleftarrow{\;h\;} & \mathrm{G}' \\ {\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle \theta'} \\ \mathscr{G}^*/\Gamma & \xrightarrow[h^*_\Gamma]{} & \mathscr{G}'^*/\Gamma' \end{array}$$

   Said differently, if θ is the Souriau cocycle associated with a moment μ of the action ρ of G, and μ' is a moment of the action ρ' of G', then θ' and $h^*_\Gamma \circ \theta \circ h$ are cohomologous.

NOTE. Thanks to the identification between the space of momenta of a diffeological group and any of its extensions by a discrete group (art. 7.13), the moment maps of the action of a group or the moment map of the restriction of this action to its identity component coincide. Said differently, the moment maps do not say anything about actions of discrete groups.

PROOF. To avoid confusion, let us denote by $\mathrm{R}(p)$ and $\mathrm{R}'(p)$ the orbit maps of G and G′ of $p \in \mathrm{Paths}(\mathrm{X})$, that is, $\mathrm{R}(p)(g) = \rho(g) \circ p$ and $\mathrm{R}'(p)(g) = \rho'(g) \circ p$. We have then, $\mathrm{R}'(p)(g) = \rho'(g) \circ p = \rho(h(g)) \circ p = (\mathrm{R}(p) \circ h)(g)$. Thus, $\mathrm{R}'(p) = \mathrm{R}(p) \circ h$.

1. By definition of the paths moment map, $\Psi'(p) = \mathrm{R}'(p)^*(\mathcal{K}\omega) = (\mathrm{R}(p) \circ h)^*(\mathcal{K}\omega) = h^*(\mathrm{R}(p)^*(\mathcal{K}\omega)) = h^*(\Psi(p))$, that is, $\Psi' = h^* \circ \Psi$.

2. Since $\Gamma' = \Psi'(\mathrm{Loops}(\mathrm{X}))$, and thanks to item 1, $\Gamma' = h^*(\Gamma)$.

3. The map h^*_Γ is defined by $\mathrm{class}_\Gamma(\alpha) \mapsto \mathrm{class}_{\Gamma'}(h^*(\alpha))$, for all $\alpha \in \mathcal{G}^*$. If $\beta = \alpha + \gamma$, with $\gamma \in \Gamma$, then $h^*(\beta) = h^*(\alpha) + \gamma'$, with $\gamma' = h^*(\gamma) \in \Gamma'$ (item 2). Thus, $\mathrm{class}_{\Gamma'}(h^*(\beta)) = \mathrm{class}_{\Gamma'}(h^*(\alpha))$ and h^*_Γ is well defined. Thanks to the linearity of h^*, h^*_Γ is clearly a homomorphism. For $\mathcal{G}^*/\Gamma$ and $\mathcal{G}'^*/\Gamma'$ equipped with the quotient diffeologies, h^*_Γ is naturally smooth.

4. With the notations above, ψ and ψ' are defined by $\mathrm{pr} \circ \Psi = \psi \circ \mathrm{ends}$ and $\mathrm{pr}' \circ \Psi' = \psi' \circ \mathrm{ends}$, where $\mathrm{ends}(p) = (p(0), p(1))$, with $p \in \mathrm{Paths}(\mathrm{X})$. Thus, by item 1 and 3, $\mathrm{pr}' \circ h^* \circ \Psi = h^*_\Gamma \circ \psi \circ \mathrm{pr}$, that is, $\mathrm{pr}' \circ \Psi' = (h^*_\Gamma \circ \psi) \circ \mathrm{pr}$. Hence, $h^*_\Gamma \circ \psi = \psi'$.

5. Let $\mu' = h^*_\Gamma \circ \mu$ and $x, y \in \mathrm{X}$. Then, $\mu'(y) - \mu'(x) = h^*_\Gamma \circ \mu(y) - h^*_\Gamma \circ \mu(x) = h^*_\Gamma(\mu(y) - \mu(x)) = h^*_\Gamma \circ \psi(y, x) = \psi'(y, x)$. Thus, μ' is a moment map for the action ρ' of G.

6. According to (art. 9.10), there exists a point $x_0 \in \mathrm{X}$ such that, for all $g' \in \mathrm{G}'$, $\theta'(g') = \psi'(x_0, \rho'(g')(x_0))$. Thus, thanks to the previous items, $\theta'(g') = (h^*_\Gamma \circ \psi)(x_0, \rho(h(g'))(x_0)) = h^*_\Gamma(\psi(x_0, \rho(h(g'))(x_0))) = h^*_\Gamma(\theta(h(g'))) = (h^*_\Gamma \circ \theta \circ h)(g')$. Hence, $\theta' = h^*_\Gamma \circ \theta \circ h$. ☐

9.13. Pushing forward moment maps. Let X and X′ be two connected diffeological spaces. Let ω and ω' be two closed 2-forms, defined respectively on X and X′. Let G be a diffeological group, let ρ be a smooth action of G on X preserving ω, and let ρ' be a smooth action, of G on X′, preserving ω'. Let $f : \mathrm{X} \to \mathrm{X}'$ be a smooth map such that $\omega = f^*(\omega')$ and $f \circ \rho(g) = \rho'(g) \circ f$ for all $g \in \mathrm{G}$. Then,

1. The paths moment maps Ψ and Ψ' relative to the actions ρ of G on (X, ω), and ρ' on (X', ω'), are related by:

$$\Psi = \Psi' \circ f_*,$$

where $f_* : \mathrm{Paths}(\mathrm{X}) \to \mathrm{Paths}(\mathrm{X}')$ is defined by $f_*(p) = f \circ p$.

The associated holonomy groups Γ and Γ' satisfy

$$\Gamma = \{\Psi'(f \circ \ell) \mid \ell \in \mathrm{Loops}(X)\} \subset \Gamma'.$$

2. Let $\phi : \mathscr{G}^*/\Gamma \to \mathscr{G}^*/\Gamma'$ be the projection induced by the inclusion $\Gamma \subset \Gamma'$. Let ψ and ψ' be the 2-points moment maps relative to the actions ρ and ρ'. Then, for any two points of X, x_1 and x_2,

$$\psi'(f(x_1), f(x_2)) = \phi(\psi(x_1, x_2)).$$

3. For every moment map μ relative to the action ρ, there exists a moment map μ' relative to the action ρ', such that

$$\mu' \circ f = \phi \circ \mu.$$

4. Let θ and θ' be two Souriau cocycles relative to the actions ρ and ρ'. Then, the map $\phi \circ \theta$ is a Souriau cocycle, cohomologous to θ' and the two Souriau classes σ and σ' satisfy $\sigma' = \phi_*(\sigma)$, where ϕ_* denotes the action of ϕ on cohomology, $\phi_*(\mathrm{class}(\theta)) = \mathrm{class}(\phi \circ \theta)$.

PROOF. 1. By definition $\Psi(p) = \hat{p}^*(\mathscr{K}\omega)$, that is, $\Psi(p) = \hat{p}^*(\mathscr{K}(f^*(\omega')))$. Thanks to the variance of the Chain-Homotopy operator $\mathscr{K} \circ f^* = (f_*)^* \circ \mathscr{K}'$ (art. 6.84), $\Psi(p) = \hat{p}^* \circ (f_*)^*(\mathscr{K}'\omega') = (f_* \circ \hat{p})^*(\mathscr{K}'\omega')$. But for all $g \in G$, $f_* \circ \hat{p}(g) = f \circ \rho(g) \circ p = \rho'(g) \circ f \circ p = \hat{p}'(g)$, where $p' = f \circ p$. Then, $\Psi(p) = \hat{p}'^*(\mathscr{K}'\omega') = \Psi'(p') = \Psi'(f_*(p))$. Therefore, $\Psi = \Psi' \circ f_*$. Now, by definition of the holonomy groups, $\Gamma = \Psi(\mathrm{Loops}(X)) = \Psi'(f_*(\mathrm{Loops}(X)))$, and since $f_*(\mathrm{Loops}(X)) \subset \mathrm{Loops}(X')$, we have $\Gamma \subset \Gamma'$.

2. Since $\Gamma \subset \Gamma'$, the map $\phi : \mathrm{class}_\Gamma(\alpha) \mapsto \mathrm{class}_{\Gamma'}(\alpha)$, from $\mathscr{G}^*/\Gamma$ to $\mathscr{G}^*/\Gamma'$, is well defined. Now, let $x'_1 = f(x_1)$ and $x'_2 = f(x_2)$. There exists then $p \in \mathrm{Paths}(X)$ connecting x_1 to x_2, and the path $f_*(p)$ connects x'_1 to x'_2. By definition of ψ', $\psi'(x'_1, x'_2) = \mathrm{class}_{\Gamma'}(\Psi'(p')) = \mathrm{class}_{\Gamma'}(\Psi' \circ f_*(p))$, and thanks to the first item, $\mathrm{class}_{\Gamma'}(\Psi'(p')) = \mathrm{class}_{\Gamma'}(\Psi(p)) = \phi(\mathrm{class}_\Gamma(\Psi(p)))$. But $\mathrm{class}_\Gamma(\Psi(p)) = \psi(x_1, x_2)$. Hence, $\psi'(x'_1, x'_2) = \phi(\psi(x_1, x_2))$, that is, $\psi'(f(x_1), f(x_2)) = \psi(x_1, x_2)$.

3. According to (art. 9.9), for every moment map μ there exist a point $x_0 \in X$ and a constant $c \in \mathscr{G}^*/\Gamma$ such that $\mu(x) = \psi(x_0, x) + c$. Let us define μ' by $\mu'(x') = \psi'(x'_0, x') + c'$, where $x'_0 = f(x_0)$ and $c' = \phi(c)$. Then, thanks to item 2, $\psi'(f(x_0), f(x)) = \phi(\psi(x_0, x))$, and $\mu'(f(x)) = \phi(\psi(x_0, x)) + \phi(c) = \phi(\psi(x_0, x) + c) = \phi(\mu(x))$. Hence, μ' satisfies $\mu' \circ f = \phi \circ \mu$.

4. Let θ be a Souriau cocycle for the action ρ. According to (art. 9.10), θ is cohomologous to $\vartheta : g \mapsto \psi(x_0, \rho(g)(x))$, where x_0 is some point of X. Then, let $x'_0 = f(x_0)$, and $\vartheta' : g \mapsto \psi'(x'_0, \rho'(g)(x'_0))$. Thus, $\vartheta'(g) = \psi'(f(x_0), \rho'(g)(f(x_0))) = \psi'(f(x_0), f(\rho(g)(x_0))) = \phi(\psi(x_0, \rho(g)(x_0))) = \phi \circ \vartheta(g)$. Now, since all Souriau

cocycles, with respect to a given action of G, are cohomologous, the cocycle θ' is cohomologous to ϑ', and then cohomologous to $\phi \circ \vartheta$, that is, cohomologous to $\phi \circ \theta$. Hence, $\sigma' = \text{class}(\theta') = \text{class}(\phi \circ \theta) = \phi_*(\text{class}(\theta)) = \phi_*(\sigma)$. ☐

The Universal Moment Maps

In this section we build the universal moment maps, and related objects, associated with the whole group of automorphisms $\mathrm{Diff}(X, \omega)$*, where* ω *is a closed* 2*-form on a diffeological space* X*. We show then how these objects, associated with a smooth automorphic action of some arbitrary diffeological group* G*, relate to the universal construction.*

9.14. Universal moment maps. Let X be a connected diffeological space, and let ω be a closed 2-form defined on X. We consider the group $\mathrm{Diff}(X, \omega)$, of all the automorphisms of (X, ω), equipped with the functional diffeology of group of diffeomorphisms. We shall also denote this group by G_ω. Every construction introduced in the previous sections — the space of momenta, the paths moment map, the holonomy group, the 2-points moment map, the moment maps, the Souriau cocycles, and the Souriau class — apply for G_ω. We shall distinguish these objects by the index ω. So, we shall denote by $\mathscr{G}^*_\omega$ the space of momenta of G_ω, by $\Psi_\omega : \mathrm{Paths}(X) \to \mathscr{G}^*_\omega$ the paths moment map, by $\Gamma_\omega = \Psi_\omega(\mathrm{Loops}(X))$ the holonomy group, by ψ_ω the 2-points moment map, by μ_ω the moment maps, by θ_ω the Souriau cocycles, and by σ_ω the Souriau class. Since G_ω and its action on X are uniquely defined by ω, these objects depend only on the 2-form ω.

Now, let G be a diffeological group, and let ρ be a smooth action of G on X, preserving ω, that is, a smooth homomorphism ρ from G to G_ω. The values of the various objects Ψ, Γ, ψ, μ, θ, with respect to the action ρ of G on X, depend only on the pullback ρ^* and on Ψ_ω, Γ_ω, ψ_ω, μ_ω, and θ_ω, as it is described in (art. 9.12),

$$\left\{ \begin{array}{lcl} \Psi & = & \rho^* \circ \Psi_\omega \\ \Gamma & = & \rho^*(\Gamma_\omega) \\ \psi & = & \rho^*_{\Gamma_\omega} \circ \psi_\omega \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{lcl} \mu & \simeq & \rho^*_{\Gamma_\omega} \circ \mu_\omega \\ \theta & \simeq & \rho^*_{\Gamma_\omega} \circ \theta_\omega \circ \rho. \end{array} \right.$$

In this sense the objects G_ω, Γ_ω, Ψ_ω, Γ_ω, ψ_ω, μ_ω, θ_ω, and σ_ω are *universal*. That is why we shall call Ψ_ω the *universal paths moment map*, Γ_ω the *universal holonomy*, ψ_ω the *universal* 2*-points moment map*, μ_ω the *universal moment maps*, θ_ω the *universal Souriau cocycles*, and σ_ω the *universal Souriau class* of ω.

NOTE. The universal holonomy leads naturally to the notion of *Hamiltonian space*, the ones for which, for one reason or another, $\Gamma_\omega = \{0\}$.

9.15. The group of Hamiltonian diffeomorphisms. Let X be a connected diffeological space, equipped with a closed 2-form ω. There exists a largest connected subgroup $\mathrm{Ham}(X,\omega) \subset \mathrm{Diff}(X,\omega)$ whose action is Hamiltonian, that is, whose holonomy is trivial. The elements of $\mathrm{Ham}(X,\omega)$ are called *Hamiltonian diffeomorphisms*. An action ρ of a diffeological group G on X is Hamiltonian if and only if, restricted to the identity component of G, ρ takes its values in $\mathrm{Ham}(X,\omega)$.

The group $\mathrm{Ham}(X,\omega)$ is precisely built as follows. Let us denote by G_ω the group $\mathrm{Diff}(X,\omega)$ and by G_ω° its identity component. Let $\pi : \tilde{G}_\omega^\circ \to G_\omega^\circ$ be the universal covering. Since the universal holonomy Γ_ω is made of closed momenta, every $\gamma \in \Gamma_\omega$ defines a unique homomorphism $k(\gamma)$ from $\tilde{G}_\omega^\circ$ to $\mathbf{R}$ such that $\pi^*(\gamma) = d[k(\gamma)]$ (art. 7.17). Let

$$\widehat{H}_\omega = \bigcap_{\gamma\in\Gamma_\omega} \ker(k(\gamma)),$$

and let $\widehat{H}_\omega^\circ$ be its identity component. Then,

$$\mathrm{Ham}(X,\omega) = \pi(\widehat{H}_\omega^\circ).$$

NOTE 1. The map $f : \tilde{G}_\omega^\circ \to \mathrm{Hom}(\pi_1(X),\mathbf{R})$, defined by $f(\tilde{g}) = [\tau \mapsto k(\gamma)(\tilde{g})]$, with $\tau = \mathrm{class}(\ell)$ and $\gamma = \Psi(\ell)$, is a homomorphism, and $\widehat{H}_\omega = \ker(f)$. In classical symplectic geometry, the image $F = \mathrm{val}(f)$ is called, by some authors, the *group of flux* of ω.

NOTE 2. Since the Hamiltonian nature of a group of automorphisms depends only on its identity component (see (art. 7.13) and (art. 7.14)), every extension $H \subset \mathrm{Diff}(X,\omega)$ of $\mathrm{Ham}(X,\omega)$ such that $H/\mathrm{Ham}(X,\omega)$ is discrete,[8] is Hamiltonian. In particular $\pi(\widehat{H}_\omega)$ is Hamiltonian, or if $\Gamma_\omega = \{0\}$, then $\mathrm{Diff}(X,\omega)$ is Hamiltonian, and $\mathrm{Ham}(X,\omega)$ is the identity component of $\mathrm{Diff}(X,\omega)$.

NOTE 3. Let us choose a point x_0 in X, and let μ be the moment map with respect to the group $\mathrm{Ham}(X,\omega)$, defined by $\mu(x_0) = 0$. Let f be a 1-parameter subgroup of $\mathrm{Ham}(X,\omega)$. Applying (art. 9.2, Note), we get the expression of $\mu(x)$, for all $x \in X$, evaluated on f

$$\mu(x)(f) = h_f(x) \times dt \text{ with } h_f(x) = -\int_{x_0}^{x} i_f(\omega).$$

The smooth function $h_f : X \to \mathbf{R}$ is the *Hamiltonian* (vanishing at x_0) of the 1-parameter subgroup f.

PROOF. Let us remark first of all that for every $\gamma \in \Gamma_\omega$, $\pi^*(\gamma) \restriction \widehat{H}_\omega = 0$. Indeed, $\pi^*(\gamma) \restriction \widehat{H}_\omega = d[k(\gamma)] \restriction \widehat{H}_\omega = d[k(\gamma) \restriction \widehat{H}_\omega]$. But, by the very definition of $\widehat{H}_\omega$, $k(\gamma) \restriction \widehat{H}_\omega = 0$, thus $\pi^*(\gamma) \restriction \widehat{H}_\omega = 0$.

[8]Where H and $\mathrm{Ham}(X,\omega)$ are equipped with the subset diffeology of the functional diffeology of $\mathrm{Diff}(X,\omega)$.

(a) Let us prove that the holonomy of $\mathrm{Ham}(X,\omega)$ is trivial. Let $H_\omega = \pi(\widehat{H}_\omega)$, and let us denote by j_{H_ω} the inclusion $H_\omega \subset G_\omega$, by $j_{\widehat{H}_\omega}$ the inclusion $\widehat{H}_\omega \subset \tilde{G}^\circ_\omega$, and by $\pi_{H_\omega} : \widehat{H}_\omega \to H_\omega$ the projection, so that $j_{H_\omega} \circ \pi_{H_\omega} = \pi \circ j_{\widehat{H}_\omega}$. Let Γ_{H_ω} be theholonomy of H_ω, then, according to (art. 9.12), $\Gamma_{H_\omega} = j^*_{H_\omega}(\Gamma_\omega)$. Thus, for every $\bar\gamma \in \Gamma_{H_\omega}$ there exists $\gamma \in \Gamma_\omega$ such that $\bar\gamma = \gamma \restriction H_\omega = j^*_{H_\omega}(\gamma)$. Hence, for all $\bar\gamma \in \Gamma_{H_\omega}$, $\pi^*_{H_\omega}(\bar\gamma) = \pi^*_{H_\omega}(j^*_{H_\omega}(\gamma)) = (j_{H_\omega} \circ \pi_{H_\omega})^*(\gamma) = (\pi \circ j_{\widehat{H}_\omega})^*(\gamma) = j^*_{\widehat{H}_\omega}(\pi^*(\gamma)) = \pi^*(\gamma) \restriction \widehat{H}_\omega$. But $\pi^*(\gamma) \restriction \widehat{H}_\omega = 0$, thus $\pi^*_{H_\omega}(\bar\gamma) = 0$, and since π_{H_ω} is a subduction, $\bar\gamma = 0$. Therefore, the holonomy of H_ω trivial, $\Gamma_{H_\omega} = \{0\}$.

(b) Let us prove now that every connected subgroup $H \subset G_\omega$ whose action is Hamiltonian is a subgroup of $\mathrm{Ham}(X,\omega)$. Let $\widehat{H} = \pi^{-1}(H)$ and $\widehat{H}^\circ$ be its identity component. Let j_H be the inclusion $H \subset G_\omega$, and $j_{\widehat{H}^\circ}$ be the inclusion $\widehat{H}^\circ \subset \tilde{G}^\circ_\omega$. Let $\pi_H = \pi \restriction \widehat{H}^\circ$, so that $j_H \circ \pi_H = \pi \circ j_{\widehat{H}^\circ}$. Let Γ_H be the holonomy of H. Since $\Gamma_H = j^*_H(\Gamma_\omega)$ and $\Gamma_H = \{0\}$, for all $\gamma \in \Gamma_\omega$, $j^*_H(\gamma) = 0$. Thus, for all $\gamma \in \Gamma_\omega$, $\pi^*_H(j^*_H(\gamma)) = 0$. But $\pi^*_H(j^*_H(\gamma)) = (j_H \circ \pi_H)^*(\gamma) = (\pi \circ j_{\widehat{H}^\circ})^*(\gamma) = j^*_{\widehat{H}^\circ}(\pi^*(\gamma)) = \pi^*(\gamma) \restriction \widehat{H}^\circ$, thus, for all $\gamma \in \Gamma_\omega$, $\pi^*(\gamma) \restriction \widehat{H}^\circ = 0$. Now, $\pi^*(\gamma) = d[k(\gamma)]$, hence $d[k(\gamma) \restriction \widehat{H}^\circ] = 0$. Then, since H° is connected, $k(\gamma)$ is constant on $\widehat{H}^\circ$, and since $k(\gamma)$ is a homomorphism to $\mathbf{R}$, this constant is necessarily 0. Thus, $\widehat{H}^\circ \subset \ker(k(\gamma))$, for all $\gamma \in \Gamma_\omega$, that is, $\widehat{H}^\circ \subset \widehat{H}_\omega$. But since H° is connected, $\widehat{H}^\circ \subset \widehat{H}^\circ_\omega \subset H_\omega$, and thus $H = \pi(\widehat{H}^\circ) \subset \mathrm{Ham}(X,\omega) = \pi(\widehat{H}^\circ_\omega)$. ☐

9.16. Time-dependent Hamiltonian. Let X be a connected diffeological space, and let ω be a closed 2-form defined on X. A diffeomorphism f of X belongs to $\mathrm{Ham}(X,\omega)$ if and only if the two following conditions are fulfilled.

1. There exists a smooth path $t \mapsto f_t$ in $\mathrm{Diff}(X,\omega)$ connecting the identity $\mathbf{1}_M = f_0$ to $f = f_1$.
2. There exists a smooth path $t \mapsto \Phi_t$ in $\mathscr{C}^\infty(X,\mathbf{R})$ such that

$$i_{F_t}(\omega) = -d\Phi_t \text{ with } F_t : s \mapsto f_t^{-1} \circ f_{t+s},$$

for all t. According to the tradition of classical symplectic geometry, the path $t \mapsto \Phi_t$ may be called a *time-dependent Hamiltonian* of the 1-parameter family of Hamiltonian diffeomorphisms $t \mapsto f_t$.

PROOF. Let us assume first that f satisfies the condition above, that there exists a smooth path $t \mapsto f_t$ in $\mathrm{Diff}(X,\omega)$ such that $f_0 = \mathbf{1}_M$, $f_1 = f$, and there exists a smooth path $t \mapsto \Phi_t$ in $\mathscr{C}^\infty(X,\mathbf{R})$ such that $i_{F_t}(\omega) = -d\Phi_t$, for all t, where $F_t : s \mapsto f_t^{-1} \circ f_{t+s}$. Let us recall that $\mathrm{Ham}(X,\omega) = \pi(\widehat{H}^\circ_\omega)$, with $\widehat{H}^\circ_\omega$ the identity component of $\widehat{H}_\omega = \bigcap_{\gamma\in\Gamma_\omega} \ker(k(\gamma))$, and let $\tilde{f} \in G^\circ_\omega$ be the homotopy class of the path $t \mapsto f_t$, notations of (art. 9.15). Then, let $\gamma \in \Gamma_\omega$, that is, $\gamma = \Psi_\omega(\ell)$, where ℓ

is some loop in M. By definition,

$$\mathrm{k}(\gamma)(\tilde{f}) = \int_{[t\mapsto f_t]} \gamma = \int_{[t\mapsto f_t]} \Psi_\omega(\ell) = \int_0^1 \Psi_\omega(\ell)([t \mapsto f_t])_t(1)\,dt\,.$$

Now, thanks to (art. 9.2, (♣)),

$$\Psi_\omega(\ell)([t \mapsto f_t])_t(1) = -\int_\ell i_{\mathrm{F}_t}(\omega) = \int_\ell d\Phi_t = \int_{\partial\ell} \Phi_t = 0.$$

Thus, $\mathrm{k}(\gamma)(\tilde{f}) = 0$ for all $\gamma \in \Gamma_\omega$, and $\tilde{f}$ belongs to $\widehat{\mathrm{H}}_\omega$, more precisely to the identity component of $\widehat{\mathrm{H}}_\omega$. Therefore, $f \in \mathrm{Ham}(\mathrm{X},\omega)$.

Conversely, let $f \in \mathrm{Ham}(\mathrm{M},\omega)$. Since $\mathrm{Ham}(\mathrm{M},\omega)$ is connected, there exists a path $t \mapsto f_t$ in $\mathrm{Ham}(\mathrm{M},\omega)$ connecting $\mathbf{1}_\mathrm{M}$ to f. And, since the projection $\pi \restriction \widehat{\mathrm{H}}^\circ_\omega : \widehat{\mathrm{H}}^\circ_\omega \to \mathrm{Ham}(\mathrm{M},\omega)$ is a covering, there exists a (unique) lift $t \mapsto \tilde{f}_t$ of $t \mapsto f$ in $\widehat{\mathrm{H}}^\circ_\omega$, along $\pi \restriction \widehat{\mathrm{H}}^\circ_\omega$, such that $\tilde{f}_0 = \mathbf{1}_{\widehat{\mathrm{H}}_\omega}$. This lift is actually given by $\tilde{f}_t = \mathrm{class}(p_t)$, with $p_t : s \mapsto f_{st}$. Thus, for all t, $\tilde{f}_t \in \widehat{\mathrm{H}}^\circ_\omega \subset \widehat{\mathrm{H}}_\omega = \bigcap_{\gamma\in\Gamma_\omega} \ker(\mathrm{k}(\gamma))$, that is, for all $\gamma \in \Gamma_\omega$, $\mathrm{k}(\gamma)(\tilde{f}_t) = 0$. In other words, for all $\ell \in \mathrm{Loops}(\mathrm{M})$, $\mathrm{k}(\Psi_\omega(\ell))(\tilde{f}_t) = 0$. But

$$\begin{aligned}
\mathrm{k}(\Psi_\omega(\ell))(\tilde{f}_t) &= \int_{p_t} \Psi_\omega(\ell) \\
&= \int_0^1 \Psi_\omega(\ell)(s \mapsto f_{st})_s(1)ds \\
&= \int_0^1 \Psi_\omega(\ell)(s \mapsto st \mapsto f_{st})_s(1)ds \\
&= \int_0^1 [\Psi_\omega(\ell)(u \mapsto f_u)]_{u=st} \left(\frac{d(st)}{ds}\right) ds \\
&= \int_0^t \Psi_\omega(\ell)(u \mapsto f_u)_u(1)\,du\,.
\end{aligned}$$

Thus, in particular $\frac{1}{t}\int_0^t \Psi_\omega(\ell)(u \mapsto f_u)_u(1)\,du = 0$, and $\Psi_\omega(\ell)(t \mapsto f_t)_t(1) = \lim_{t\to 0} \frac{1}{t}\int_0^t \Psi_\omega(\ell)(u \mapsto f_u)_u(1)\,du = 0$, and since $\Psi_\omega(\ell)([t \mapsto f_t])_t(1) = -\int_\ell i_{\mathrm{F}_t}(\omega)$ (art. 9.2, (♣)), for all t and all $\ell \in \mathrm{Loops}(\mathrm{X})$, $\int_\ell i_{\mathrm{F}_t}(\omega) = 0$. Then, since F_t is a path in $\mathrm{Diff}(\mathrm{X},\omega)$ centered at the identity, the Lie derivative of ω by F_t vanishes, and by application of the Cartan formula (art. 6.72), we get $\pounds_{\mathrm{F}_t}\omega = 0$, which implies $d[i_{\mathrm{F}_t}(\omega)] + i_{\mathrm{F}_t}(d\omega) = d[i_{\mathrm{F}_t}(\omega)] = 0$. Thus, the 1-form $i_{\mathrm{F}_t}(\omega)$ is closed and its integral on any loop ℓ in X vanishes, hence $i_{\mathrm{F}_t}(\omega)$ is exact (art. 6.89). Therefore, for all real number t, there exists a real function $\Phi_t \in \mathscr{C}^\infty(\mathrm{X},\mathbf{R})$ such that $i_{\mathrm{F}_t}(\omega) = -d\Phi_t$. The fact that $t \mapsto \Phi_t$ is a smooth map from $\mathbf{R}$ to $\mathscr{C}^\infty(\mathrm{X},\mathbf{R})$, for the functional diffeology, is a consequence of the construction of Φ_t by integration along the paths. ☐

9.17. The characteristics of the moment maps. Let X be a connected diffeological space equipped with a closed 2-form ω. Let G be a diffeological group, and let ρ be a smooth action of G on X, preserving ω. Let ψ be the 2-points moment map (art. 9.8). Thanks to the additive property of ψ, the relation $\mathscr{R}$, defined on X by

$$x \,\mathscr{R}\, x', \text{ if } \psi(x,x') = 0_{\mathscr{G}^*/\Gamma},$$

is an equivalence relation. The classes of this equivalence relation are the preimages of the values of a moment map μ, solution of $\psi(x,x') = \mu(x') - \mu(x)$ (art. 9.9).

DEFINITION. *We shall define the* characteristics of the moment map μ *(or ψ) as the connected components of the equivalence classes of $\mathscr{R}$, that is, the connected components of the preimages of* μ.

NOTE 1. This definition applies obviously to the universal moment map μ_ω. Since, in this case, the characteristics depend only on ω, it is tempting to call them the *characteristics of the* 2*-form* ω, especially when we have in mind the particular case of homogeneous manifolds, treated in (art. 9.26).

DEFINITION. *We shall define the* characteristics *of the* 2*-form* ω *as the characteristics of the universal moment map* μ_ω.

We get then a general picture: by equivariance, the image of the universal moment map μ_ω is a union of coadjoint orbits of Diff(X, ω), images of its orbits in X. The moment map factorizes then through the space of characteristics of ω, denoted here by Chars(X, ω), by a map $\bar\mu_\omega : \mathrm{Chars}(\mathrm{X},\omega) \to \mathscr{G}^*_\omega/\Gamma_\omega$. The preimages by $\bar\mu_\omega$ are the connected components of the preimages by μ_ω, that is, $\bar\mu_\omega^{-1}(m) = \pi_0(\mu_\omega^{-1}(m))$. The projection char_ω, associating with each $x \in \mathrm{X}$ the characteristic passing through x, is a kind of *symplectic reduction*. But we do not know if, in general, the 2-form ω passes to the quotient, especially when the group of automorphisms has more than one orbit. This is still an open question, but the framework is here.

$$\begin{array}{ccc} \mathrm{X} & \xrightarrow{\quad\mu_\omega\quad} & \mathscr{G}^*_\omega/\Gamma_\omega \\ & {\scriptstyle \mathrm{char}_\omega}\searrow \qquad \swarrow{\scriptstyle \bar\mu_\omega} & \\ & \mathrm{Chars}(\mathrm{X},\omega) & \end{array}$$

This construction is reminiscent of the Marsden-Weinstein symplectic reduction [MaWe74], when it is applied to some subspaces $\mathrm{W} \subset \mathrm{X}$ for the restriction $\omega \restriction \mathrm{W}$. There is, however, a small difference: we reduce first by the characteristics of ω and then by the moment map, which can be interpreted as a *regularization* of the reduction by the characteristics.[9]

[9]It is not impossible that in particular, for the two-bodies problem [Sou83], it be precisely the universal moment map which regularizes the space of motions.

NOTE 2. We also can think of characteristics as arrows of the groupoid **Chars**, with objects X and an arrow (x, x') only if $\psi(x, x') = 0$. Therefore, the characteristics are the connected components of the transitivity components of this groupoid.

The Homogeneous Case

Because of its elementary character, the case of a homogeneous action of a diffeological group G *on a space* X*, preserving a closed* 2*-form* ω*, deserves a special attention. We shall see in* (art. 9.23) *how this applies to classical symplectic geometry.*

9.18. The homogeneous case. Let X be a connected diffeological space equipped with a closed 2-form ω. Let ρ be a smooth action of a diffeological group G on X, preserving ω. Let us assume that X is homogeneous for this action (art. 7.8). Let Γ be the holonomy of the action ρ, let μ be a moment map, and let θ be the cocycle associated to μ. Let x_0 be any point of X, and let $\mu_0 = \mu(x_0)$. Let $\mathrm{St}_{\mathrm{Ad}_*^{\Gamma,\theta}}(\mu_0)$ be the stabilizer of μ_0 for the affine coadjoint action of G on $\mathscr{G}^*/\Gamma$. Thanks to the equivariance of the moment map μ, with respect to the θ-affine coadjoint action of G on $\mathscr{G}^*/\Gamma$, $\mu \circ \rho(g) = \mathrm{Ad}_*^{\Gamma,\theta}(g) \circ \mu$, the image $\mathscr{O} = \mu(\mathrm{X})$ is a (Γ, θ)-orbit of G, and $\mathrm{St}_\rho(x_0) \subset \mathrm{St}_{\mathrm{Ad}_*^{\Gamma,\theta}}(\mu_0)$. Let us equip $\mathscr{O}$ with the pushforward of the diffeology of G by the orbit map $\hat{\mu}_0 : g \mapsto \mathrm{Ad}_*^{\Gamma,\theta}(g)(\mu_0)$. Then, the orbit map $\hat{x}_0 : \mathrm{G} \to \mathrm{X}$ is a principal fibration with structure group $\mathrm{St}_\rho(x_0)$, the orbit map $\hat{\mu}_0 : \mathrm{G} \to \mathscr{O}$ is a principal fibration with structure group $\mathrm{St}_{\mathrm{Ad}_*^{\Gamma,\theta}}(\mu_0)$, and the moment map $\mu : \mathrm{X} \to \mathscr{O}$ is a fibration, with fiber the homogeneous space $\mathrm{St}_{\mathrm{Ad}_*^{\Gamma,\theta}}(\mu_0)/\mathrm{St}_\rho(x_0)$.

NOTE 1. The moment maps μ are defined up to a constant. But that does not affect the characteristics of μ, which are the connected components of the subspaces defined by $\mu(x) = \mathrm{cst}$ (art. 9.17).

NOTE 2. Let us consider the simple case $d\alpha$, where $\alpha \in \mathscr{G}^*$. Its moment map $\mu : \mathrm{G} \to \mathscr{G}^*$ is the coadjoint orbit map $\mu : g \mapsto \mathrm{Ad}_*(g)(\alpha)$. A natural question is then, is there a closed 2-form ω, defined on the coadjoint orbit $\mathscr{O}_\alpha = \mu(\mathrm{G})$, such that $\mu^*(\omega) = d\alpha$? I don't have a definitive answer to this question yet; the best I can say is contained in the following proposition. Let us consider $\mathscr{O}_\alpha$, equipped with the quotient diffeology of G.

PROPOSITION. *There exists a closed* 2*-form* ω *on* $\mathscr{O}_\alpha$ *such that* $\mu^*(\omega) = d\alpha$*, if and only if* $\alpha \upharpoonright \mathrm{St}_{\mathrm{Ad}_*}(\alpha)$ *is closed. In that case,* ω *is the canonical symplectic structure of coadjoint orbit of diffeological group.*

There are particular cases, or special group diffeologies, for which the restriction of α on its stabilizer $\mathrm{St}_{\mathrm{Ad}_*}(\alpha)$ is closed. For example, if G is a Lie group, then we can use the Cartan formula to check it. But even simpler, if the stabilizer $\mathrm{St}_{\mathrm{Ad}_*}(\alpha)$ is discrete or 1-dimensional, then every 1-form is closed. Paul Donato has given in [Don94] an interesting example of a discrete stabilizer. Let us give here the diffeological version of his construction. We consider $\mathrm{G} = \mathrm{Diff}(\mathrm{S}^1)$ and its universal covering $\tilde{\mathrm{G}}$, described in Exercise 28, p. 33, as the subgroup of diffeomorphisms φ of $\mathbf{R}$ such that $\varphi(x+2\pi) = \varphi(x) + 2\pi$. The kernel of the projection $\pi : \tilde{\mathrm{G}} \to \mathrm{G}$ is the subgroup $2\pi\mathbf{Z}$ of the translations $\mathrm{T}_{2\pi k} : x \mapsto x + 2\pi k$, commuting with all $\varphi \in \tilde{\mathrm{G}}$. Let $\mathrm{P} : \mathrm{U} \to \tilde{\mathrm{G}}$ be an n-plot, and let $r \in \mathrm{U}$, $\delta r \in \mathbf{R}^n$, and α defined by

$$\alpha(\mathrm{P})_r(\delta r) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\partial}{\partial s} \Big\{ \mathrm{P}(r)^{-1} \circ \mathrm{P}(s)(n) \Big\}_{s=r} (\delta r).$$

One can check that α is a left invariant 1-form, and by the way an element of $\mathscr{G}^*$. The moment map μ is then given by

$$\mu(\varphi)(\mathrm{P})_r(\delta r) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\partial}{\partial s} \Big\{ \varphi^{-1} \circ \mathrm{P}(r)^{-1} \circ \mathrm{P}(s) \circ \varphi(n) \Big\}_{s=r} (\delta r).$$

Because a momentum is characterized by its values on arcs centered at the identity, and because every arc γ centered at the identity in $\tilde{\mathrm{G}}$ is tangential to some ray $h \in \mathrm{Hom}^\infty(\mathbf{R}, \tilde{\mathrm{G}})$, the computation of the stabilizer of α, for the coadjoint action, is reduced to Donato's computation in his paper, and it coincides with the orbits of $2\pi\mathbf{Z}$. Thus, $d\alpha$ passes to the coadjoint orbit $\mathscr{O}_\alpha$, which is actually diffeomorphic to $\mathrm{Diff}(\mathrm{S}^1)$ itself, and symplectic according to the meaning we define below (art. 9.19).

PROOF. The triple fibration is an application of (art. 8.15).

$$\begin{array}{ccc} & \mathrm{G} & \\ {\scriptstyle \hat{x}_0} \swarrow & & \searrow {\scriptstyle \hat{\mu}_0} \\ \mathrm{X} & \xrightarrow[\mu]{} & \mathscr{O} \end{array} \qquad \begin{array}{ccc} & \mathrm{G} & \\ {\scriptstyle \mathrm{St}_\rho(x_0)} \swarrow & & \searrow {\scriptstyle \mathrm{St}_{\mathrm{Ad}_*^{\Gamma,\theta}}(\mu_0)} \\ \mathrm{X} & \xrightarrow[\mathrm{St}_{\mathrm{Ad}_*^{\Gamma,\theta}}(\mu_0)/\mathrm{St}_\rho(x_0)]{} & \mathscr{O} \end{array}$$

Let us focus on Note 2. There exists a (closed) 2-form ω on $\mathscr{O}_\alpha$ such that $\mu^*(\omega) = d\alpha$ if and only if, for two plots P and P′ of G such that $\mu \circ \mathrm{P} = \mu \circ \mathrm{P}'$, then $d\alpha(\mathrm{P}) = d\alpha(\mathrm{P}')$ (art. 6.38). But $\mu \circ \mathrm{P} = \mu \circ \mathrm{P}'$ means that there exists a plot $\varkappa$ of $\mathrm{St}_{\mathscr{G}^*}(\alpha)$ such that $\mathrm{P}'(r) = \mathrm{P}(r) \cdot \varkappa(r)$. Then, thanks to Exercise 127, p. 274, $\alpha[r \mapsto \mathrm{P}(r) \cdot \varkappa(r)]_r = [\mathrm{L}(\mathrm{P}(r))^*(\alpha)](\varkappa)_r + [\mathrm{R}(\varkappa(r))^*(\alpha)](\mathrm{P})_r = \alpha(\varkappa)_r + [\mathrm{Ad}_*(\varkappa(r))(\alpha)](\mathrm{P})_r$, but since $\varkappa$ is a plot of the stabilizer of α for the coadjoint action, $\mathrm{Ad}_*(\varkappa(r))(\alpha) = \alpha$ and $\alpha(\mathrm{P}') = \alpha(\mathrm{P}) + \alpha(\varkappa)$. Thus, $d\alpha$ passes to the quotient $\mathscr{O}_\alpha \simeq \mathrm{G}/\mathrm{St}_{\mathscr{G}^*}(\alpha)$ if and only if $d\alpha(\varkappa) = 0$ for all plots $\varkappa$ of $\mathrm{St}_{\mathscr{G}^*}(\alpha)$, *i.e.*, if and only if $d[\alpha \upharpoonright \mathrm{St}_{\mathscr{G}^*}(\alpha)] = 0$. ☐

9.19. Symplectic homogeneous diffeological spaces. Let X be a connected diffeological space and ω be a closed 2-form defined on X.[10]

[10] To shorten the vocabulary, I chose now to call *parasymplectic* any closed 2-form.

DEFINITION. *We say that* (X, ω) *is a* homogeneous symplectic space *if it is homogeneous under the action of* $\mathrm{Diff}(X, \omega)$*, and if the characteristics of the universal moment map* μ_ω *are reduced to points* (art. 9.17).

For example, the 2-form $d\alpha$ on $\mathrm{Diff}(S^1)$ described in the previous article (art. 9.18) is symplectic in this sense. Let us recall that being homogeneous under the action of $\mathrm{Diff}(X, \omega)$ means that the orbit map $R(x) : \mathrm{Diff}(X, \omega) \to X$, defined by $R(x)(\varphi) = \varphi(x)$, is a subduction, where $x \in X$ and $\mathrm{Diff}(X, \omega)$ is equipped with the functional diffeology (art. 1.61).

NOTE 1. The characteristics of the universal moment map μ_ω are reduced to points if and only if it is a covering onto its image, equipped with the quotient diffeology. If it is the case for one universal moment map, then it is the case for every one.

NOTE 2. The homogeneous situation where the moment map μ_ω is not a covering onto its image can be regarded as the *presymplectic homogeneous case*, as suggested by (art. 9.26).

NOTE 3. Let G be a diffeological group, and let ρ be a smooth action of G on X, preserving ω. If the action ρ of G on X is homogeneous, then X is a homogeneous space of $\mathrm{Diff}(X, \omega)$. And, if a moment map $\mu : X \to \mathscr{G}^*/\Gamma$ is a covering onto its image, then any universal moment map $\mu_\omega : X \to \mathscr{G}^*_\omega/\Gamma_\omega$ is. Thus, to check that a homogeneous pair (X, ω) is symplectic it is sufficient to find a homogeneous smooth action, of some diffeological group G, for which a moment map is a covering onto its image.

NOTE 4. It would be possible to relax the homogeneity hypothesis to have an acceptable definition of a *symplectic diffeological space* :

DEFINITION. *A closed* 2*-form* ω *on a diffeological space* X *would be said symplectic if* $\mathrm{Diff}_{\mathrm{loc}}(X, \omega)$ *is transitive on* X*, and if the characteristics of the universal moment map* μ_ω *are reduced to points.*

The transitivity of local automorphisms is the diffeological version of Darboux's theorem, and the injectivity of the universal moment map μ_ω, the non-degeneracy of ω. This definition should perhaps be refined as to the role of the universal moment map, a desired universal "local moment map" would be more appropriate.

PROOF. Note 1 is obvious, by definition of the characteristics (art. 9.17) and by homogeneity (art. 9.18). Let us prove Note 3. To be homogeneous under the action of G means that, for some point (and thus for every point) $x \in X$, the orbit map $\hat{x} : G \to X$, defined by $\hat{x}(g) = \rho(g)(x)$, is a subduction. So, $\hat{x}$ is surjective and, for any plot $P : U \to X$, for any $r_0 \in U$, there exist an open neighborhood V of r_0 and a plot $Q : V \to G$ such that $P \restriction V = \hat{x} \circ Q$, that is, $P(r) = \rho(Q(r))(x)$ for all

$r \in \mathrm{V}$. Since ρ is smooth, $\bar{\mathrm{Q}} = \rho \circ \mathrm{Q}$ is a plot of $\mathrm{Diff}(\mathrm{X}, \omega)$, and $\mathrm{P} \restriction \mathrm{V} = \hat{x} \circ \bar{\mathrm{Q}}$. Since, $\hat{x} : \mathrm{Diff}(\mathrm{X}, \omega) \to \mathrm{X}$ is surjective, it is a subduction and X is a homogeneous space of $\mathrm{Diff}(\mathrm{X}, \omega)$. Now, let us remark that, since the moment map is unique up to a constant, if a moment map μ is a covering onto its image $\mathscr{O}$ equipped with the quotient diffeology of G, then every other moment map $\mu' = \mu + \mathrm{cst}$ is a covering onto its image $\mathscr{O}' = \mathscr{O} + \mathrm{cst}$. Then, let x_0 be a point of X, and let $\mu(x) = \psi(x_0, x)$, where ψ is the 2-points moment map. Let $\mu_\omega = \psi_\omega(x_0, x)$. According to (art. 9.14), $\mu = \rho^*_{\Gamma_\omega} \circ \mu_\omega$. Let $\mathscr{O} = \mu(\mathrm{X}) = \rho^*_{\Gamma_\omega}(\mu_\omega(\mathrm{X})) = \rho^*_{\Gamma_\omega}(\mathscr{O}_\omega)$, with $\mathscr{O}_\omega = \mu_\omega(\mathrm{X})$. Let $m_\omega \in \mathscr{O}_\omega$ and $m = \rho^*_{\Gamma_\omega}(m_\omega)$. If $\mu_\omega(x) = m_\omega$, then $\rho^*_{\Gamma_\omega}(\mu_\omega(x)) = \rho^*_{\Gamma_\omega}(m_\omega)$, that is, $\mu(x) = m$. Thus, $\mu_\omega^{-1}(m_\omega) \subset \mu^{-1}(m)$. Therefore, if $\mu^{-1}(m)$ is discrete, then $\mu_\omega^{-1}(m_\omega)$ is discrete, *a fortiori*, and if μ is injective, then μ_ω is injective. ☐

Exercise

✎ EXERCISE 144 (Compact supported real functions III). Let us consider the space (X, ω) as defined in Exercise 142, p. 368. Show that (X, ω) is a homogeneous symplectic space.

About Symplectic Manifolds

The case of classical symplectic manifolds (M, ω) *deserves special care. We shall see in this section that, in this case, any universal moment map* μ_ω *is injective and therefore identifies* M *with a coadjoint orbit of* $\mathrm{Diff}(\mathrm{M}, \omega)$*, in the meaning of* (art. 7.16).

9.20. Value of the moment maps for manifolds. Let M be a connected manifold equipped with a closed 2-form ω. In this context, the paths moment map Ψ_ω takes a special expression. Let p be a path in M and $\mathrm{F} : \mathrm{U} \to \mathrm{Diff}(\mathrm{M}, \omega)$ be an n-plot, then

$$\Psi_\omega(p)(\mathrm{F})_r(\delta r) = \int_0^1 \omega_{p(t)}(\dot{p}(t), \delta p(t))\, dt\,, \qquad (\diamondsuit)$$

for all $r \in \mathrm{U}$ and $\delta r \in \mathbf{R}^n$, where δp is the lift in the tangent space TM of the path p, defined by

$$\delta p(t) = [\mathrm{D}(\mathrm{F}(r))(p(t))]^{-1} \frac{\partial \mathrm{F}(r)(p(t))}{\partial r}(\delta r). \qquad (\heartsuit)$$

PROOF. By definition, $\Psi(p)(\mathrm{F}) = \hat{p}^*(\mathscr{K}\omega)(\mathrm{F}) = \mathscr{K}\omega(\hat{p} \circ \mathrm{F})$. The expression of the operator $\mathscr{K}$ (art. 6.83), applied to the plot $\hat{p} \circ \mathrm{F} : r \mapsto \mathrm{F}(r) \circ p$ of Paths(X), gives

$$(\mathscr{K}\omega)(\hat{p} \circ \mathrm{F})_r(\delta r) = \int_0^1 \omega\left[\begin{pmatrix} s \\ u \end{pmatrix} \mapsto (\hat{p} \circ \mathrm{F})(u)(s+t)\right]_{\begin{pmatrix} s=0 \\ u=r \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \delta r \end{pmatrix} dt\,.$$

But $(\hat{p} \circ \mathrm{F})(u)(s+t) = \mathrm{F}(u)(p(s+t))$, let us denote temporarily by Φ_t the plot $(s, u) \mapsto \mathrm{F}(u)(p(s+t))$, so $\mathrm{F}(u)(p(s+t))$ writes $\Phi_t(s, u)$. Now, let us denote by $\mathscr{I}$

the integrand of the right term of this expression. We have

$$\begin{aligned}
\mathscr{I} &= \omega\left[\begin{pmatrix} s \\ u \end{pmatrix} \mapsto \Phi_t(s,u)\right]_{\binom{s=0}{u=r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \delta r \end{pmatrix} \\
&= \Phi_t^*(\omega)_{\binom{0}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \delta r \end{pmatrix} \\
&= \omega_{\Phi_t\binom{0}{r}}\left(\mathrm{D}(\Phi_t)_{\binom{0}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathrm{D}(\Phi_t)_{\binom{0}{r}} \begin{pmatrix} 0 \\ \delta r \end{pmatrix}\right) \\
&= \omega_{\mathrm{F}(r)(p(t))}\left(\frac{\partial}{\partial s}\Big\{\mathrm{F}(r)(p(s+t))\Big\}_{s=0}, \frac{\partial}{\partial r}\Big\{\mathrm{F}(r)(p(t))\Big\}(\delta r)\right).
\end{aligned}$$

But,

$$\begin{aligned}
\frac{\partial}{\partial s}\Big\{\mathrm{F}(r)(p(s+t))\Big\}_{s=0} &= \mathrm{D}(\mathrm{F}(r))(p(t))\left(\left.\frac{\partial p(s+t)}{\partial s}\right|_{s=0}\right) \\
&= \mathrm{D}(\mathrm{F}(r))(p(t))(\dot p(t)).
\end{aligned}$$

Then, using this last expression and the fact that F is a plot of $\mathrm{Diff}(\mathrm{M},\omega)$, that is, for all r in U, $\mathrm{F}(r)^*\omega = \omega$, we have

$$\begin{aligned}
\mathscr{I} &= \omega_{\mathrm{F}(r)(p(t))}\left(\mathrm{D}(\mathrm{F}(r))(p(t))(\dot p(t)), \frac{\partial \mathrm{F}(r)(p(t))}{\partial r}(\delta r)\right) \\
&= \omega_{p(t)}\left(\dot p(t), [\mathrm{D}(\mathrm{F}(r))(p(t))]^{-1} \frac{\partial \mathrm{F}(r)(p(t))}{\partial r}(\delta r)\right) \\
&= \omega_{p(t)}(\dot p(t), \delta p(t)).
\end{aligned}$$

Therefore, $\Psi_\omega(p)(\mathrm{F})_r(\delta r) = \mathscr{K}\omega(\hat p \circ \mathrm{F})_r(\delta r) = \displaystyle\int_0^1 \omega_{p(t)}(\dot p(t), \delta p(t))\, dt.$ ☐

9.21. The paths moment map for symplectic manifolds. Let M be a Hausdorff manifold, and let ω be a nondegenerate closed 2-form defined on M. Let m_0 and m_1 be two points of M connected by a path p. Let $f \in \mathscr{C}^\infty(\mathrm{M}, \mathbf{R})$ with compact support. Let F be the exponential of the symplectic gradient $\mathrm{grad}_\omega(f)$,[11] F is a 1-plot of $\mathrm{Diff}(\mathrm{M},\omega)$, and precisely a 1-parameter subgroup. Then, the universal paths moment map Ψ_ω, computed at the path p, evaluated on the 1-plot F, is the constant 1-form of $\mathbf{R}$

$$\Psi_\omega(p)(\mathrm{F}) = [f(m_1) - f(m_0)] \times dt \text{ with } \mathrm{F} : t \mapsto e^{t\,\mathrm{grad}_\omega(f)},$$

where dt is the standard 1-form of $\mathbf{R}$. Note that we are in the special case where F is actually a 1-parameter subgroup of $\mathrm{Ham}(\mathrm{M},\omega) \subset \mathrm{Diff}(\mathrm{M},\omega)$, and the function f is a *Hamiltonian* of F.

[11]Let us recall that the symplectic gradient is defined by $\omega(\mathrm{grad}_\omega(f), \cdot) = -df$.

PROOF. Let us remark that, in our case, the lift δp defined by (art. 9.20, (♡)) writes simply

$$\delta p(t) = [\mathrm{D}(e^{r\xi})(p(t))]^{-1} \frac{\partial e^{r\xi}(p(t))}{\partial r}(\delta r) = \xi(p(t)) \times \delta r,$$

with $\xi = \mathrm{grad}_\omega(f)$, and where r and δr are real numbers. Thus, the expression (art. 9.20, (◇)) becomes

$$\begin{aligned} \Psi_\omega(p)(\mathrm{F})_r(\delta r) &= \int_0^1 \omega_{p(t)}(\dot p(t), \xi(p(t)))\,dt \times \delta r \\ &= \int_0^1 \omega_{p(t)}(\dot p(t), \mathrm{grad}_\omega(f)(p(t)))\,dt \times \delta r \\ &= \int_0^1 df\left(\frac{dp(t)}{dt}\right) dt \times \delta r \\ &= [f(p(1)) - f(p(0))] \times \delta r, \end{aligned}$$

that is, $\Psi_\omega(p)(\mathrm{F}) = [f(m_1) - f(m_0)] \times dt$. □

9.22. Hamiltonian diffeomorphisms of symplectic manifolds. Let (M, ω) be a connected Hausdorff symplectic manifold. According to Banayaga [Ban78], a diffeomorphism f is said to be *Hamiltonian* if it can be connected to the identity $\mathbf{1}_\mathrm{M}$ by a smooth path $t \mapsto f_t$ in $\mathrm{Diff}(\mathrm{M}, \omega)$ such that

$$\omega(\dot f_t, \cdot) = d\phi_t \text{ with } \dot f_t(x) = \frac{d}{ds}\left\{f_s \circ f_t^{-1}(x)\right\}_{s=t},$$

where $(t, x) \mapsto \phi_t(x)$ is a smooth real function. If f is Hamiltonian according to this definition, then it belongs to $\mathrm{Ham}(\mathrm{M}, \omega)$, as defined in (art. 9.15). Conversely, any element f of $\mathrm{Ham}(\mathrm{M}, \omega)$ satisfies the above Banyaga's condition. Thus, the definition of Hamiltonian diffeomorphisms given in (art. 9.15) is a faithful generalization of the usual definition for symplectic manifolds.

PROOF. This proposition is a direct consequence of the general statement given in (art. 9.16) and the following comparison between the above 1-parameter family of vector fields $\dot f_t$ and the family F_t of the (art. 9.16). Since $f_{t'} \circ f_t^{-1} = f_t \circ (f_t^{-1} \circ f_{t'}) \circ f_t^{-1}$, the vector fields $\dot f_t$ and F_t are conjugated by f_t, precisely

$$\dot f_t = (f_t)_*(\mathrm{F}_t), \text{ or } \dot f_t(x) = \mathrm{D}(f_t)(f_t^{-1}(x))(\mathrm{F}_t(f_t^{-1}(x))).$$

This implies in particular that if the vector field $\dot f_t$ satisfies Banyaga's condition for the function ϕ_t, then the vector field F_t satisfies Banyaga's condition for the function $\Phi_t = -\phi_t \circ f_t$, and conversely, that is,

$$\omega(\dot f_t, \cdot) = d\phi_t, \text{ if and only if } \omega(\mathrm{F}_t, \cdot) = -d\Phi_t \text{ with } \Phi_t = -\phi_t \circ f_t.$$

Let us check it. Let $x \in \mathrm{M}$, $x' = f_t(x)$, $\delta x \in \mathrm{T}_x\mathrm{M}$, and $\delta x' = \mathrm{D}(f_t)(x)(\delta x)$, then

$$\begin{aligned}
\omega_{x'}(\dot{f}_t(x'), \delta x') &= [d\phi_t]_{x'}(\delta x') \\
\omega_{f_t(x)}(\dot{f}_t(f_t(x)), \mathrm{D}(f_t)(x)(\delta x)) &= [d\phi_t]_{f_t(x)}(\mathrm{D}(f_t)(x)(\delta x)) \\
\omega_{f_t(x)}(\mathrm{D}(f_t)(x)(\mathrm{F}_t(x)), \mathrm{D}(f_t)(x)(\delta x)) &= [f_t^*(d\phi_t)]_x(\delta x) \\
[f_t^*(\omega)]_x(\mathrm{F}_t(x), \delta x) &= d[f_t^*(\phi_t)]_x(\delta x) \\
\omega_x(\mathrm{F}_t(x), \delta x) &= d[\phi_t \circ f_t]_x(\delta x).
\end{aligned}$$

Thus, $\Phi_t = -\phi_t \circ f_t$. ☐

9.23. Symplectic manifolds are coadjoint orbits. Let M be a manifold, and let ω be a closed 2-form defined on M. We assume M connected and Hausdorff. Then:

THEOREM . *The form ω is nondegenerate, thus symplectic, if and only if:*

A. *The manifold* M *is homogeneous under* $\mathrm{Diff}(\mathrm{M}, \omega)$.
B. *The universal moment map* $\mu_\omega : \mathrm{M} \to \mathscr{G}_\omega^*/\Gamma_\omega$ *is injective.*

In that case, the image $\mathscr{O}_\omega = \mu_\omega(\mathrm{M}) \in \mathscr{G}_\omega^*/\Gamma_\omega$ of the universal moment map[12] (art. 9.14) is a $(\Gamma_\omega, \theta_\omega)$-coadjoint orbit of $\mathrm{Diff}(\mathrm{M}, \omega)$ (art. 7.16), and μ_ω identifies M with $\mathscr{O}_\omega$, where $\mathscr{O}_\omega$ is equipped with the quotient diffeology of $\mathrm{Diff}(\mathrm{M}, \omega)$. In other words, *every symplectic manifold is a coadjoint orbit*.

NOTE 1. Let $\mathrm{Ham}(\mathrm{M}, \omega)$ be the group of Hamiltonian diffeomorphisms, and let $\mathscr{H}_\omega^*$ be the space of its momenta. Let $\mu_\omega^\star : \mathrm{M} \to \mathscr{H}_\omega^*$ be the moment map associated with the action of $\mathrm{Ham}(\mathrm{M}, \omega)$, and let $\theta_\omega^\star$ be the associated Souriau cocycle. Then, $\mu_\omega^\star$ is also injective, and identifies M to a $\theta_\omega^\star$-coadjoint orbit $\mathscr{O}_\omega^\star \subset \mathscr{H}_\omega^*$ of $\mathrm{Ham}(\mathrm{M}, \omega)$.

NOTE 2. Let us consider the example $\mathrm{M} = \mathbf{R}^2$ and $\omega = (x^2 + y^2)\, dx \wedge dy$. Since $\mathbf{R}^2$ is contractible, the holonomy Γ_ω is trivial (art. 9.7). Next, ω is nondegenerate on $\mathbf{R}^2 - \{0\}$, but degenerates at the point $(0, 0)$. Thus, $(0, 0)$ is an orbit of the group $\mathrm{Diff}(\mathbf{R}^2, \omega)$, and actually $\mathbf{R}^2 - \{0\}$ is the other orbit. Hence, the universal moment map μ_ω such that $\mu_\omega(0, 0) = 0_{\mathscr{G}_\omega^*}$ is equivariant (art. 9.10, Note 2). Moreover, μ_ω is injective. The closed 2-form ω not being symplectic, with an injective universal moment map, shows that the hypothesis of transitivity of $\mathrm{Diff}(\mathrm{M}, \omega)$ on M is not superfluous is this proposition.

NOTE 3. Every symplectic manifold is a coadjoint orbit of its group of automorphisms, or Hamiltonian diffeomorphisms, maybe affine when Souriau's cocycle θ_ω is not trivial; see also [PIZ16]. This theorem has been improved,

[12]The universal moment maps are defined up to a constant, but if one is injective, then they are all injectives.

THEOREM . *Every symplectic manifold is a (linear) coadjoint orbit of the group of automorphisms of its integration bundle.*

Here, the term linear refers to the orbits of the linear coadjoint action of $\mathrm{Aut}(Y, \lambda)$, where $\pi : Y \to M$ is the (T_ω-principal) integration bundle over M, with T_ω being the torus of periods of ω, and λ is the connection form of curvature ω [Igl95]. The details can be found in [DIZ22]. The situation is sumarized by the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\ \mu_Y\ } & \mathscr{A}^* \\ {\scriptstyle \pi}\big\downarrow & \overset{\bar\mu_M}{\nearrow} & \big\uparrow{\scriptstyle \eta^*} \\ M & \xrightarrow[\ \mu_M\ ]{} & \mathscr{H}^* \end{array}$$

where $\mathscr{A}^*$ denotes the space of momenta of $\mathrm{Aut}(Y, \lambda)$ whose identity component projects onto $\mathrm{Ham}(M, \omega)$, and $\mathscr{H}^*$ is the space of momenta of $\mathrm{Ham}(M, \omega)$. Then, μ_Y is the moment map for $\mathrm{Aut}(Y, \lambda)$ relative to the closed 2-form $d\lambda$, μ_M is the moment map for $\mathrm{Ham}(M, \omega)$ and $\bar\mu_M$ is the projection of μ_Y on M — because μ_Y is invariant by T_ω — and η^* is the transpose of the projection $\eta : \mathrm{Aut}(Y, \lambda)^\circ \to \mathrm{Ham}(M, \omega)$.

PROOF. Let us assume first that ω is nondegenerate, that is, symplectic. Then, the group $\mathrm{Diff}(M, \omega)$ is transitive on M [Boo69]. Moreover, for every $m \in M$, the orbit map $\hat m : \varphi \mapsto \varphi(m)$ is a subduction [Don84]. Thus, the image of the moment map μ_ω is one orbit $\mathscr{O}_\omega$ of the affine coadjoint action of G_ω on $\mathscr{G}^*_\omega/\Gamma_\omega$, associated with the cocycle θ_ω. Hence, the orbit $\mathscr{O}_\omega$ being equipped with the quotient diffeology of G_ω, the moment map μ_ω is a subduction.

Now, let m_0 and m_1 be two points of M such that $\mu_\omega(m_0) = \mu_\omega(m_1)$, that is, $\psi_\omega(m_0, m_1) = \mu_\omega(m_1) - \mu_\omega(m_0) = 0$. Let $p \in \mathrm{Paths}(M)$ such that $p(0) = m_0$ and $p(1) = m_1$. Thus, $\psi_\omega(m_0, m_1) = 0$ is equivalent to $\Psi_\omega(p) = \Psi_\omega(\ell)$, where ℓ is some loop in M, we can choose $\ell(0) = \ell(1) = m_0$. Now, let us assume that $m_0 \neq m_1$. Since M is Hausdorff there exists a smooth real function $f \in \mathscr{C}^\infty(M, \mathbf{R})$, with compact support, such that $f(m_0) = 0$ and $f(m_1) = 1$. Let us denote by ξ the symplectic gradient field associated with f and by F the exponential of ξ. Thanks to (art. 9.21), on the one hand we have $\Psi(p)(F) = [f(m_1) - f(m_0)]\,dt = dt$, and on the other hand $\Psi_\omega(\ell)(F) = [f(m_0) - f(m_0)]\,dt = 0$. But $dt \neq 0$, thus $\psi_\omega(m_0, m_1) \neq 0$, and the moment map μ_ω is injective. Therefore, μ_ω is an injective subduction on $\mathscr{O}_\omega$, that is, a diffeomorphism.

Conversely, let us assume that M is a homogeneous space of $\mathrm{Diff}(M, \omega)$ and that μ_ω is injective. Let us notice first that since $\mathrm{Diff}(M, \omega)$ is transitive, the rank of ω is constant. In other words, $\dim(\ker(\omega)) = \mathrm{cst}$. Now, let us assume ω is degenerate, that is, $\dim(\ker(\omega)) \geq 1$. Since $m \mapsto \ker(\omega_m)$ is a smooth foliation, for every point

m of M there exists a smooth path p in M such that $p(0) = m$ and for t belonging to a small interval around $0 \in \mathbf{R}$, $\dot{p}(t) \neq 0$ and $\dot{p}(t) \in \ker(\omega_{p(t)})$ for all t in this interval. Then, we can reparametrize the path p and assume now that p is defined on the whole $\mathbf{R}$ and satisfies $p(0) = m$, $p(1) = m'$ with $m \neq m'$, and $\dot{p}(t) \in \ker(\omega_{p(t)})$ for all t. Next, since $\dot{p}(t) \in \ker(\omega_{p(t)})$ for all t, using the expression (art. 9.20, ($\diamondsuit$)), we get $\Psi_\omega(p) = 0_{\mathscr{G}^*_\omega}$, that is, $\mu_\omega(m) = \mu_\omega(m')$. But since $m \neq m'$ and we assumed μ_ω injective, this is a contradiction. Thus, the kernel of ω is reduced to $\{0\}$, and ω is nondegenerate, that is, symplectic.

Let us prove Note 1. According to a Boothby's theorem, the group $\mathrm{Ham}(\mathrm{M}, \omega)$ acts transitively on M [Boo69]. With respect to this group, and by construction, the holonomy is trivial: the associated paths moment map $\Psi^\star_\omega$ and the moment maps $\mu^\star_\omega$ take their values in $\mathscr{H}^*_\omega$. Let $j : \mathrm{Ham}(\mathrm{M}, \omega) \to \mathrm{Diff}(\mathrm{M}, \omega)$ be the inclusion, thus the universal holonomy Γ_ω is in the kernel of j^*, and we get a natural mapping $j^*_{\Gamma_\omega} : \mathscr{G}^*_\omega/\Gamma_\omega \to \mathscr{H}^*_\omega$. Now, the paths moment maps satisfy $\Psi^\star_\omega = j^*_{\Gamma_\omega} \circ \Psi_\omega$, and $\mu^\star_\omega = j^*_{\Gamma_\omega} \circ \mu_\omega$ (art. 9.14). Then, since (art. 9.21) involves only plots of $\mathrm{Ham}(\mathrm{X}, \omega)$, the proof above applies *mutatis mutandis* to the Hamiltonian case, and we deduce that the moment maps $\mu^\star_\omega$ are injective. By transitivity, they identify M with a $\theta^\star_\omega$-coadjoint orbit of $\mathrm{Ham}(\mathrm{M}, \omega)$.

Let us finish by proving the second note, that is, the universal moment map μ_ω of $\omega = (x^2 + y^2)\, dx \wedge dy$ is injective. First of all $\mu_\omega(0,0) = 0_{\mathscr{G}^*}$. Now, if $z = (x, y)$ and $z' = (x', y')$ are two different points of $\mathbf{R}^2$ and different from $(0,0)$, then there is a smooth function with compact support contained in a small ball, not containing $(0,0)$ nor z, such that $f(z') = 1$. Then, the 1-parameter group generated by $\mathrm{grad}_\omega(f)$ belongs to $\mathrm{Diff}(\mathbf{R}^2, \omega)$, and a similar argument as the one of the proof above shows that $\mu_\omega(z) \neq \mu_\omega(z')$. We still need to prove that if $z \neq (0,0)$, then $\mu_\omega(z) \neq 0_{\mathscr{G}^*}$. Let us consider $p(t) = tz$ and $\mathrm{F}(r)$ be the positive rotation of angle $2\pi r$, where $r \in \mathbf{R}$. The application of (art. 9.20, ($\diamondsuit$)), computed at the point $r = 0$ and applied to the vector $\delta r = 1$, gives $(\pi/2)(x^2 + y^2)^2$ which is not zero. Therefore, the moment map μ_ω is injective. ☐

9.24. The classical homogeneous case. Let (M, ω) be a symplectic manifold. Let G be a Lie group together with a homogeneous Hamiltonian action on (M, ω), that is, the holonomy Γ of G is trivial. For the sake of simplicity we assume M connected and G a Lie subgroup of $\mathrm{Diff}(\mathrm{M}, \omega)$. By functoriality of the moment maps (art. 9.12), we know that if a moment map μ of G is injective, then every universal moment map μ_ω is injective (art. 9.23). But we are now in the opposite case, since ω is symplectic every universal moment map μ_ω is injective, but what about μ? This is actually the original case treated by Souriau in [Sou70]. He showed that the moment map μ is a covering onto its image, which is some coadjoint orbit $\mathscr{O} \subset \mathscr{G}^*$

(affine or not) of G. We give here the proof of this theorem, according to the present framework. This case is illustrated by Exercise 146, p. 394.

PROOF. Let p be a path in M such that $\mu \circ p = \mathrm{cst}$, that is, $\Psi(p) = 0_{\mathscr{G}^*}$, where Ψ is the paths moment map of G. Then, for any 1-parameter subgroup $\mathrm{F} \in \mathrm{Hom}^\infty(\mathbf{R}, \mathrm{G})$, $\Psi(p)(\mathrm{F})_r(\delta r) = 0$, for all r and all δr belonging to $\mathbf{R}$. Adapting to our case the expression of Ψ given in (art. 9.20), we get

$$\int_0^1 \omega_{p(t)}(\dot{p}(t), \mathrm{Z}(p(t)))\,dt = 0, \text{ where } \mathrm{Z}(m) = \left.\frac{\partial \mathrm{F}(t)(m)}{\partial t}\right|_{t=0}.$$

Now, considering the 1-parameter family of paths $p_s : t \mapsto p(st)$, the derivative of the above expression gives $\omega_{p(0)}(\dot{p}(0), \mathrm{Z}(p(0))) = 0$. But since G is transitive, by running over all the 1-parameter subgroups F of G we describe the whole tangent space $\mathrm{T}_m\mathrm{M}$, where $m = p(0)$. And since ω is nondegenerate, $\dot{p}(0) = 0$. The path p is thus constant, $p(t) = m$ for all t in $\mathbf{R}$. Therefore the preimages of the values of the moment map μ are discrete. But, μ is a fibration (see (art. 9.18)), thus μ is a covering (art. 8.24) onto its image which is, by transitivity, a coadjoint orbit. ☐

9.25. The Souriau-Nœther theorem. Let M be a manifold, and let ω be a closed 2-form on M. We say that two points m and m' are on the same *characteristic*[13] of ω if there exists a path p connecting m to m' such that $\dot{p}(t) \in \ker(\omega_{p(t)})$ for all t. Then, the universal moment map μ_ω is constant on the characteristics of ω.

NOTE 1. In particular, for any smooth action of a diffeological group G, preserving ω, the moment map μ is constant on the characteristics of ω.

NOTE 2. This is an analogue to the first Nœther theorem, relating symmetries to conserved quantities for Lagrangian systems. For a comprehensive presentation on the subject, see the book of Y. Kosmann-Schwarzbach [YKS10].

PROOF. We have $\mu_\omega(m') - \mu_\omega(m) = \psi_\omega(m, m') = \mathrm{class}(\Psi_\omega(p)) \in \mathscr{G}_\omega^*/\Gamma_\omega$, where p is any (smooth) path connecting m to m'. Then, thanks to the hypothesis, we can use a path p contained in the characteristic of ω containing m and m', that is, $\dot{p}(t) \in \ker(\omega_{p(t)})$ for all t. Now, thanks to the explicit formula of (art. 9.20), for every n-plot F of $\mathrm{Diff}(\mathrm{M}, \omega)$, $n \in \mathbf{N}$, for every $r \in \mathrm{def}(\mathrm{F})$, for every $\delta r \in \mathbf{R}^n$,

$$\Psi_\omega(p)(\mathrm{F})_r(\delta r) = \int_0^1 \omega_{p(t)}(\dot{p}(t), \delta p(t))\,dt = \int_0^1 0 \times dt = 0.$$

Thus, $\Psi_\omega(p) = 0$ and therefore $\mu_\omega(m') = \mu_\omega(m)$. The note is a consequence of the functoriality of the moment map (art. 9.14). Indeed, let ρ be the morphism from G

[13] This is the classical definition of the characteristics in the case of a closed 2-form ω defined on a manifold M. They are the integral submanifolds of the distribution $m \mapsto \ker(\omega_m)$.

to Diff(M, ω), then the paths moment map Ψ, relative to G, satisfies $\Psi = \rho^* \circ \Psi_\omega$. Thus, $\Psi_\omega(p) = 0$ implies $\Psi(p) = 0$ which implies $\mu(m) = \mu(m')$, for any moment map μ relative to G. □

9.26. Presymplectic homogeneous manifolds. Let M be a connected Hausdorff manifold, and let ω be a closed 2-form on M. Let $G \subset \mathrm{Diff}(M, \omega)$ be a connected subgroup. If M is a homogeneous space of G, then the characteristics of ω are the connected components of the preimages of the moment maps μ.

NOTE. In particular, if M is a homogeneous space of Diff(M, ω), and thus of its identity component (art. 7.9), then:

THEOREM . *The characteristics of* ω *are the connected components of the preimages of the values of the universal moment map* μ_ω.

This justifies *a posteriori* the definition of the characteristics of moment maps, for the general case of homogeneous diffeological spaces, in (art. 9.17).

PROOF. The Souriau-Nœther theorem states that if m and m' are on the same characteristic, then $\mu(m) = \mu(m')$ (art. 9.25). We shall prove the converse in a few steps.

(a) Let us consider first the case when the holonomy Γ is trivial, $\Gamma = \{0\}$. Let us assume m and m' connected by a path p such that $\mu(p(t)) = \mu(m)$ for all t. Then, let $s \mapsto p_s$ be defined by $p_s(t) = p(st)$, for all s and t. We have $\mu(p_s(1)) = \mu(p_s(0))$, that is, $\Psi(p_s) = 0_{\mathscr{G}^*}$, for all s. Thus, for all n-plots F of G, for all $r \in \mathrm{def}(F)$ and all $\delta r \in \mathbf{R}^n$, $\Psi(p_s)(F)_r(\delta r) = 0$, and hence

$$0 = \frac{\partial \Psi(p_s)(F)_r(\delta r)}{\partial s} = \frac{\partial}{\partial s}\int_0^1 \omega_{p_s(t)}(\dot p_s(t), \delta p_s(t))\, dt = \omega_{p(s)}(\dot p(s), \delta p(s)),$$

where $\delta p(t)$ is given by (art. 9.20, (♡)). Next, let $v \in T_{p(t)}(M)$, then there exists a path c of M such that $c(0) = p(t)$ and $dc(s)/ds|_{s=0} = v$. Since M is assumed homogeneous under the action of G, there exists a 1-plot $s \mapsto F(s)$ centered at the identity, that is, $F(0) = \mathbf{1}_M$, such that $F(s)(p(t)) = c(s)$. Then, for $s = 0$ and $\delta s = 1$, we get from (art. 9.20, (♡)),

$$\delta p(t) = \mathbf{1}_{T_{p(t)}M} \left.\frac{dF(s)(p(t))}{ds}\right|_{s=0} = \left.\frac{dc(s)}{ds}\right|_{s=0} = v.$$

Hence, for every $v \in T_{p(t)}M$, $\omega(\dot p(t), v) = 0$, *i.e.*, $\dot p(t) \in \ker(\omega_{p(t)})$ for all t. Therefore, the connected components of the preimages of the values of the moment map μ are the characteristics of ω.

(b) Let us consider the general case. Let $\widetilde{M}$ be the universal covering of M, $\pi : \widetilde{M} \to M$ the projection, and let $\tilde{\omega} = \pi^*(\omega)$. Let $\widehat{G}$ be the group defined by

$$\widehat{G} = \{\hat{g} \in \mathrm{Diff}(\widetilde{M}, \tilde{\omega}) \mid \exists g \in G \text{ and } \pi \circ \hat{g} = g \circ \pi\}.$$

Let $\rho : \widehat{G} \to G$ be the morphism $\hat{g} \mapsto g$. The group $\widehat{G}$ is an extension of G by the homotopy group $\pi_1(M)$

$$\mathbf{1} \longrightarrow \pi_1(M) \longrightarrow \widehat{G} \xrightarrow{\ \rho\ } G \longrightarrow \mathbf{1}.$$

1. The morphism ρ is surjective. Let $g \in G$, let $t \mapsto g_t$ be a smooth path in G connecting $\mathbf{1}_G$ to g. Let $\tilde{m} \in \widetilde{M}$ and $m = \pi(\tilde{m})$, the path $t \mapsto g_t(m)$ has a unique lift $t \mapsto \tilde{m}_t$ in $\widetilde{M}$ starting at $\tilde{m}$ (art. 8.25). We can check that, $\tilde{g} : \tilde{m} \mapsto \tilde{m}_1$ is a diffeomorphism of $\widetilde{M}$, satisfying by construction $\pi \circ \tilde{g} = g \circ \pi$. Next, $\tilde{g}^*(\tilde{\omega}) = \tilde{g}^*(\pi^*(\omega)) = \tilde{g}^* \circ \pi^*(\omega) = (\pi \circ \tilde{g})^*(\omega) = (g \circ \pi)^*(\omega) = \pi^*(g^*(\omega)) = \pi^*(\omega) = \tilde{\omega}$. Thus, $\tilde{g} \in \widehat{G}$.

2. The group $\widehat{G}$ is transitive on $\widetilde{M}$. Let $\tilde{m}$ and $\tilde{m}'$ be two points of $\widetilde{M}$, let $m = \pi(\tilde{m})$ and $m' = \pi(\tilde{m}')$. Since G is transitive on M there exists $g \in G$ such that $g(m) = m'$. The lift $\tilde{g}$ defined in part 1 maps $\tilde{m}$ to $\tilde{m}_1$, and we have $\pi(\tilde{m}_1) = \pi(\tilde{m}') = m'$. So there exists an element $k \in \pi_1(M)$ such that $k_{\widetilde{M}}(\tilde{m}_1) = \tilde{m}'$ (art. 8.26). Let $\hat{g} = k_{\widetilde{M}} \circ \tilde{g}$, since $\pi \circ \hat{g} = g \circ \pi$ and $\hat{g}^*(\tilde{\omega}) = (k_{\widetilde{M}} \circ \tilde{g})^*(\tilde{\omega}) = \tilde{g}^*(k^*_{\widetilde{M}}(\tilde{\omega})) = \tilde{g}^*(\tilde{\omega}) = \tilde{\omega}$, $\hat{g}$ belongs to $\widehat{G}$, and maps $\tilde{m}$ to $\tilde{m}'$.

3. The kernel of ρ is reduced to $\pi_1(M)$. Let $\tilde{g} \in \widehat{G}$ such that $\rho(\tilde{g}) = \mathbf{1}_M$, that is, $\pi \circ \tilde{g} = \pi$. Thus, for every $\tilde{m} \in M$ there exists $\varkappa(\tilde{m}) \in \pi_1(M)$ such that $\tilde{g}(\tilde{m}) = \varkappa(\tilde{m})_M(\tilde{m})$. But the map $\varkappa : \widetilde{M} \to \pi_1(M)$ is smooth, and $\pi_1(M)$ discrete, so $\varkappa$ is constant. Therefore, there exists $k \in \pi_1(M)$ such that $\tilde{g}(\tilde{m}) = k_M(\tilde{m})$, for all $\tilde{m} \in \widetilde{M}$.

4. Since $\widetilde{M}$ is simply connected, $\widehat{G}$ has no holonomy. Let $\hat{\mu}$ be a moment map of the action of $\widehat{G}$ on $\widetilde{M}$. Since $\widehat{G}$ is a discrete extension of G, their space of momenta coincide (art. 7.13), thus $\hat{\mu}$ takes its values in $\mathscr{G}^*$, and since the action of G on M is the image by the morphism ρ of the action of $\widehat{G}$ on $\widetilde{M}$, for every $\tilde{m} \in \widetilde{M}$, $\mu(\pi(\tilde{m})) = \mathrm{class}(\hat{\mu}(\tilde{m})) \in \mathscr{G}^*/\Gamma$ (art. 9.13). Next, let $c = \mu(m) = \mathrm{class}(\hat{\mu}(\tilde{m})) \in \mathscr{G}^*/\Gamma$, $m = \pi(\tilde{m})$, and $C = \mu^{-1}(c)$. The preimage $\pi^{-1}(C) = \pi^{-1}(\mu^{-1}(c))$ is equal to $(\mu \circ \pi)^{-1}(c) = (\mathrm{class} \circ \hat{\mu})^{-1}(c) = \hat{\mu}^{-1}(\mathrm{class}^{-1}(c)) = \hat{\mu}^{-1}(\hat{\mu}(\tilde{m}) + \Gamma)$. Thus,

$$\mu^{-1}(c) = \pi\Big(\bigcup_{\gamma \in \Gamma} \hat{\mu}^{-1}(\alpha + \gamma)\Big) \text{ with } \alpha = \hat{\mu}(\tilde{m}) \text{ and } c = \mathrm{class}(\alpha).$$

Since $\widehat{G}$ is transitive on $\widetilde{M}$ and since there is no holonomy, we can apply the result of part (a): for every $\gamma \in \Gamma$, $\hat{\mu}^{-1}(\alpha + \gamma)$ is a union of characteristics of $\tilde{\omega}$. Thus, the union over all the $\gamma \in \Gamma$ is still a union of characteristics of ω. Hence, $\mu^{-1}(c)$ is the π-projection of a union of characteristics of $\tilde{\omega}$. But, since $\pi : \widetilde{M} \to M$ is a covering and since $\tilde{\omega} = \pi^*(\omega)$, the π-projection of a characteristic of $\tilde{\omega}$ is a characteristic of

ω. Thus $\mu^{-1}(c)$ is a union of characteristics of ω, and the connected components of the preimages of the values of μ are the characteristics of ω. ☐

Exercises

✎ EXERCISE 145 (The classical moment map). Let M be a connected Hausdorff manifold equipped with a closed 2-form ω. Let G be a Lie group: a diffeological group which is a manifold. Let $\rho : g \mapsto g_M$ be a Hamiltonian action of G on M. Let us recall that a 1-parameter subgroup $F \in \mathrm{Hom}^\infty(\mathbf{R}, G)$ is uniquely defined by its derivative at the identity:

$$Z = \left.\frac{dF(t)}{dt}\right|_{t=0} \text{ and } Z \in \mathscr{G} = T_{1_G}(G).$$

For Lie groups, the spaces $\mathrm{Hom}^\infty(\mathbf{R}, G)$, $T_{1_G}(G)$ and the space of invariant vector fields on G are identified and called the *Lie algebra* of G.

We denote $F(t) = e^{tZ}$. The *fundamental vector field* Z_M is defined on M by

$$Z_M(m) = \left.\frac{\partial e_M^{tZ}(m)}{\partial t}\right|_{t=0} \in T_m(M).$$

Let μ be a moment map of the action of G on M, we shall denote

$$\mu_Z(m) = \mu(m)(t \mapsto e^{tZ})_0(1).$$

1) Show that the moment map μ is defined, up to a constant, by the differential equation

$$i_{Z_M}(\omega) = -d\mu_Z, \text{ that is, } \omega_m(Z_M(m), \delta m) = -d[\mu_Z]_m(\delta m),$$

for all $m \in M$ and all $\delta m \in T_m(M)$.

2. Show then, using a basis of $\mathscr{G}$, that $Z \mapsto \mu_Z$ is linear.

✎ EXERCISE 146 (The cylinder and SL(2, R)). We consider the space $\mathbf{R}^2$, equipped with the standard symplectic form $\omega = dx \wedge dy$, with $X = (x, y) \in \mathbf{R}^2$. Check that the special linear group $SL(2, \mathbf{R})$ preserves ω, and that its action on $\mathbf{R}^2$ has two orbits, the origin $0 \in \mathbf{R}^2$ and the *cylinder* $M = \mathbf{R}^2 - \{0\}$.

1) Justify, without computation, that the action of $SL(2, \mathbf{R})$ on $\mathbf{R}^2$ is Hamiltonian and exact.

2) For every $X \in \mathbf{R}^2$, let $\gamma_X = [t \mapsto tX] \in \mathrm{Paths}(\mathbf{R}^2)$ connecting 0 to X. Use the general expression of the paths moment map given in (art. 9.20), for $p = \gamma_X$ and $F_\sigma = [s \mapsto e^{s\sigma}]$, with $\sigma \in \mathfrak{sl}(2, \mathbf{R})$ — the Lie algebra of $SL(2, \mathbf{R})$, that is, the space of 2×2 traceless matrices — to show that

$$\mu(X)(F_\sigma) = \frac{1}{2}\omega(X, \sigma X) \times dt.$$

3) Deduce that the moment map $\mu_{\mathrm{M}} = \mu \restriction \mathrm{M}$ of $\mathrm{SL}(2, \mathbf{R})$ on M is a nontrivial double sheet covering onto its image.

Examples of Moment Maps in Diffeology

The few following examples want to illustrate how the theory of moment maps in diffeology can be applied to the field of infinite dimensional situations, but also to the less familiar case of singular spaces. It is, at the same time, the opportunity to familiarize ourselves with the computational techniques in diffeology.

9.27. On the intersection 2-form of a surface, I. Let Σ be a closed surface oriented by a 2-form Surf, chosen once and for all. Let us consider $\Omega^1(\Sigma)$, the infinite dimensional vector space of 1-forms of Σ, equipped with functional diffeology. Let us consider the antisymmetric bilinear map defined on $\Omega^1(\Sigma)$ by

$$(\alpha, \beta) \mapsto \int_\Sigma \alpha \wedge \beta,$$

for all α, β in $\Omega^1(\Sigma)$. Since the wedge-product $\alpha \wedge \beta$ is a 2-form of Σ, there exists a real smooth function $\varphi \in \mathscr{C}^\infty(\Sigma, \mathbf{R})$ such that $\alpha \wedge \beta = \varphi \times \mathrm{Surf}$. Thus, by definition, $\int_\Sigma \alpha \wedge \beta = \int_\Sigma \varphi \times \mathrm{Surf}$.

1. A well defined differential 2-form ω of $\Omega^1(\mathrm{X})$ is naturally associated with the above bilinear form. For every n-plot $\mathrm{P} : \mathrm{U} \to \mathrm{X}$, for all $r \in \mathrm{U}$, δr and $\delta' r$ in $\mathbf{R}^n$,

$$\omega(\mathrm{P})_r(\delta r, \delta' r) = \int_\Sigma \frac{\partial \mathrm{P}(r)}{\partial r}(\delta r) \wedge \frac{\partial \mathrm{P}(r)}{\partial r}(\delta' r).$$

2. The 2-form ω is the differential of the 1-form λ defined on $\Omega^1(\Sigma)$ by

$$\lambda(\mathrm{P})_r(\delta r) = \frac{1}{2}\int_\Sigma \mathrm{P}(r) \wedge \frac{\partial \mathrm{P}(r)}{\partial r}(\delta r), \text{ and } \omega = d\lambda.$$

3. Consider now the additive group $(\mathscr{C}^\infty(\Sigma, \mathbf{R}), +)$ of smooth real functions on Σ, and let us define the following action of $\mathscr{C}^\infty(\Sigma, \mathbf{R})$ on $\Omega^1(\Sigma)$,

$$\text{for all } f \in \mathscr{C}^\infty(\Sigma, \mathbf{R}),\ f \mapsto \bar{f} = [\alpha \mapsto \alpha + df].$$

Then, the additive group $\mathscr{C}^\infty(\Sigma, \mathbf{R})$ acts by automorphisms on the pair $(\Omega^1(\Sigma), \omega)$,

$$\text{for all } f \text{ in } \mathscr{C}^\infty(\Sigma, \mathbf{R}),\ f^*(\omega) = \omega.$$

Note that the kernel of the action $f \mapsto \bar{f}$ is the subgroup of constant maps, and the image of $\mathscr{C}^\infty(\Sigma, \mathbf{R})$ is the group $\mathrm{B}^1_{\mathrm{dR}}(\Sigma)$ of exact 1-forms of Σ.

4. Let $p \in \mathrm{Paths}(\Omega^1(\Sigma))$ be a path connecting α_0 to α_1. The paths moment map $\Psi(p)$ is then given by

$$\Psi(p) = \left(\hat{\alpha}_1^*(\lambda) + d\left[f \mapsto \frac{1}{2}\int_\Sigma f \times d\alpha_1\right]\right) - \left(\hat{\alpha}_0^*(\lambda) + d\left[f \mapsto \frac{1}{2}\int_\Sigma f \times d\alpha_0\right]\right).$$

On this expression, we get immediately the 2-points moment map $\psi(\alpha_0, \alpha_1) = \Psi(p)$, for any path p connecting α_0 to α_1. Note that, since $\Omega^1(\Sigma)$ is contractible, the holonomy of the action of $\mathscr{C}^\infty(\Sigma, \mathbf{R})$ is trivial, $\Gamma = \{0\}$, and the action of $\mathscr{C}^\infty(\Sigma, \mathbf{R})$ is Hamiltonian.

5. The moment maps of this action of $\mathscr{C}^\infty(\Sigma, \mathbf{R})$ on $\Omega^1(\Sigma)$ are, up to a constant, equal to

$$\mu : \alpha \mapsto d\Big[f \mapsto \int_\Sigma f \times d\alpha\Big].$$

Moreover, the moment map μ is equivariant, that is, invariant, since the group $\mathscr{C}^\infty(\Sigma, \mathbf{R})$ is Abelian,

$$\text{for all } f \in \mathscr{C}^\infty(\Sigma, \mathbf{R}),\ \mu \circ \bar{f} = \mu.$$

In summary, the action of $\mathscr{C}^\infty(\Sigma, \mathbf{R})$ on $\Omega^1(\Sigma)$ is exact and Hamiltonian.

NOTE. The moment map $\mu(\alpha)$ is fully characterized by $d\alpha$. This is why we find in the mathematical literature on the subject that the moment map for this action is the exterior derivative (or curvature, depending on the authors) $\alpha \mapsto d\alpha$. But as we see again on this example, the diffeological framework gives to this statement a precise meaning. Let us also remark that the moment map μ is linear, for all real numbers t and s, and for all α and β in $\Omega^1(\Sigma)$, $\mu(t\,\alpha + s\,\beta) = t\,\mu(\alpha) + s\,\mu(\beta)$. The kernel of μ is the subspace of closed 1-forms,

$$\ker(\mu) = Z^1_{\rm dR}(\Sigma) = \big\{\alpha \in \Omega^1(\Sigma) \mid d\alpha = 0\big\}.$$

The orbit of the zero form $0 \in \Omega^1(\Sigma)$ by $\mathscr{C}^\infty(\Sigma, \mathbf{R})$ is just the subspace $B^1_{\rm dR}(\Sigma) \subset Z^1_{\rm dR}(\Sigma)$, see (art. 9.29, Note 3) for a discussion about that.

PROOF. 1. Let us check that ω defines a differential 1-form on $\Omega^1(\Sigma)$. Note that, for any $r \in U = \mathrm{def}(P)$, $P(r)$ is a section of the ordinary cotangent bundle $T^*\Sigma$, $P(r) = [x \mapsto P(r)(x)] \in \mathscr{C}^\infty(\Sigma, T^*\Sigma)$, where $P(r)(x) \in T^*_x(\Sigma)$. Thus,

$$\frac{\partial P(r)}{\partial r}(\delta r) = [x \mapsto \frac{\partial P(r)(x)}{\partial r}(\delta r)], \text{ and } \frac{\partial P(r)(x)}{\partial r}(\delta r) \in T^*_x(\Sigma),$$

where $\partial P(r)(x)/\partial r$ denotes the tangent linear map $D(r \mapsto P(r)(x))(r)$. The formula giving ω is then well defined. Now, $\omega(P)_r$ is clearly antisymmetric and depends smoothly on r. Hence, $\omega(P)$ is a smooth 2-form of U. Let us check that $P \mapsto \omega(P)$ defines a 2-form on $\Omega^1(\Sigma)$, that is, satisfies the compatibility condition $\omega(P \circ F) = F^*(\omega(P))$, for all $F \in \mathscr{C}^\infty(V, U)$, where V is a real domain. Let $s \in V$, δs and $\delta' s$ two tangent vectors to V at s, let $r = F(s)$, and compute

$$\omega(P \circ F)_s(\delta s, \delta' s) \quad = \quad \int_\Sigma \frac{\partial P \circ F(s)}{\partial s}(\delta s) \wedge \frac{\partial P \circ F(s)}{\partial s}(\delta' s)$$

$$\begin{aligned}
&= \int_\Sigma \frac{\partial P(r)}{\partial r}\frac{\partial F(s)}{\partial s}(\delta s) \wedge \frac{\partial P(r)}{\partial r}\frac{\partial F(s)}{\partial s}(\delta' s) \\
&= \omega(P)_{F(s)}(DF_s(\delta s), DF_s(\delta' s)) \\
&= F^*(\omega(P))_s(\delta s, \delta' s).
\end{aligned}$$

Thus, $\omega(P \circ F) = F^*(\omega(P))$, and ω is a well defined 2-form on $\Omega^1(\Sigma)$.

2. First of all, the proof that the map $P \mapsto \lambda(P)$ is a well defined differential 1-form of $\Omega^1(\Sigma)$ is analogous to the proof of the first item. Now, let us recall that $\omega = d\lambda$ means $d(\lambda(P)) = \omega(P)$, for all plots P of $\Omega^1(\Sigma)$. Let us apply the usual formula of differentiation of 1-forms on real domains,

$$d\varepsilon_r(\delta r, \delta' r) = \delta(\varepsilon_r(\delta' r)) - \delta'(\varepsilon_r(\delta r)),$$

where δ and δ' are two independent variations. For the sake of simplicity let us denote

$$\alpha = P(r),\ \delta\alpha = \frac{\partial P(r)}{\partial r}(\delta r),\ \delta'\alpha = \frac{\partial P(r)}{\partial r}(\delta' r).$$

Then,

$$\begin{aligned}
d(\lambda(P))_r(\delta r, \delta' r) &= \frac{1}{2}\left[\delta \int_\Sigma \alpha \wedge \delta'\alpha - \delta' \int_\Sigma \alpha \wedge \delta\alpha\right] \\
&= \frac{1}{2}\left[\int_\Sigma \delta\alpha \wedge \delta'\alpha + \alpha \wedge \delta\delta'\alpha - \int_\Sigma \delta'\alpha \wedge \delta\alpha + \alpha \wedge \delta'\delta\alpha\right],
\end{aligned}$$

but $\delta\delta'\alpha = \delta'\delta\alpha$, thus

$$\begin{aligned}
d(\lambda(P))_r(\delta r, \delta' r) &= \frac{1}{2}\left[\int_\Sigma \delta\alpha \wedge \delta'\alpha - \int_\Sigma \delta'\alpha \wedge \delta\alpha\right] \\
&= \frac{1}{2}\left[\int_\Sigma \delta\alpha \wedge \delta'\alpha + \int_\Sigma \delta\alpha \wedge \delta'\alpha\right] \\
&= \int_\Sigma \delta\alpha \wedge \delta'\alpha \\
&= \omega_r(\delta r, \delta' r).
\end{aligned}$$

3. Let us compute the pullback of λ by the action of $f \in \mathscr{C}^\infty(\Sigma, \mathbf{R})$. Let $P : U \to \Omega^1(\Sigma)$ be an n-plot, and let $r \in U$ and $\delta r \in \mathbf{R}^n$,

$$\begin{aligned}
\bar f^*(\lambda)(P)_r(\delta r) &= \lambda(\bar f \circ P)_r(\delta r) \\
&= \lambda(r \mapsto P(r) + df)_r(\delta r) \\
&= \frac{1}{2}\int_\Sigma (P(r) + df) \wedge \frac{\partial P(r)}{\partial r}(\delta r) \\
&= \frac{1}{2}\int_\Sigma P(r) \wedge \frac{\partial P(r)}{\partial r}(\delta r) + \frac{1}{2}\int_\Sigma df \wedge \frac{\partial P(r)}{\partial r}(\delta r)
\end{aligned}$$

$$\begin{aligned} &= \lambda(\mathrm{P})_r(\delta r) + \frac{\partial}{\partial r}\left\{\frac{1}{2}\int_\Sigma df \wedge \mathrm{P}(r)\right\}(\delta r) \\ &= \lambda(\mathrm{P})_r(\delta r) - \frac{\partial}{\partial r}\left\{\frac{1}{2}\int_\Sigma f \times d(\mathrm{P}(r))\right\}(\delta r). \end{aligned}$$

Next, we define, for every $f \in \mathscr{C}^\infty(\Sigma, \mathbf{R})$, $\varphi(f) : \Omega^1(\Sigma) \to \mathbf{R}$ by

$$\varphi(f) : \alpha \mapsto \frac{1}{2}\int_\Sigma f \times d\alpha.$$

Then,

$$d(\varphi(f))(\mathrm{P})_r(\delta r) = \frac{\partial}{\partial r}\left\{\frac{1}{2}\int_\Sigma f \times d(\mathrm{P}(r))\right\}(\delta r).$$

Thus,

$$\bar{f}^*(\lambda)(\mathrm{P})_r(\delta r) = \lambda(\mathrm{P})_r(\delta r) - (d\varphi(f))(\mathrm{P})_r(\delta r),$$

that is,

$$\bar{f}^*(\lambda) = \lambda - d(\varphi(f)).$$

Hence, $d[\bar{f}^*(\lambda)] = d\lambda$, and $\omega = d\lambda$ is invariant by the action of $\mathscr{C}^\infty(\Sigma, \mathbf{R})$.

4. Let p be a path in $\Omega^1(\Sigma)$ connecting α_0 to α_1. By definition $\Psi(p) = \hat{p}^*(\mathscr{K}\omega)$. Applying the property of the Chain-Homotopy operator $d \circ \mathscr{K} + \mathscr{K} \circ d = \hat{1}^* - \hat{0}^*$ to $\omega = d\lambda$, we get

$$\begin{aligned} \Psi(p) &= \hat{p}^*(\mathscr{K} d\lambda) \\ &= \hat{p}^*(\hat{1}^*(\lambda) - \hat{0}^*(\lambda) - d(\mathscr{K}\lambda)) \\ &= (\hat{1} \circ \hat{p})^*(\lambda) - (\hat{0} \circ \hat{p})^*(\lambda) - d[(\mathscr{K}\lambda) \circ \hat{p}] \\ &= \hat{\alpha}_1^*(\lambda) - \hat{\alpha}_0^*(\lambda) - d[f \mapsto \mathscr{K}\lambda(\hat{p}(f))]. \end{aligned}$$

But $\mathscr{K}\lambda(\hat{p}(f)) = \mathscr{K}\lambda(\bar{f} \circ p) = \int_{\bar{f}\circ p} \lambda = \int_p \bar{f}^*(\lambda)$, and since $\bar{f}^*(\lambda) = \lambda - d(\varphi(f))$, $\mathscr{K}\lambda(\hat{p}(f)) = \int_p \lambda - \int_p d(\varphi(f)) = \int_p \lambda - \varphi(f)(\alpha_1) + \varphi(f)(\alpha_0)$. Therefore,

$$\begin{aligned} \Psi(p) &= \hat{\alpha}_1^*(\lambda) - \hat{\alpha}_0^*(\lambda) - d[f \mapsto -\varphi(f)(\alpha_1) + \varphi(f)(\alpha_0)] \\ &= \hat{\alpha}_1^*(\lambda) - \hat{\alpha}_0^*(\lambda) + d\left[f \mapsto \frac{1}{2}\int_\Sigma f \times d\alpha_1 - \frac{1}{2}\int_\Sigma f \times d\alpha_0\right]. \end{aligned}$$

We get then the paths moment map Ψ,

$$\Psi(p) = \left(\hat{\alpha}_1^*(\lambda) + d\left[f \mapsto \frac{1}{2}\int_\Sigma f \times d\alpha_1\right]\right) - \left(\hat{\alpha}_0^*(\lambda) + d\left[f \mapsto \frac{1}{2}\int_\Sigma f \times d\alpha_0\right]\right).$$

Concerning the 2-points moment map ψ, we clearly have $\psi(\alpha_0, \alpha_1) = \Psi(p)$, for any path connecting α_0 to α_1.

5. The 1-point moment maps are given by $\mu(\alpha) = \psi(\alpha_0, \alpha)$ for any origin α_0. Let us choose $\alpha_0 = 0$. So,

$$\mu(\alpha) = \hat{\alpha}^*(\lambda) + d\left[f \mapsto \frac{1}{2}\int_\Sigma f \times d\alpha\right] - \hat{0}^*(\lambda).$$

But $\hat{0}^*(\lambda)$ is not necessarily zero. Let us compute generally $\hat{\alpha}^*(\lambda)$. Let $\mathrm{P} : \mathrm{U} \to \Omega^1(\Sigma)$ be an n-plot, $\hat{\alpha}^*(\lambda)(\mathrm{P}) = \lambda(\hat{\alpha} \circ \mathrm{P}) = \lambda(r \mapsto \hat{\alpha}(\mathrm{P}(r)) = \lambda(r \mapsto \alpha + d(\mathrm{P}(r)))$, and

$$\begin{aligned}
\lambda(r \mapsto \alpha + d(\mathrm{P}(r))) &= \frac{1}{2}\int_\Sigma (\alpha + \mathrm{P}(r)) \wedge \frac{\partial}{\partial r}(\alpha + d(\mathrm{P}(r))) \\
&= \frac{1}{2}\int_\Sigma (\alpha + \mathrm{P}(r)) \wedge \frac{\partial d(\mathrm{P}(r))}{\partial r} \\
&= \frac{1}{2}\int_\Sigma \alpha \wedge \frac{\partial d(\mathrm{P}(r))}{\partial r} + \frac{1}{2}\int_\Sigma \mathrm{P}(r) \wedge \frac{\partial d(\mathrm{P}(r))}{\partial r}.
\end{aligned}$$

Then,

$$(\hat{\alpha}^*(\lambda) - \hat{0}^*(\lambda))(\mathrm{P}) = \frac{1}{2}\int_\Sigma \alpha \wedge \frac{\partial d(\mathrm{P}(r))}{\partial r}.$$

Therefore,

$$\begin{aligned}
\mu(\alpha)(\mathrm{P})_r &= (\hat{\alpha}^*(\lambda) - \hat{0}^*(\lambda))(\mathrm{P})_r + d\left[f \mapsto \frac{1}{2}\int_\Sigma f \times d\alpha\right](\mathrm{P})_r \\
&= \frac{1}{2}\int_\Sigma \alpha \wedge \frac{\partial d(\mathrm{P}(r))}{\partial r} + \frac{\partial}{\partial r}\left\{\frac{1}{2}\int_\Sigma \mathrm{P}(r) \times d\alpha\right\} \\
&= \frac{1}{2}\frac{\partial}{\partial r}\left\{\int_\Sigma \alpha \wedge d(\mathrm{P}(r)) + \mathrm{P}(r) \times d\alpha\right\} \\
&= \frac{\partial}{\partial r}\left\{\int_\Sigma \mathrm{P}(r) \times d\alpha\right\},
\end{aligned}$$

which gives finally

$$\mu(\alpha) = d\left[f \mapsto \int_\Sigma f \times d\alpha\right].$$

Now, let us express the variance of μ. Let $f \in \mathscr{C}^\infty(\Sigma, \mathbf{R})$ and $\mathrm{F}(\alpha)$ be the real function $\mathrm{F}(\alpha) : f \mapsto \int_\Sigma f \times d\alpha$, such that $\mu(\alpha) = d\mathrm{F}(\alpha)$. We have $\mu(\bar{f}(\alpha)) = \mu(\alpha + df) = d\mathrm{F}(\alpha + df)$ but, for every $h \in \mathscr{C}^\infty(\Sigma, \mathbf{R})$, $\mathrm{F}(\alpha + df)(h) = \int_\Sigma h \times d(\alpha + df) = \int_\Sigma h \times d\alpha = \mathrm{F}(\alpha)(h)$. Then, for all $f \in \mathscr{C}^\infty(\Sigma, \mathbf{R})$, $\mu \circ \hat{f} = \mu$. The moment map μ is invariant by the group $\mathscr{C}^\infty(\Sigma, \mathbf{R})$. The Souriau class is zero, the action of $\mathscr{C}^\infty(\Sigma, \mathbf{R})$ is then exact and Hamiltonian.

Let us end with the computation of the kernel of the moment map μ. Clearly, $\mu(\alpha) = 0$ if and only if $d\mathrm{F}(\alpha) = 0$. But since $\mathscr{C}^\infty(\Sigma, \mathbf{R})$ is connected (actually contractible as a diffeological vector space), $d\mathrm{F}(\alpha) = 0$ if and only if $\mathrm{F}(\alpha) = \mathrm{cst} = \mathrm{F}(\alpha)(0) = 0$.

But $\mathrm{F}(\alpha) = 0$ if and only if, for all $f \in \mathscr{C}^\infty(\Sigma, \mathbf{R})$, $\int_\Sigma f \times d\alpha = 0$, that is, if and only if $d\alpha = 0$. ☐

9.28. On the intersection 2-form of a surface, II. We continue with the example of (art. 9.27) using the same notations. Let us introduce the group G of positive diffeomorphisms of (Σ, Surf), that is,

$$\mathrm{G} = \left\{ g \in \mathrm{Diff}(\Sigma) \;\middle|\; \frac{g^*(\mathrm{Surf})}{\mathrm{Surf}} > 0 \right\}.$$

The group G acts by pushforward on $\Omega^1(\Sigma)$. For all $g \in \mathrm{G}$, for all $\alpha \in \Omega^1(\Sigma)$, $g_*(\alpha) \in \Omega^1(\Sigma)$, and for all pairs g, g' of elements of G, $(g \circ g')_* = g_* \circ g'_*$, and this action is smooth.

1. The pushforward action of G on $\Omega^1(\Sigma)$ preserves the 1-form λ, and thus the 2-form ω. For all $g \in \mathrm{G}$, $(g_*)^*(\lambda) = \lambda$, and $(g_*)^*(\omega) = \omega$. Thus, the action of G is exact, $\sigma = 0$, and Hamiltonian, $\Gamma = \{0\}$.

2. The moment maps are, up to a constant, equal to the moment μ,

$$\mu(\alpha)(\mathrm{P})_r(\delta r) = \frac{1}{2} \int_\Sigma \alpha \wedge \mathrm{P}(r)^* \left(\frac{\partial \mathrm{P}(r)_*(\alpha)}{\partial r}(\delta r) \right),$$

for all $\alpha \in \Omega^1(\Sigma)$, for all n-plots P, where $r \in \mathrm{def}(\mathrm{P})$ and $\delta r \in \mathbf{R}^n$. In particular, applied to any 1-plot F centered at the identity $\mathbf{1}_\mathrm{G}$, that is, $\mathrm{F}(0) = \mathbf{1}_\mathrm{G}$, we get the special expression

$$\mu(\alpha)(\mathrm{F})_0(1) = -\frac{1}{2} \int_\Sigma \alpha \wedge \pounds_\mathrm{F}(\alpha) = -\int_\Sigma i_\mathrm{F}(\alpha) \times d\alpha,$$

where $\pounds_\mathrm{F}(\alpha)$ and $i_\mathrm{F}(\alpha)$ are the Lie derivative and the contraction of α by F. Note that it is not surprising that the Lie derivative of α is closely associated with the moment map of the action of the group of diffeomorphisms.

PROOF. 1. Let us compute the pullback of λ by the action of $g \in \mathrm{G}$, that is, $(g_*)^*(\lambda)$. Let $\mathrm{P} : \mathrm{U} \to \Omega^1(\Sigma)$ be an n-plot, let $r \in \mathrm{U}$, and $\delta r \in \mathbf{R}^n$, then

$$\begin{aligned}
(g_*)^*(\lambda)(\mathrm{P})_r(\delta r) &= \lambda(g_* \circ \mathrm{P})_r(\delta r) \\
&= \frac{1}{2} \int_\Sigma g_*(\mathrm{P}(r)) \wedge \frac{\partial g_*(\mathrm{P}(r))}{\partial r}(\delta r) \\
&= \frac{1}{2} \int_\Sigma g_*(\mathrm{P}(r)) \wedge g_* \left(\frac{\partial \mathrm{P}(r)}{\partial r}(\delta r) \right) \\
&= \frac{1}{2} \int_\Sigma g_* \left(\mathrm{P}(r) \wedge \frac{\partial \mathrm{P}(r)}{\partial r}(\delta r) \right) \\
&= \frac{1}{2} \int_{g^*(\Sigma)} \mathrm{P}(r) \wedge \frac{\partial \mathrm{P}(r)}{\partial r}(\delta r)
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}\int_\Sigma \mathrm{P}(r)\wedge \frac{\partial \mathrm{P}(r)}{\partial r}(\delta r) \\ &= \lambda(\mathrm{P})_r(\delta r). \end{aligned}$$

Thus, λ is invariant by G, and so is $\omega = d\lambda$.

2. Since the 1-form λ is invariant by the action of G, we can use directly the results of the exact case detailed in (art. 9.11). The moment map is, up to a constant, $\mu : \alpha \mapsto \hat{\alpha}^*(\lambda)$. Then, let $\mathrm{P} : \mathrm{U} \to \mathrm{G}$ be an n-plot, let $r \in \mathrm{U}$, $\delta r \in \mathbf{R}^n$. We have:

$$\begin{aligned} \mu(\alpha)(\mathrm{P})_r(\delta r) &= \alpha^*(\lambda)(\mathrm{P})_r(\delta r) \\ &= \lambda(\hat{\alpha}\circ \mathrm{P})_r(\delta r) \\ &= \lambda(r \mapsto \mathrm{P}(r)_*(\alpha))_r(\delta r) \\ &= \frac{1}{2}\int_\Sigma \mathrm{P}(r)_*(\alpha)\wedge \frac{\partial \mathrm{P}(r)_*(\alpha)}{\partial r}(\delta r) \\ &= \frac{1}{2}\int_\Sigma \alpha\wedge \mathrm{P}(r)^*\Big(\frac{\partial \mathrm{P}(r)_*(\alpha)}{\partial r}(\delta r)\Big). \end{aligned}$$

Now, let $\mathrm{P} = \mathrm{F}$ be a 1-plot centered at the identity, $\mathrm{F}(0) = \mathbf{1}_\mathrm{G}$. Let us change the variable r for the variable t. The previous expression, computed at $t = 0$ and applied to the vector $\delta t = 1$ gives immediately

$$\mu(\alpha)(\mathrm{F})_0(1) = \frac{1}{2}\int_\Sigma \alpha\wedge \left.\frac{\partial \mathrm{F}(t)_*(\alpha)}{\partial t}\right|_{t=0}.$$

But, by definition of the Lie derivative,

$$\left\{\frac{\partial \mathrm{F}(t)_*(\alpha)}{\partial t}\right\}_{t=0} = \left\{\frac{\partial (\mathrm{F}(t)^{-1})^*(\alpha)}{\partial t}\right\}_{t=0} = -£_\mathrm{F}(\alpha).$$

Thus, we get the first expression of the moment map μ applied to F, that is,

$$\mu(\alpha)(\mathrm{F})_0(1) = -\frac{1}{2}\int_\Sigma \alpha\wedge £_\mathrm{F}(\alpha).$$

Now, on a surface $\alpha\wedge d\alpha = 0$, but $i_\mathrm{F}(\alpha\wedge d\alpha) = i_\mathrm{F}(\alpha)\times d\alpha - \alpha\wedge i_\mathrm{F}(d\alpha)$, thus $i_\mathrm{F}(\alpha)\times d\alpha = \alpha\wedge i_\mathrm{F}(d\alpha)$. Then, using the Cartan-Lie formula $£_\mathrm{F}(\alpha) = i_\mathrm{F}(d\alpha) + d(i_\mathrm{F}(\alpha))$,

$$\begin{aligned} \int_\Sigma \alpha\wedge £_\mathrm{F}(\alpha) &= \int_\Sigma \alpha\wedge [i_\mathrm{F}(d\alpha) + d(i_\mathrm{F}(\alpha))] \\ &= \int_\Sigma i_\mathrm{F}(\alpha)d\alpha + \int_\Sigma \alpha\wedge d(i_\mathrm{F}(\alpha)) \\ &= \int_\Sigma i_\mathrm{F}(\alpha)d\alpha + \int_\Sigma i_\mathrm{F}(\alpha)d\alpha - \int_\Sigma d[\alpha\wedge i_\mathrm{F}(\alpha)] \\ &= 2\int_\Sigma i_\mathrm{F}(\alpha)d\alpha. \end{aligned}$$

And finally, we get the second expression for the moment map, that is,

$$\mu(\alpha)(\mathrm{F})_0(1) = -\int_\Sigma i_{\mathrm{F}}(\alpha) \times d\alpha,$$

for any 1-plot of the group of positive diffeomorphisms of the surface Σ, centered at the identity. □

9.29. On the intersection 2-form of a surface, III. We continue with the example of (art. 9.27), using the same notations. Let us consider the space $\Omega^1(\Sigma)$ as an additive group acting onto itself by translations. Let us denote by t_β the translation $t_\beta : \alpha \mapsto \alpha + \beta$, where α and β belong to $\Omega^1(\Sigma)$.

1. The 2-form ω is invariant by translation, that is, $t_\alpha^*(\omega) = \omega$ for all $\alpha \in \Omega^1(\Sigma)$. This action of $\Omega^1(\Sigma)$ onto itself is Hamiltonian but not exact.

2. The moment maps of the additive action of $\Omega^1(\Sigma)$ onto itself are, up to a constant, equal to

$$\mu : \alpha \mapsto d\left[\beta \mapsto \int_\Sigma \alpha \wedge \beta\right].$$

In other words, $\mu(\alpha) = d[\omega(\alpha)]$, where ω is regarded as the smooth linear function $\omega(\alpha) : \beta \mapsto \omega(\alpha, \beta)$, defined on $\Omega^1(\Sigma)$. Moreover, the moment map μ is linear and injective.

3. The moment map μ is its own Souriau cocycle, $\theta = \mu$. The moment map μ identifies $\Omega^1(\Sigma)$ with the θ-affine coadjoint orbit of $0 \in \Omega^1(\Sigma)^*$. Be aware that $\Omega^1(\Sigma)^*$ denotes the space of invariant 1-forms of the Abelian group $\Omega^1(\Sigma)$, and not its algebraic dual.

NOTE 1. This example is analogous to finite dimension symplectic vector spaces. The 2-form ω can be regarded as a real 2-cocycle of the additive group $\Omega^1(\Sigma)$. This cocycle builds up a central extension by **R**,

$$(\alpha, t) \cdot (\alpha', t') = \left(\alpha + \alpha', t + t' + \int_\Sigma \alpha \wedge \alpha'\right)$$

for all (α, t) and (α', t') in $\Omega^1(\Sigma) \times \mathbf{R}$. This central extension acts on $\Omega^1(\Sigma)$, preserving ω. This action is Hamiltonian, and now exact. The lack of equivariance, characterized by the Souriau class, has been absorbed in the extension. This group could be named, by analogy, the *Heisenberg group* of the oriented surface (Σ, Surf).

NOTE 2. According to (art. 9.19), the space $\Omega^1(\Sigma)$, equipped with the 2-form ω, is a homogeneous symplectic space. This is the first simple example of an infinite dimensional symplectic space, where diffeology avoids losing us in considerations about the kernel of ω and on the relations between algebraic duals.

NOTE 3. The preimage of zero by the moment map of the Abelian group $\mathscr{C}^\infty(\Sigma, \mathbf{R})$ on $\Omega^1(\Sigma)$ in (art. 9.27) is the subgroup of closed forms $Z^1_{\rm dR}(\Sigma) \subset \Omega^1(\Sigma)$. This group acts homogeneously on itself and its moment map for the restriction of the 2-form ω is the projection of the moment map of the action of $\mathscr{C}^\infty(\Sigma, \mathbf{R})$, that is, $\mu' : \alpha \mapsto d\left[\beta \mapsto \int_\Sigma \alpha \wedge \beta\right]$, but now α and β are closed. The characteristics of this moment map are the orbits of the subgroup $\mu'^{-1}(0)$, that is, the subgroup of $\alpha \in Z^1_{\rm dR}(\Sigma)$ such that $\int_\Sigma \alpha \wedge \beta = 0$ for all $\beta \in Z^1_{\rm dR}(\Sigma)$. This is the subgroup of exact 1-forms $B^1_{\rm dR}(\Sigma)$. Then, the image of μ' identifies naturally with the quotient space $H^1_{\rm dR}(\Sigma) = Z^1_{\rm dR}(\Sigma)/B^1_{\rm dR}(\Sigma)$. Moreover, the closed 2-form passes to this quotient, it is the well known *intersection form*, denoted here by $\omega_{\rm int}$, $\omega \upharpoonright Z^1_{\rm dR}(\Sigma) = \mu'^*(\omega_{\rm int})$. This is an example of symplectic reduction in diffeology (see also (art. 9.17, Note)), the full general case will be addressed in a future work.

PROOF. Let us compute the pullback of λ by a translation. Let $P : U \to X$ be an n-plot, let $r \in U$, and let $\delta r \in \mathbf{R}^n$,

$$\begin{aligned}
t^*_\alpha(\lambda)(P)_r(\delta r) &= \lambda(t_\alpha \circ P)_r(\delta r) \\
&= \lambda[r \mapsto P(r) + \alpha]_r(\delta r) \\
&= \frac{1}{2}\int_\Sigma (P(r) + \alpha) \wedge \frac{\partial(P(r)+\alpha)}{\partial r}(\delta r) \\
&= \frac{1}{2}\int_\Sigma P(r) \wedge \frac{\partial P(r)}{\partial r}(\delta r) + \frac{1}{2}\int_\Sigma \alpha \wedge \frac{\partial P(r)}{\partial r}(\delta r) \\
&= \lambda(P)_r(\delta r) + d\left[\beta \mapsto \frac{1}{2}\int_\Sigma \alpha \wedge \beta\right](P)_r(\delta r).
\end{aligned}$$

Let us define next, for all $\alpha \in \Omega^1(\Sigma)$, the smooth real function $F(\alpha)$ by

$$F(\alpha) : \beta \mapsto \frac{1}{2}\int_\Sigma \alpha \wedge \beta.$$

Then,

$$t^*_\alpha(\lambda) = \lambda + d(F(\alpha)) \text{ and } t^*_\alpha(\omega) = \omega.$$

Next, $\Omega^1(\Sigma)$, as an additive group, acts on itself by automorphisms. Let us compute the moment maps. Let p be a path in $\Omega^1(\Sigma)$, connecting α_0 to α_1, then

$$\begin{aligned}
\Psi(p) &= \hat{\alpha}^*_1(\lambda) - \hat{\alpha}^*_0(\lambda) - d\left[\beta \mapsto \int_p d(F(\beta))\right] \\
&= \hat{\alpha}^*_1(\lambda) - \hat{\alpha}^*_0(\lambda) - d[\beta \mapsto F(\beta)(\alpha_1) - F(\beta)(\alpha_0)] \\
&= \{\alpha^*_1(\lambda) - d[\beta \mapsto F(\beta)(\alpha_1)]\} - \{\alpha^*_0(\lambda) - d[\beta \mapsto F(\beta)(\alpha_0)]\} \\
&= \{\hat{\alpha}^*_1(\lambda) + d(F(\alpha_1))\} - \{\hat{\alpha}^*_0(\lambda) + d(F(\alpha_0))\}.
\end{aligned}$$

Thus, the 2-points moment map is clearly given by $\psi(\alpha_0, \alpha_1) = \Psi(p)$. Now, the moment maps are, up to a constant, equal to

$$\mu(\alpha) = \psi(0, \alpha) = \hat{\alpha}_1^*(\lambda) + d(\mathrm{F}(\alpha)) - \hat{0}^*(\lambda).$$

But for any plot $\mathrm{P} : \mathrm{U} \to \Omega^1(\Sigma)$,

$$\begin{aligned} \hat{\alpha}^*(\lambda)(\mathrm{P}) - \hat{0}^*(\lambda)(\mathrm{P}) &= \lambda(\hat{\alpha} \circ \mathrm{P}) - \lambda(\hat{0} \circ \mathrm{P}) \\ &= \lambda(r \mapsto \mathrm{P}(r) + \alpha) - \lambda(r \mapsto \mathrm{P}(r)) \\ &= d\left[\beta \mapsto \frac{1}{2}\int_\Sigma \alpha \wedge \beta\right](\mathrm{P}) \\ &= d(\mathrm{F}(\alpha))(\mathrm{P}). \end{aligned}$$

Hence, $\hat{\alpha}^*(\lambda)(\mathrm{P}) - \hat{0}^*(\lambda) = d(\mathrm{F}(\alpha))$ and μ is finally given by

$$\mu(\alpha) = 2d(\mathrm{F}(\alpha)) = d\left[\beta \mapsto \int_\Sigma \alpha \wedge \beta\right].$$

The moment map μ is not equivariant, the Souriau cocycle θ is given by

$$\mu(\mathrm{t}_\alpha^*(\beta)) = \mu(\alpha + \beta) = \mu(\beta) + \theta(\alpha), \text{ with } \theta(\alpha) = \mu(\alpha).$$

Considering the Note, the moment map μ is clearly smooth and linear. Let $\alpha \in \ker(\mu)$, $\mu(\alpha) = 0$ if and only if $d(\mathrm{F}(\alpha)) = 0$, that is, if and only if $\mathrm{F}(\alpha) = \mathrm{cst} = \mathrm{F}(\alpha)(0) = 0$. But $\mathrm{F}(\alpha)(\beta) = 0$, for all $\beta \in \Omega^1(\Sigma)$, implies $\alpha = 0$. Therefore, μ is injective. ☐

9.30. On symplectic irrational tori. Consider the smooth space $\mathbf{R}^n$, for some integer n. For all $u \in \mathbf{R}^n$, let t_u be the translation by u, that is, $\mathrm{t}_u : x \mapsto x + u$. Let ω be a 2-form of $\mathbf{R}^n$ invariant by translation, that is, for all $u \in \mathbf{R}^n$, $\mathrm{t}_u^*(\omega) = \omega$. Thus, ω is a constant bilinear 2-form, thus closed, $d\omega = 0$. Let us consider the moment maps associated with the translations $(\mathbf{R}^n, +)$. Since $\mathbf{R}^n$ is simply connected, the holonomy is trivial, $\Gamma = \{0\}$. Let p be a path in $\mathbf{R}^n$ connecting $x = p(0)$ to $y = p(1)$, the paths moment map $\Psi(p)$, and the 2-points moment map $\psi(p)$ are given by

$$\Psi(p) = \psi(x, y) = \omega(y - x),$$

where $\omega(u)$ is regarded as the linear 1-form $\omega(u) : v \mapsto \omega(u, v)$. The moment maps are, up to constant, equal to the linear map

$$\mu : x \mapsto \omega(x),$$

and the Souriau cocycle θ associated with μ is equal to μ. For all $u \in \mathbf{R}^n$,

$$\theta(u) = \mu(u) = \omega(u).$$

Consider now a discrete diffeological subgroup $\mathrm{K} \subset \mathbf{R}^n$. Let us denote by Q the quotient $\mathrm{Q} = \mathbf{R}^n/\mathrm{K}$ and by $\pi : \mathbf{R}^n \to \mathrm{Q}$ the projection. Let us continue to denote by t_u the translation on Q, by $u \in \mathbf{R}^n$, that is, $\mathrm{t}_u(q) = \pi(x + u)$, for any x such that

$q = \pi(x)$. Now, since ω is invariant by translation, ω is invariant by K, and since K is discrete, ω projects on Q as a $\mathbf{R}^n$-invariant closed 2-form denoted by ω_Q, that is,

$$\omega_Q = \pi_*(\omega), \text{ or } \omega = \pi^*(\omega_Q).$$

Note that the translation by any vector u of $\mathbf{R}^n$ on Q is still an automorphism of ω_Q, that is, $t_u^*(\omega_Q) = \omega_Q$.

1. The holonomy Γ_Q of the action of $(\mathbf{R}^n, +)$ on (Q, ω_Q) is the image of the subgroup K by μ,

$$\Gamma_Q = \mu(K), \ \Gamma_Q \subset \mathbf{R}^{n*}.$$

Thus, if $\omega \neq 0$ and if K is not reduced to $\{0\}$, then the action of $(\mathbf{R}^n, +)$ on (Q, ω_Q) is not Hamiltonian and not exact.

2. The moment map $\mu : \mathbf{R}^n \to \mathbf{R}^{n*}$ projects on a moment μ_Q such that the following diagram commutes.

$$\begin{array}{ccc} \mathbf{R}^n & \xrightarrow{\ \mu\ } & \mathbf{R}^{n*} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}} \\ Q = \mathbf{R}^n/K & \xrightarrow[\ \mu_Q\ ]{} & \mathbf{R}^{n*}/\mu(K) \end{array}$$

For all $q \in Q$, $\mu_Q(q) = \mathrm{pr}(\omega(x))$ for any x such that $q = \pi(x)$. The Souriau cocycle θ_Q associated with μ_Q, for all $u \in \mathbf{R}^n$, is given by

$$\theta_Q(u) = \mu_Q(\pi(u)).$$

Hence, if we consider the space Q as an additive group acting on itself by translations, then the moment map μ_Q coincides with its Souriau cocycle θ_Q.

3. The map μ is a fibration onto its image whose fiber is the kernel of μ, that is, $\mathrm{val}(\mu) \simeq \mathbf{R}^n/E$, $E = \ker(\mu)$. And, the map μ_Q is a fibration onto its image $\mu(\mathbf{R}^n)/\mu(K)$ whose fiber is $\ker(\mu_Q) = E/(K \cap E)$. If $\omega : \mathbf{R}^n \to \mathbf{R}^{n*}$ is injective (which implies that n is even), then the moment map μ_Q is a diffeomorphism which identifies Q with its image $\mathbf{R}^{n*}/\mu(K)$.

NOTE 1. Regarded as a group, $Q = \mathbf{R}^n/K$ acts onto itself by projection of the translations of $\mathbf{R}^n$. Since the pullback by $\pi : \mathbf{R}^n \to Q$ is an isomorphism from $\mathcal{Q}^*$ to $\mathbf{R}^{n*}$ ($\mathbf{R}^n$ is the universal covering of Q), the moment maps computed above give the moment maps associated with this action.

NOTE 2. This construction applies to the torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$. The action of $(\mathbf{R}^2, +)$, is obviously not Hamiltonian, but the moment map μ_{T^2} is well defined. And, μ_{T^2} identifies T^2 with the quotient of $\mathbf{R}^{2*}$ — the (Γ_Q, θ_Q)-coadjoint orbit of the point 0 — by the holonomy $\Gamma_Q = \omega(\mathbf{Z}^2) \subset \mathbf{R}^{2*}$, according to the meaning of a coadjoint orbit we gave in (art. 7.16). As strange as it may sound, the torus T^2, equipped with the standard symplectic form ω, is a coadjoint orbit of $\mathbf{R}^2$, or even a coadjoint orbit of itself. This is a special case of the (art. 9.23) discussion.

NOTE 3. All this construction above also applies to situations regarded as more singular that the simple quotient of $\mathbf{R}^n$ by a lattice. It applies, for example, to the product of any irrational tori. An (n-dimensional) irrational torus T_K is the quotient of $\mathbf{R}^n$ by any generating discrete strict subgroup K of $\mathbf{R}^n$. See for example [IgLa90] for an analysis of 1-dimensional irrational tori. For example, we can consider the product of two 1-dimensional irrational tori $\mathrm{Q} = \mathrm{T}_\mathrm{H} \times \mathrm{T}_\mathrm{K}$, quotient of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ by the discrete subgroup $\alpha_\mathrm{H}(\mathbf{Z}^p) \times \alpha_\mathrm{K}(\mathbf{Z}^q)$, where $\alpha_\mathrm{H} : \mathbf{R}^p \to \mathbf{R}$ and $\alpha_\mathrm{K} : \mathbf{R}^q \to \mathbf{R}$ are two linear 1-forms with coefficients independent over $\mathbf{Q}$. In this case, the moment map μ_Q will also identify $\mathrm{T}_\mathrm{H} \times \mathrm{T}_\mathrm{K}$ with the quotient of $\mathbf{R}^{2*}$ — $(\Gamma_\mathrm{Q}, \theta_\mathrm{Q})$-coadjoint orbit of 0 — by $\Gamma_\mathrm{Q} = \omega(\alpha_\mathrm{H}(\mathbf{Z}^p) \times \alpha_\mathrm{K}(\mathbf{Z}^q))$. This is the simplest example of *totally irrational symplectic space*, and *totally irrational coadjoint orbit*. Note that these cases escape completely the usual analysis, of course, but also the analysis in terms of Sikorski or Frölicher spaces; see Exercise 80, p. 123.

PROOF. First of all, the fact that there exists a closed 2-form ω_Q on $\mathbf{R}/\mathrm{K}$ such that $\pi^*(\omega_\mathrm{Q}) = \omega$ is an application of the criterion of pushing forward forms, in the special case of a covering (art. 8.27). Now, the computation of the moment map of a linear antisymmetric form ω on $\mathbf{R}^n$ is well known, and independently of the method gives the same result $\mu(x) = \omega(x)$. The additive constant is set here by the condition $\mu(0) = 0$. But the value of the paths moment map $\Psi(p)$ can be computed as well by the general method, applying the particular expression

$$\mathscr{K}\omega_p(\delta p) = \int_0^1 \omega_{p(t)}(\dot{p}(t), \delta p(t))\, dt\,, \text{ with } \dot{p}(t) = \frac{dp(t)}{dt},$$

of the Chain-Homotopy operator for manifold, where p is a path and δp is a *variation* of p. Then, since the result depends only on the ends of the path, choose, for any two points x and y in $\mathbf{R}^n$, the connecting path $p : t \mapsto x + t(y - x)$. Let us recall that $\Psi(p) = \hat{p}^*(\mathscr{K}\omega)$, let u and δu in $\mathbf{R}^n$, note that $\hat{p}_*(\mathrm{t}_u) = \mathrm{t}_u \circ p = [t \mapsto p(t) + u]$. Then,

$$\begin{aligned}
\Psi(p)_u(\delta u) &= \hat{p}^*(\mathscr{K}\omega)_u(\delta u) \\
&= (\mathscr{K}\omega)_{\mathrm{t}_u \circ p}(\delta(\mathrm{t}_u \circ p)), \text{ with } \delta p = 0 \\
&= \int_0^1 \omega(\dot{p}(t), \delta u)\; d\,t \\
&= \omega(y - x, \delta u).
\end{aligned}$$

Thus, $\Psi(p) = \psi(x, y) = \omega(y - x) = \omega(y) - \omega(x)$, and $\mu : x \mapsto \omega(x)$, $x \in \mathbf{R}^n$. Consider now ω_Q, since $\mathbf{R}^n$ is the universal covering of Q, every loop $\ell \in \mathrm{Loops}(\mathrm{Q}, 0)$

can be lifted into a path p in $\mathbf{R}^n$ starting at 0 and ending in K. In other words,

$$\Gamma = \{\Psi(\ell) \mid \ell \in \mathrm{Loops}(\mathrm{Q})\} = \{\Psi(t \mapsto tk) \mid k \in \mathrm{K}\} = \omega(\mathrm{K}).$$

The other propositions are then a direct application of the functoriality of the moment map described in (art. 9.13), and standard analysis on quotients and fibrations. ☐

9.31. The corner orbifold. Let us consider the quotient $\mathcal{Q}$ of $\mathbf{R}^2$ by the action of the finite subgroup $\mathrm{K} \simeq \{\pm 1\}^2$, embedded in $\mathrm{GL}(2, \mathbf{R})$ by

$$\mathrm{K} = \left\{ \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{pmatrix} \middle| \, \varepsilon, \varepsilon' \in \{\pm 1\} \right\}.$$

The space $\mathcal{Q} = \mathbf{R}^2/\mathrm{K}$ (Figure 9.1) is an orbifold, according to [IKZ10]. It is diffeomorphic to the quarter space $[0, \infty[\times [0, \infty[\subset \mathbf{R}^2$, equipped with the pushforward of the standard diffeology of $\mathbf{R}^2$ by the map $\pi : \mathbf{R}^2 \to [0, \infty[\times [0, \infty[$, defined by,

$$\pi(x, y) = (x^2, y^2) \text{ and } \mathcal{Q} \simeq \pi_*(\mathbf{R}^2).$$

The letter $\mathcal{Q}$ will denote indifferently the quotient $\mathbf{R}^2/\mathrm{K}$ or the quarter space $\pi_*(\mathbf{R}^2)$, and the meaning of the letter π follows. Be aware that the corner orbifold is not a manifold with boundary, $\mathcal{Q}$ is not diffeomorphic to the corner equipped with the induced diffeology of $\mathbf{R}^2$. That said, we remark that the decomposition of $\mathcal{Q}$ in terms of point's structure is given by

$$\mathrm{Str}(0,0) = \{\pm 1\}^2, \ \mathrm{Str}(x, 0) = \mathrm{Str}(0, y) = \{\pm 1\}, \text{ and } \mathrm{Str}(x, y) = \{1\},$$

where x and y are positive real numbers. Then, since the structure group of a point is preserved by diffeomorphisms [IKZ10], there are at least three orbits of $\mathrm{Diff}(\mathcal{Q})$, the point $0_{\mathcal{Q}} = (0, 0)$, the regular stratum $\dot{\mathcal{Q}} =]0, \infty[^2$ and the union of the two axes, ox and oy. In particular, any diffeomorphism of $\mathcal{Q}$ preserves the origin $0_{\mathcal{Q}}$. Actually, these are exactly the orbits of $\mathrm{Diff}(\mathcal{Q})$. Let us remark that, since $\dim(\mathcal{Q}) = 2$ (art. 1.78), every 2-form is closed.

1. Every 2-form of $\mathcal{Q}$ is proportional to the 2-form ω defined on $\mathcal{Q}$ by

$$\pi^*(\omega) : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto 4xy \times dx \wedge dy,$$

that is, for any other 2-form ω' there exists a smooth function $\phi \in \mathcal{C}^\infty(\mathcal{Q}, \mathbf{R})$ such that $\omega' = \phi \times \omega$.

2. The space $(\mathcal{Q}, \omega)$ is Hamiltonian, $\Gamma_\omega = \{0\}$, and the action of G_ω is exact, that is, $\sigma_\omega = 0$. In particular, the universal moment map μ_ω defined by $\mu_\omega(0_{\mathcal{Q}}) = 0$, is equivariant.

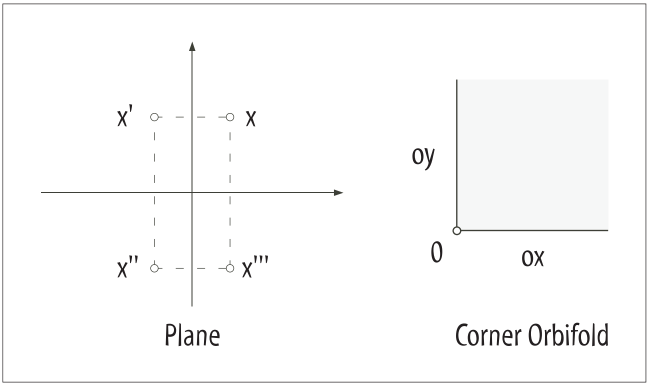

FIGURE 9.1. The corner orbifold $\mathcal{Q}$.

3. The universal equivariant moment map μ_ω vanishes on the singular strata $\{0\}$, ox and oy, and is injective on the regular stratum $\dot{\mathcal{Q}}$. Therefore, the image $\mu_\omega(\mathcal{Q})$ is diffeomorphic to an open disc with a point somehow attached on the boundary.

PROOF. 1. Let ω' be a 2-form on $\mathcal{Q}$, and let $\tilde{\omega}'$ be its pullback by π, $\tilde{\omega}' = \pi^*(\omega')$. Thus, there exists a smooth real function F such that $\tilde{\omega}' = \mathrm{F} \times dx \wedge dy$. But since $\pi \circ k = \pi$, for all $k \in \mathrm{K}$, we get $\varepsilon\varepsilon'\mathrm{F}(\varepsilon x, \varepsilon' y) = \mathrm{F}(x, y)$, for all $(x, y) \in \mathbf{R}^2$ and all ε, ε' in $\{\pm 1\}$. Thus, $\mathrm{F}(-x, y) = -\mathrm{F}(x, y)$ and $\mathrm{F}(x, -y) = -\mathrm{F}(x, y)$. In particular, $\mathrm{F}(0, y) = 0$ and $\mathrm{F}(x, 0) = 0$. Therefore, since F is smooth, there exists $f \in \mathcal{C}^\infty(\mathbf{R}^2, \mathbf{R})$ such that $\mathrm{F}(x, y) = 4xyf(x, y)$, with $f(\varepsilon x, \varepsilon' y) = f(x, y)$. Therefore, $\tilde{\omega}' = f \times \tilde{\omega}$, with $\tilde{\omega} = 4xy \times dx \wedge dy$, that is, $\tilde{\omega} = d(x^2) \wedge d(y^2)$, but $x \mapsto x^2$ and $y \mapsto y^2$ are invariant by K so, they are the pullback by π of some smooth real functions on $\mathcal{Q}$. Thus, $d(x^2)$ and $d(y^2)$ are the pullback of 1-forms on $\mathcal{Q}$, let us say $d(x^2) = \pi^*(ds)$ and $d(y^2) = \pi^*(dt)$, so $\tilde{\omega} = \pi^*(\omega)$, where $\omega = ds \wedge dt$ is a well defined 2-form on $\mathcal{Q}$. Now, since $f(\varepsilon x, \varepsilon' y) = f(x, y)$ means just that f is the pullback of a smooth real function ϕ on $\mathcal{Q}$, it follows that any 2-form ω' on $\mathcal{Q}$ is proportional to ω, that is, $\omega' = \phi \times \omega$, with $\phi \in \mathcal{C}^\infty(\mathcal{Q}, \mathbf{R})$.

2. The orbifold is contractible. The deformation retraction $(s, x, y) \mapsto (sx, sy)$ of $\mathbf{R}^2$ to $\{(0, 0)\}$ projects on a smooth deformation retraction of $\mathcal{Q}$. Thus, the holonomy is trivial, $\Gamma = \{0\}$. Now, since the origin $0_{\mathcal{Q}}$ is the only point with structure $\{\pm 1\}$, every diffeomorphism of $\mathcal{Q}$ preserves the origin $0_{\mathcal{Q}}$. Then, the 2-point moment map is exact, see (art. 9.10, Note 2), the Souriau class is zero, $\sigma_\omega = 0$. Let q be any point of $\mathcal{Q}$ and let $\mu_\omega(q) = \psi(0_{\mathcal{Q}}, q)$. This is an equivariant moment map and $\mu_\omega(0_{\mathcal{Q}}) = \psi(0_{\mathcal{Q}}, 0_{\mathcal{Q}}) = 0$.

3. Let $q \in \mathcal{Q}$, thus $\mu_\omega(q) = \Psi(p)$ for any path p connecting $0_{\mathcal{Q}}$ to q. Let q belong to a semi-axis ox or oy, and let us choose $p = t \mapsto \lambda(t)q$, where λ is a smashing

function equal to 0 on $]-\infty, 0]$ and equal to 1 on $[1, +\infty[$. Thus, for all $t \in \mathbf{R}$, $p(t)$ belongs to the same semi-axis as q. Thanks to (art. 9.2, (♡)), for any 1-plot ϕ of $\mathrm{Diff}(\mathcal{Q}, \omega_\omega)$ centered at the identity,

$$\Psi(p)(\phi)_0(1) = \int_0^1 \omega\left[\begin{pmatrix} s \\ r \end{pmatrix} \mapsto \phi(r)(\lambda(s+t)q)\right]_{\binom{0}{0}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} dt\,.$$

But $(s, r) \mapsto \phi(r)(\lambda(s+t)q)$ is a plot of the semi-axis, and thanks to item 1, the form ω vanishes on the semi-axis. Thus, the integrand vanishes and $\Psi(p)(\phi)_0(1) = 0$. Then, since 1-forms are characterized by 1-plots and since momenta are characterized by centered plots, $\mu_\omega(q) = 0$ for all $q \in \mathcal{Q}$ belonging to any semi-axis.

On the other hand, let q and q' be two points of the regular stratum $\dot{\mathcal{Q}}$. Since $\pi \restriction \{(x, y) \mid x > 0 \text{ and } y > 0\}$ is a diffeomorphism, and since $\tilde{\omega} \restriction \{(x, y) \mid x > 0$ and $y > 0\}$ is a symplectic Hausdorff manifold, there exists always a symplectomorphism ϕ with compact support $\mathcal{S} \subset \{(x, y) \mid x > 0 \text{ and } y > 0\}$ which exchanges q and q'. Then, the image of this diffeomorphism on $\dot{\mathcal{Q}}$ can be extended by the identity on the whole $\mathcal{Q}$. Therefore, the automorphisms of ω are transitive on the regular stratum. The fact that the universal moment map is injective on the regular stratum comes from what we know already on symplectic manifolds (art. 9.23). □

9.32. The cone orbifold. Let $\mathcal{Q}_m$ be the quotient of the smooth complex plane $\mathbf{C}$ by the multiplicative action of the cyclic subgroup Z_m, mth-roots of unity. The space $\mathcal{Q}_m$ (Figure 9.2) is an orbifold, according to [IKZ10]. We identify $\mathcal{Q}_m$ to the complex plane $\mathbf{C}$, equipped with the pushforward of the standard diffeology by the map $\pi_m : z \mapsto z^m$.

$$\mathcal{Q}_m = \mathbf{C}/\mathrm{Z}_m \text{ with } \mathrm{Z}_m = \{\zeta \in \mathbf{C} \mid \zeta^m = 1\} \text{ with } m > 1.$$

The plots of $\mathcal{Q}_m$ are the parametrizations P of $\mathbf{C}$ which write locally $\mathrm{P}(r) = \phi(r)^m$, where ϕ is a smooth parametrization in $\mathbf{C}$. Let us remark first that the decomposition of $\mathcal{Q}_m$, in terms of structure group, is given by

$$\mathrm{Str}(0) = \mathrm{Z}_m, \text{ and } \mathrm{Str}(z) = \{1\}, \text{ if } z \neq 0.$$

And secondly that there are two orbits of $\mathrm{Diff}(\mathcal{Q}_m)$, the point 0 and the regular stratum $\dot{\mathcal{Q}}_m = \mathbf{C} - \{0\}$. In particular any diffeomorphism of $\mathcal{Q}_m$ preserves the origin 0. It is not difficult to check that $\{\pi_m\}$ is a minimal generating family, thus $\dim(\mathcal{Q}_m) = 2$ (art. 1.78), and every 2-form on $\mathcal{Q}_m$ is closed.

1. Every 2-form of $\mathcal{Q}_m$ is proportional to the 2-form ω defined by

$$\pi_m^*(\omega) : z \mapsto dx \wedge dy \text{ with } z = x + iy.$$

For every other 2-form ω' there exists a smooth function $f \in \mathscr{C}^\infty(\mathcal{Q}_m, \mathbf{R})$ such that $\omega' = f \times \omega$.

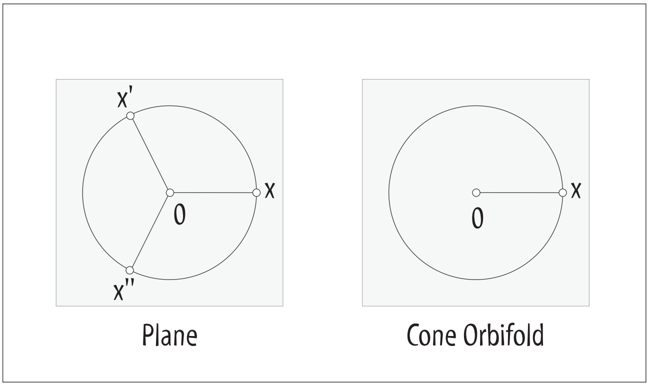

FIGURE 9.2. The cone orbifold $\mathcal{Q}_3$.

2. The space $(\mathcal{Q}, \omega)$ is Hamiltonian, $\Gamma_\omega = \{0\}$, and the action of G_ω is exact, that is, $\sigma_\omega = 0$. In particular, the universal moment map μ_ω, defined by $\mu_\omega(0) = 0$, is equivariant.

3. The universal moment map μ_ω is injective. Its image is the reunion of two coadjoint orbits, the point $0 \in \mathcal{G}^*_\omega$, value of the origin of $\mathcal{Q}_m$, and the image of the regular stratum $\dot{\mathcal{Q}}_m$.

PROOF. Let us first prove that the usual surface form $\mathrm{Surf} = dx \wedge dy$ is the pullback of a 2-form ω defined on $\mathcal{Q}_m$. We shall apply the standard criterion and prove that for any two plots ϕ_1 and ϕ_2 of $\mathbf{C}$ such that $\pi_m \circ \phi_1 = \pi_m \circ \phi_2$, we have $\mathrm{Surf}(\phi_1) = \mathrm{Surf}(\phi_2)$, that is, $\phi_1(r)^m = \phi_2(r)^m$ implies $\mathrm{Surf}(\phi_1) = \mathrm{Surf}(\phi_2)$. First of all let us recall that, since we are dealing with 2-forms, is is sufficient to consider 2-plots. So, let the ϕ_i be defined on some real domain $U \subset \mathbf{R}^2$. Let $r_0 \in U$, we split the problem into two cases.

1. $\phi_1(r_0) \neq 0$ — Thus, $\phi_2(r_0) \neq 0$, there exists an open disc B centered at r_0 on which the ϕ_i do not vanish. Thus, the map $r \mapsto \zeta(r) = \phi_2(r)/\phi_1(r)$ defined on B is smooth with values in Z_m. But, since Z_m is discrete there exists $\zeta \in Z_m$ such that $\phi_2(r) = \zeta \times \phi_1(r)$ on B. Now, Surf is invariant by $U(1) \supset Z_m$. Therefore $\mathrm{Surf}(\phi_1) = \mathrm{Surf}(\phi_2)$ on B.

2. $\phi_1(r_0) = 0$ — Thus, $\phi_2(r_0) = 0$. Now, we have $\mathrm{Surf}(\phi_i) = \det(D(\phi_i)) \times \mathrm{Surf}$, where $D(\phi_i)$ denotes the tangent map of ϕ_i. We split this case in two subcases.

2.a. $D(\phi_1)_{r_0}$ is not degenerate — Thus, thanks to the implicit function theorem, there exists a small open disc B around r_0 where ϕ_1 is a local diffeomorphism onto its image. Since $\phi_1(r)^m = \phi_2(r)^m$, the common zero r_0 of both ϕ_1 and ϕ_2 is isolated. Thus, the map $r \mapsto \zeta(r) = \phi_2(r)/\phi_1(r)$ defined on $B - \{r_0\}$ is smooth, and for the same reason as in the first case, ζ is constant. Then, $\phi_2(r) = \zeta \times \phi_1(r)$ on $B - \{r_0\}$. But since $\phi_i(r_0) = 0$, this extends on B. Therefore, $\mathrm{Surf}(\phi_1) = \mathrm{Surf}(\phi_2)$ on B.

2.b. $D(\phi_1)_{r_0}$ is degenerate — Let u be in the kernel of $D(\phi_1)_{r_0}$. Then, $\phi_1(r_0+su)^m = \phi_2(r_0+su)^m$ for s small enough. Hence, differentiating this equality m times with respect to s, we get at $s=0$, $D(\phi_1)_{r_0}(u)^m = D(\phi_2)_{r_0}(u)^m = 0$. Therefore, $D(\phi_2)_{r_0}$ is also degenerate at r_0 and thus $\mathrm{Surf}(\phi_1)_{r_0} = \mathrm{Surf}(\phi_2)_{r_0} = 0$.

Thus, we proved that for all $r \in U$, $\mathrm{Surf}(\phi_1)_r = \mathrm{Surf}(\phi_2)_r$. Therefore, there exists a 2-form ω on $\mathcal{Q}_m$ such that $\pi_m^*(\omega) = \mathrm{Surf}$, and this form ω is completely defined by its pullback. Now, since the pullback by π_m of any other 2-form ω' on $\mathcal{Q}_m$ is proportional to Surf, the form ω' is proportional to ω. Now, for the same reasons as in (art. 9.31), the universal holonomy Γ_ω is trivial, the Souriau class σ_ω is zero, and the universal moment map μ_ω defined by $\mu_\omega(0) = 0_{\mathscr{G}^*}$ is equivariant. Moreover, the regular stratum $\dot{\mathcal{Q}}$ is just a symplectic manifold for the restriction of ω. Every symplectomorphism with compact support which does not contain 0 can be extended to an automorphism of $(\mathcal{Q}, \omega)$. Thus, since the compactly supported symplectomorphisms of a connected symplectic manifold are transitive [Boo69], the regular stratum $\dot{\mathcal{Q}}$ is an orbit of $\mathrm{Diff}(\mathcal{Q}, \omega)$ and, the universal moment map is injective on this stratum (art. 9.23). Therefore, the moment map μ_ω maps injectively $\mathcal{Q}$ onto the two orbits, $\{0_{\mathscr{G}^*}\}$ and $\mu_\omega(\dot{\mathcal{Q}})$. ☐

9.33. The infinite projective space. Let $\mathscr{H}$ be the Hilbert space of the square summable complex series

$$\mathscr{H} = \left\{ Z = (Z_i)_{i=1}^{\infty} \;\middle|\; \sum_{i=1}^{n} Z_i \cdot Z_i < \infty \right\},$$

where the dot denotes the Hermitian product. The space $\mathscr{H}$ is equipped with the *fine structure* of complex diffeological vector space (art. 3.15). Let $P : U \to \mathscr{H}$ be a plot, then for every $r_0 \in U$ there exist an integer n, an open neighborhood $V \subset U$ of r_0 and a finite family $\mathscr{F} = \{(\lambda_a, Z_a)\}_{a \in A}$, where the $Z_a \in \mathscr{H}$, and the $\lambda_a \in \mathscr{C}^\infty(V, \mathbf{C}^n)$, such that $P \restriction V : r \mapsto \sum_{a\in A} \lambda_a(r) \times Z_a$. Such a family $\{(\lambda_a, Z_a)\}_{a\in A}$ is called a *local family* of P at the point r_0. We introduced the symbol dZ in Exercise 99, p. 194, which associates every local family $\mathscr{F} = \{(\lambda_a, Z_a)\}_{a\in A}$, defined on the domain V, with the complex valued 1-form of V

$$dZ(\mathscr{F}) : r \mapsto \sum_{a\in A} d\lambda_a(r) Z_a.$$

For every $\lambda_a = x_a + iy_a$, where x_a and y_a are real smooth parametrizations, $d\lambda_a = dx_a + idy_a$. There exists on $\mathscr{H}$ a 1-form α defined by

$$\alpha = \frac{1}{2i}[Z \cdot dZ - dZ \cdot Z].$$

1. As an additive group $(\mathscr{H}, +)$ acts on itself, preserving $d\alpha$. Let $Z \in \mathscr{H}$, and let t_Z be the translation by Z, then $t_Z^*(d\alpha) = d\alpha$. This action is Hamiltonian but not exact. Let μ be the moment map of the translations $(\mathscr{H}, +)$, defined by $\mu(0_{\mathscr{H}}) = 0$,

$$\mu(Z) = 2d[w(Z)] \text{ with } w(\zeta) : Z \mapsto \frac{1}{2i}[\zeta \cdot Z - Z \cdot \zeta] \in \mathscr{C}^\infty(\mathscr{H}, \mathbf{R}).$$

The moment map μ is injective and $(\mathscr{H}, d\alpha)$ is a homogeneous symplectic space.

2. Let $\mathbf{U}(\mathscr{H})$ be the group of unitary transformations of $\mathscr{H}$, equipped with the functional diffeology. The group $\mathbf{U}(\mathscr{H})$ acts on $\mathscr{H}$ preserving α. The action of $\mathbf{U}(\mathscr{H})$ on $(\mathscr{H}, d\alpha)$ is exact and Hamiltonian. Let $P : U \to \mathbf{U}(\mathscr{H})$ be an n-plot. The value of the moment map μ, for the action of $\mathbf{U}(\mathscr{H})$ on $(\mathscr{H}, d\alpha)$, evaluated on P, is given by

$$\mu(Z)(P)_r(\delta r) = \frac{1}{2i}\left[P(r)(Z) \cdot \frac{\partial P(r)(Z)}{\partial r}(\delta r) - \frac{\partial P(r)(Z)}{\partial r}(\delta r) \cdot P(r)(Z)\right],$$

where, $r \in U$, $\delta r \in \mathbf{R}^n$ and if

$$P(r)(Z) =_{\text{loc}} \sum_{\alpha \in A} \lambda_\alpha(r) Z_\alpha, \text{ then } \frac{\partial P(r)(Z)}{\partial r}(\delta r) =_{\text{loc}} \sum_{\alpha \in A} \frac{\partial \lambda_\alpha(r)}{\partial r}(\delta r) Z_\alpha.$$

3. The unit sphere $\mathscr{S} \subset \mathscr{H}$ is a homogeneous space of $\mathbf{U}(\mathscr{H})$; see Exercise 126, p. 267. The fibers of the equivariant moment map μ of the action of $\mathbf{U}(\mathscr{H})$ on $(\mathscr{S}, d\alpha \upharpoonright \mathscr{S})$ are the fibers of the infinite Hopf fibration $\pi : \mathscr{S} \to \mathscr{P} = \mathscr{S}/S^1$, where $S^1 \in \mathbf{C}$ acts multiplicatively on $\mathscr{S}$. There exists a symplectic form ω on $\mathscr{P}$, called the *infinite projective space*, such that $\pi^*(\omega) = d\alpha \upharpoonright \mathscr{S}$; see Exercise 100, p. 194. The equivariant moment map of the induced action of $\mathbf{U}(\mathscr{H})$ on $\mathscr{P}$ is injective. Therefore, equipped with the Fubini-Study form, $\mathscr{P}$ is a homogeneous symplectic space and can be regarded as a coadjoint orbit of $\mathbf{U}(\mathscr{H})$.

PROOF. Let us prove what is claimed here and has not been already proved in a previous paragraph or exercise.

1. Since $\mathscr{H}$ is contractible, there is no holonomy. Now, let $\zeta \in \mathscr{H}$, and let t_ζ be the translation $t_\zeta(Z) = Z + \zeta$. A direct computation shows that $t_\zeta^*(\alpha) = \alpha + d[w(\zeta)]$. Thus, $d\alpha$ is invariant by translation $t_\zeta^*(d\alpha) = d\alpha$. Now, let p be a path connecting $0_{\mathscr{H}}$ to Z, we have $\mu(Z) = \Psi(p) = \hat{p}^*\mathscr{K}(d\alpha) = \hat{Z}^*(\alpha) - \hat{0}_{\mathscr{H}}^*(\alpha) - d[\mathscr{K}\alpha \circ \hat{p}]$. But on the one hand we have $\hat{Z} = t_Z$, thus $\hat{Z}^*(\alpha) - \hat{0}_{\mathscr{H}}^*(\alpha) = t_Z^*(\alpha) - \mathbf{1}_{\mathscr{H}}^*(\alpha) = \alpha + d[w(Z)] - \alpha = d[w(Z)]$, and on the other hand, $\hat{p}(\zeta) = t_\zeta \circ p$. Thus, $\mathscr{K}\alpha \circ \hat{p} = \int_{t_\zeta \circ p} \alpha = \int_p t_\zeta^*(\alpha) = \int_p \alpha + \int_p d[w(\zeta)] = \int_p \alpha + w(\zeta)(Z)$, since $w(\zeta)(0_{\mathscr{H}}) = 0$. Hence, $\mu(Z) = d[w(Z)] - d[\zeta \mapsto w(\zeta)(Z)]$. But $w(\zeta)(Z) = -w(Z)(\zeta)$, then $\mu(Z) = d[w(Z)] - d[\zeta \mapsto -w(Z)(\zeta)] = 2d[w(Z)]$. Next, let Z be in the kernel of μ, thus $w(Z) = \text{cst} = w(0_{\mathscr{H}}) = 0$. But $w(Z)(Z') = 0$ for all $Z' \in \mathscr{H}$ if and only if $Z = 0_{\mathscr{H}}$,

we have just to decompose Z into real and imaginary parts and use the fact that the Hermitian norm on $\mathscr{H}$ is nondegenerate. Therefore, μ is injective.

2. Since the 1-form α is invariant by $\mathbf{U}(\mathscr{H})$, this statement is a direct application of (art. 9.11). ☐

9.34. The Virasoro coadjoint orbits. Let $\mathrm{Imm}(\mathrm{S}^1, \mathbf{R}^2)$ be the space of all the immersions of the circle $\mathrm{S}^1 = \mathbf{R}/2\pi\mathbf{Z}$ into $\mathbf{R}^2$, equipped with the functional diffeology.

1. For all integers n and all n-plots $\mathrm{P} : \mathrm{U} \to \mathrm{Imm}(\mathrm{S}^1, \mathbf{R}^2)$, let $\alpha(\mathrm{P})$ be the 1-form defined on U by

$$\alpha(\mathrm{P})_r(\delta r) = \int_0^{2\pi} \frac{1}{\|\mathrm{P}(r)'(t)\|^2} \left\langle \mathrm{P}(r)''(t) \,\middle|\, \frac{\partial \mathrm{P}(r)'(t)}{\partial r}(\delta r) \right\rangle dt\,,$$

where $r \in \mathrm{U}$ and $\delta r \in \mathbf{R}^n$. The prime denotes the derivative with respect to the parameter t, and the brackets $\langle \cdot \mid \cdot \rangle$ denote the ordinary scalar product. Then, α is a 1-form on $\mathrm{Imm}(\mathrm{S}^1, \mathbf{R}^2)$.

2. We consider now the group $\mathrm{Diff}_+(\mathrm{S}^1)$, of positive diffeomorphisms of the circle, and its action on $\mathrm{Imm}(\mathrm{S}^1, \mathbf{R}^2)$ by reparametrization. For all φ in $\mathrm{Diff}_+(\mathrm{S}^1)$, for all x in $\mathrm{Imm}(\mathrm{S}^1, \mathbf{R}^1)$, we denote by $\bar{\varphi}(x)$ the pushforward of x by φ, that is,

$$\bar{\varphi}(x) = \varphi_*(x) = x \circ \varphi^{-1}.$$

Let $\mathrm{F} : \mathrm{Diff}_+(\mathrm{S}^1) \to \mathscr{C}^\infty(\mathrm{Imm}(\mathrm{S}^1, \mathbf{R}^2), \mathbf{R})$ defined, for all $\varphi \in \mathrm{Diff}_+(\mathrm{S}^1)$, by

$$\mathrm{F}(\varphi) : x \mapsto \int_0^{2\pi} \log \|x'(t)\| \; d \log(\varphi'(t)).$$

Then, the map F is smooth and for every $\varphi \in \mathrm{Diff}(\mathrm{S}^1)$,

$$\bar{\varphi}^*(\alpha) = \alpha - d[\mathrm{F}(\varphi)].$$

It follows that the exact 2-form $\omega = d\alpha$, defined on $\mathrm{Imm}(\mathrm{S}^1, \mathbf{R}^2)$, is invariant by the action of $\mathrm{Diff}(\mathrm{S}^1)$. Moreover, the action of $\mathrm{Diff}(\mathrm{S}^1)$ is Hamiltonian.

3. Let $x_0 : \mathrm{class}(t) \mapsto (\cos(t), \sin(t))$ be the *standard immersion* from S^1 to $\mathbf{R}^2$. The moment maps for ω restricted to the connected component of $x_0 \in \mathrm{Imm}(\mathrm{S}^1, \mathbf{R}^2)$, relative to $\mathrm{Diff}_+(\mathrm{S}^1)$, are given by

$$\mu(x)(r \mapsto \varphi)_r(\delta r) = \int_0^{2\pi} \left\{ \frac{\|x''(u)\|^2}{\|x'(u)\|^2} - \frac{d^2}{du^2} \log \|x'(u)\|^2 \right\} \delta u \; du \; + \; \mathrm{cst},$$

where $r \mapsto \varphi$ is any plot of $\mathrm{Diff}_+(\mathrm{S}^1)$ defined on some n-domain U, r is a point of U, $\delta r \in \mathbf{R}^n$, $u = \varphi^{-1}(t)$, and $\delta u = \mathrm{D}(r \mapsto u)(r)(\delta r)$.

4. With the same conventions as in item 3, the Souriau cocycles of the $\mathrm{Diff}_+(\mathrm{S}^1)$ action on $\mathrm{Imm}(\mathrm{S}^1, \mathbf{R}^2)$ are cohomologous to

$$\theta(g)(r \mapsto \varphi)_r(\delta r) = \int_0^{2\pi} \frac{3\gamma''(u)^2 - 2\gamma'''(u)\gamma'(u)}{\gamma'(u)^2}\, \delta u\, du,$$

where $g \in \mathrm{Diff}_+(\mathrm{S}^1)$ and $\gamma = g^{-1}$. We recognize in the integrand of the right hand side the Schwartzian derivative of γ.

5. Let β be the function defined, for all g and h in $\mathrm{Diff}_+(\mathrm{S}^1)$, by

$$\beta(g,h) = \int_0^{2\pi} \log(g \circ h)'(t)\, d\log h'(t).$$

Then, for all g and h in $\mathrm{Diff}_+(\mathrm{S}^1)$,

$$\mathrm{F}(g \circ g') = \mathrm{F}(g) \circ \bar{g}' + \mathrm{F}(g') - \beta(g, g').$$

This function β is known as the *Bott cocycle* [Bot78]. The central extension of $\mathrm{Diff}_+(\mathrm{S}^1)$ by β is the Virasoro group. Its action on $\mathrm{Imm}(\mathrm{S}^1, \mathbf{R}^2)$, through $\mathrm{Diff}_+(\mathrm{S}^1)$, is still Hamiltonian, and now exact. This is a well known construction which will not be more developed here.

NOTE. This example, built on purpose [Igl95], gathers the main ingredients found in the literature on the construction of Virasoro's group. It illustrates the whole theory, by linking objects that originally appeared in disorder.

PROOF. The proof is actually a long and tedious series of computations. To make it as clear as possible, we shall split the computations into a few steps.

The 1*-form* α. We prove first that α is a well defined 1-form on $\mathrm{Imm}(\mathrm{S}^1, \mathbf{R}^2)$. Let $\mathrm{F} : \mathrm{U} \to \mathrm{U}$ be a smooth m-parametrization. Let $s \in \mathrm{V}$, $\delta s \in \mathbf{R}^m$. Denoting by r the point $\mathrm{F}(s)$, we have

$$\begin{aligned}
\alpha(\mathrm{P} \circ \mathrm{F})_s(\delta s) &= \int_0^{2\pi} \frac{1}{\|(\mathrm{P} \circ \mathrm{F})(s)'(t)\|^2} \left\langle (\mathrm{P} \circ \mathrm{F})(s)''(t) \,\middle|\, \frac{\partial (\mathrm{P} \circ \mathrm{F})(s)'(t)}{\partial s}(\delta s) \right\rangle dt \\
&= \int_0^{2\pi} \frac{1}{\|\mathrm{P}(\mathrm{F}(s))'(t)\|^2} \left\langle \mathrm{P}(\mathrm{F}(s))''(t) \,\middle|\, \frac{\partial \mathrm{P}(\mathrm{F}(s))'(t)}{\partial s}(\delta s) \right\rangle dt \\
&= \int_0^{2\pi} \frac{1}{\|\mathrm{P}(r)'(t)\|^2} \left\langle \mathrm{P}(r)''(t) \,\middle|\, \frac{\partial \mathrm{P}(r)'(t)}{\partial r}\left(\frac{\partial \mathrm{F}(s)}{\partial s}(\delta s) \right) \right\rangle dt \\
&= \alpha(\mathrm{P})_{r=\mathrm{F}(s)}\left(\frac{\partial \mathrm{F}(s)}{\partial s}(\delta s) \right) \\
&= \mathrm{F}^*(\alpha(\mathrm{P}))_s(\delta s).
\end{aligned}$$

Thus, $\alpha(\mathrm{P} \circ \mathrm{F}) = \mathrm{F}^*(\alpha(\mathrm{P}))$, α satisfies the compatibility condition and is then a differential 1-form on $\mathrm{Imm}(\mathrm{S}^1, \mathbf{R}^2)$.

Let us consider now the action of $\mathrm{Diff}_+(\mathrm{S}^1)$ on $\mathrm{Imm}(\mathrm{S}^1, \mathbf{R}^2)$. This action is obviously smooth from the very definition of the functional diffeology of $\mathrm{Diff}_+(\mathrm{S}^1)$. Let us denote φ^{-1} by ϕ such that

$$\bar{\varphi}^*(\alpha)(\mathrm{P}) = \alpha(\bar{\varphi} \circ \mathrm{P}) = \alpha[r \mapsto \mathrm{P}(r) \circ \varphi^{-1}] = \alpha[r \mapsto \mathrm{P}(r) \circ \phi].$$

Note that $\mathrm{Diff}_+(\mathrm{S}^1)$ acts on *speed* and *acceleration* of any immersion x, by

$$\begin{aligned} (x \circ \phi)'(t) &= x'(\phi(t)) \cdot \phi'(t), \\ (x \circ \phi)''(t) &= x''(\phi(t)) \cdot \phi'(t)^2 + x'(\phi(t)) \cdot \phi''(t). \end{aligned} \tag{$\heartsuit$}$$

Let us denote by Q the plot $\bar{\varphi} \circ \mathrm{P}$, that is, $\mathrm{Q} = [r \mapsto \mathrm{P}(r) \circ \phi]$, such that

$$\alpha(\bar{\varphi} \circ \mathrm{P})_r(\delta r) = \int_0^{2\pi} \frac{1}{\|\mathrm{Q}(r)'(t)\|^2} \left\langle \mathrm{Q}(r)''(t) \,\middle|\, \frac{\partial \mathrm{Q}(r)'(t)}{\partial r}(\delta r) \right\rangle dt$$

for all $r \in \mathrm{U}$ and all $\delta r \in \mathbf{R}^n$. Now, from ($\heartsuit$) we have

$$\begin{aligned} \mathrm{Q}(r)'(t) &= (\mathrm{P}(r) \circ \phi)'(t) = \mathrm{P}(r)'(\phi(t)) \cdot \phi'(t), \\ \mathrm{Q}(r)''(t) &= (\mathrm{P}(r) \circ \phi)''(t) = \mathrm{P}(r)''(\phi(t)) \cdot \phi'(t)^2 + \mathrm{P}(r)'(\phi(t)) \cdot \phi''(t). \end{aligned}$$

Let us write $\alpha(\bar{\varphi} \circ \mathrm{P})_r(\delta r)$ according to this decomposition,

$$\alpha(\bar{\varphi} \circ \mathrm{P})_r(\delta r) = \int_0^{2\pi} \mathrm{A}\, dt + \int_0^{2\pi} \mathrm{B}\, dt.$$

One has first,

$$\mathrm{A} = \frac{1}{\|\mathrm{P}(r)'(\phi(t)) \cdot \phi'(t)\|^2} \left\langle \mathrm{P}(r)''(\phi(t)) \cdot \phi'(t)^2 \,\middle|\, \frac{\partial \mathrm{P}(r)'(\phi(t)) \cdot \phi'(t)}{\partial r}(\delta r) \right\rangle,$$

that is,

$$\mathrm{A} = \frac{1}{\|\mathrm{P}(r)'(\phi(t))\|^2} \left\langle \mathrm{P}(r)''(\phi(t)) \,\middle|\, \frac{\partial \mathrm{P}(r)'(\phi(t))}{\partial r}(\delta r) \right\rangle \phi'(t).$$

Since φ, and thus ϕ, is a positive diffeomorphism, after the change of variable $t \mapsto \phi(t)$ under the integral, we get already

$$\int_0^{2\pi} \mathrm{A}\, dt = \alpha(\mathrm{P})_r(\delta r).$$

Next,

$$\mathrm{B} = \frac{1}{\|\mathrm{P}(r)'(\phi(t)) \cdot \phi'(t)\|^2} \left\langle \mathrm{P}(r)'(\phi(t)) \cdot \phi''(t) \,\middle|\, \frac{\partial \mathrm{P}(r)'(\phi(t)) \cdot \phi'(t)}{\partial r}(\delta r) \right\rangle,$$

then,

$$\int_0^{2\pi} \mathrm{B}\, dt = \int_0^{2\pi} \frac{1}{\|\mathrm{P}(r)'(\phi(t))\|^2} \left\langle \mathrm{P}(r)'(\phi(t)) \,\middle|\, \frac{\partial \mathrm{P}(r)'(\phi(t))}{\partial r}(\delta r) \right\rangle \frac{\phi''(t)}{\phi'(t)}\, dt.$$

Let us then denote for short,

$$x = \mathrm{P}(r),\ x' = \mathrm{P}(r)',\ \text{and}\ \delta x' = \left[t \mapsto \frac{\partial \mathrm{P}'(r)(t)}{\partial r}(\delta r)\right].$$

Using that, for any variation δ, we have the identities

$$\delta\|v\| = \frac{1}{\|v\|}\langle v \mid \delta v\rangle \text{ and } \delta \log\|v\| = \frac{1}{\|v\|}\delta\|v\| = \frac{1}{\|v\|^2}\langle v \mid \delta v\rangle,$$

we get, with a change of variable $s = \varphi^{-1}(t)$ in the middle,

$$\begin{aligned}
\int_0^{2\pi} \mathrm{B}\,dt &= \int_0^{2\pi} \frac{1}{\|x'(\phi(t))\|^2}\langle x'(\phi(t)) \mid \delta x'(\phi(t))\rangle \frac{\phi''(t)}{\phi'(t)}\,dt \\
&= \int_0^{2\pi} \delta \log\|x'(\phi(t))\|\, d\log(\phi'(t)) \\
&= \delta\int_0^{2\pi} \log\|x'(\phi(t))\|\, d\log(\phi'(t)) \\
&= \delta\int_0^{2\pi} \log\|x'(\varphi^{-1}(t))\|\, d\log((\varphi^{-1})'(t)) \\
&= \delta\int_0^{2\pi} \log\|x'(s)\|\, d\log[(\varphi^{-1})'(\varphi(s))] \\
&= -\,\delta\int_0^{2\pi} \log\|x'(s)\|\, d\log(\varphi'(s)) \\
&= -\frac{\partial}{\partial r}\left\{\int_0^{2\pi} \log\|\mathrm{P}(r)'(s)\|\, d\log(\varphi'(s))\right\}(\delta r) \\
&= \int_0^{2\pi} \delta \log\|x'(\phi(t))\|\, d\log(\phi'(t)) \\
&= -\frac{\partial}{\partial r}\Big\{\mathrm{F}(\varphi)(\mathrm{P}(r))\Big\}(\delta r) \\
&= -\,d[\mathrm{F}(\varphi)](\mathrm{P})_r(\delta r).
\end{aligned}$$

Coming back to $\alpha(\bar\varphi \circ \mathrm{P})_r(\delta r)$, we get finally

$$\alpha(\bar\varphi \circ \mathrm{P})_r(\delta r) = \alpha(\mathrm{P})_r(\delta r) - d[\mathrm{F}(\varphi)](\mathrm{P})_r(\delta r),$$

that is,

$$\bar\varphi^*(\alpha) = \alpha - d[\mathrm{F}(\varphi)].$$

Hence, the exterior derivative $\omega = d\alpha$ is invariant by the action of $\mathrm{Diff}_+(\mathrm{S}^1)$, and since the difference $\bar\varphi^*(\alpha) - \alpha$ is exact, this action is Hamiltonian.

The 2*-points moment map.* Now, let us compute the 2-points moment map ψ of the action of $\mathrm{Diff}_+(\mathrm{S}^1)$ on $(\mathrm{Imm}(\mathrm{S}^1, \mathbf{R}^2), \omega)$. Let p be a path connecting two immersions

x_0 and x_1. Thus, $\Psi(p) = \hat{p}^*(\mathcal{K}\omega) = \hat{p}^*(\mathcal{K} d\alpha) = \hat{p}^*(\hat{1}^*(\alpha) - \hat{0}^*(\alpha) - d(\mathcal{K}\alpha)) = \hat{x}_1^*(\alpha) - \hat{x}_0^*(\alpha) - d(\mathcal{K}\alpha \circ \hat{p})$. But, for all $\varphi \in \mathrm{Diff}_+(\mathrm{S}^1)$,

$$\mathcal{K}\alpha \circ \hat{p}(\varphi) = \int_{\bar{\varphi}(p)} \alpha = \int_p \bar{\varphi}^*(\alpha)$$
$$= \int_p \alpha - \int_p d\mathrm{F}(\varphi) = \int_p \alpha - \mathrm{F}(\varphi)(x_1) + \mathrm{F}(\varphi)(x_0).$$

We get from there

$$\begin{aligned} \Psi(p) &= \psi(x_0, x_1) \\ &= \{\hat{x}_1^*(\alpha) + d[\varphi \mapsto \mathrm{F}(\varphi)(x_1)]\} - \{\hat{x}_0^*(\alpha) + d[\varphi \mapsto \mathrm{F}(\varphi)(x_0)]\}. \end{aligned}$$

But note that $\hat{x}^*(\alpha) + d[\varphi \mapsto \mathrm{F}(\varphi)(x)]$ is not a momentum of $\mathrm{Diff}_+(\mathrm{S}^1)$.

The 1*-point moment maps*. Let us compute the moment map $\psi(x_0, x)$. Let

$$m = \{\hat{x}^*(\alpha) + d[\varphi \mapsto \mathrm{F}(\varphi)(x)]\}(r \mapsto \varphi)_r(\delta r).$$

Let us denote for short

$$\begin{aligned} \mathrm{A} &= \hat{x}^*(\alpha)(r \mapsto \varphi)_r(\delta r), \\ \mathrm{B} &= d[\varphi \mapsto \mathrm{F}(\varphi)(x)](r \mapsto \varphi)_r(\delta r) = \frac{\partial \mathrm{F}(\varphi)(x)}{\partial r}\delta r. \end{aligned}$$

We shall use the notation m_0, A_0 and B_0 for the immersion x_0, thus

$$\psi(x_0, x)(r \mapsto \varphi)_r(\delta r) = m - m_0 = \mathrm{A} + \mathrm{B} - \mathrm{A}_0 - \mathrm{B}_0.$$

But $\hat{x}^*(\alpha)(r \mapsto \varphi) = \alpha(\hat{x} \circ [r \mapsto \varphi]) = \alpha(r \mapsto x \circ \varphi^{-1})$, then let $\phi = \varphi^{-1}$, we get

$$\mathrm{A} = \int_0^{2\pi} \frac{1}{\|(x\circ\phi)'(t)\|^2} \left\langle (x\circ\phi)''(t) \,\middle|\, \frac{\partial (x\circ\phi)'(t)}{\partial r}(\delta r) \right\rangle.$$

Let us now introduce

$$u = \phi(t),\ u' = \phi'(t),\ \text{and } u'' = \phi''(t).$$

Then, the decomposition given by (♡) writes

$$(x\circ\phi)'(t) = x'(u)\cdot u' \text{ and } (x\circ\phi)''(t) = x''(u)\cdot u'^2 + x'(u)\cdot u''.$$

Next, let us use the prefix δ for the various variations associated with δr, that is, $\delta\star = \mathrm{D}(r \mapsto \star)(r)(\delta r)$. Then,

$$\frac{\partial (x\circ\phi)'(t)}{\partial r}(\delta r) = \delta[x'(u)\cdot u'] = x''(u)\cdot\delta u\cdot u' + x'(u)\cdot\delta u'.$$

Thus,

$$\mathrm{A} = \int_0^{2\pi} \frac{1}{\|x'(u)\|^2 u'^2} \langle x''(u)u'^2 + x'(u)u'' \mid x''(u)u'\delta u + x'(u)\delta u'\rangle \, dt$$

$$= \int_0^{2\pi} \frac{\|x''(u)\|^2}{\|x'(u)\|^2}\, \delta u\, u'\, dt + \int_0^{2\pi} \frac{\langle x'(u), x''(u)\rangle}{\|x'(u)\|^2}\Big[\delta u' + \frac{u''}{u'}\delta u\Big] dt$$
$$+ \int_0^{2\pi} \frac{u''}{u'}\delta u'\, dt\,.$$

Now,

$$\mathrm{B} = \frac{\partial \mathrm{F}(\varphi)(x)}{\partial r}\delta r = -\frac{\partial \bar{\mathrm{F}}(\phi)(x)}{\partial r}\delta r = -\delta[\bar{\mathrm{F}}(\phi)(x)],$$

with

$$\bar{\mathrm{F}}(\phi)(x) = \int_0^{2\pi} \log\|x'(\phi(t))\|\; d\log\phi'(t) = \int_0^{2\pi} \log\|x'(u)\|\; d\log(u').$$

Then, after the variation with respect to δr and an integration by parts, we get

$$\mathrm{B} = -\int_0^{2\pi} \frac{\langle x'(u), x''(u)\rangle}{\|x'(u)\|^2}\delta u \frac{u''}{u'}\, dt - \int_0^{2\pi} \log\|x'(u)\|\; \delta d\log(u')$$
$$= -\int_0^{2\pi} \frac{\langle x'(u), x''(u)\rangle}{\|x'(u)\|^2}\delta u \frac{u''}{u'}\, dt + \int_0^{2\pi} \frac{\langle x'(u), x''(u)\rangle}{\|x'(u)\|^2} u'\; \delta\log(u')\, dt$$
$$= -\int_0^{2\pi} \frac{\langle x'(u), x''(u)\rangle}{\|x'(u)\|^2}\delta u \frac{u''}{u'}\, dt + \int_0^{2\pi} \frac{\langle x'(u), x''(u)\rangle}{\|x'(u)\|^2}\; \delta u'\, dt\,.$$

Therefore, grouping the terms and integrating again by parts, we get

$$\mathrm{A}+\mathrm{B} = \int_0^{2\pi} \frac{\|x''(u)\|^2}{\|x'(u)\|^2}\delta u\, du + 2\int_0^{2\pi} \frac{\langle x'(u), x''(u)\rangle}{\|x'(u)\|^2}\delta u'\, dt + \int_0^{2\pi} \frac{u''}{u'}\delta u'\, dt$$
$$= \int_0^{2\pi} \frac{\|x''(u)\|^2}{\|x'(u)\|^2}\delta u\, du - 2\int_0^{2\pi} \frac{d^2}{du^2}\log\|x'(u)\|\delta u\, du + \int_0^{2\pi} \frac{u''}{u'}\; \delta u'\, dt$$
$$= \int_0^{2\pi} \left\{\frac{\|x''(u)\|^2}{\|x'(u)\|^2} - \frac{d^2}{du^2}\log\|x'(u)\|^2\right\}\delta u\, du + \int_0^{2\pi} \frac{u''}{u'}\; \delta u'\, dt\,.$$

Now, since $\|x_0'(t)\| = 1$ we get the value of the 2-points moment map,

$$\begin{aligned}\psi(x_0, x)(r \mapsto \varphi)_r(\delta r) \;&=\; \int_0^{2\pi} \left\{\frac{\|x''(u)\|^2}{\|x'(u)\|^2} - \frac{d^2}{du^2}\log\|x'(u)\|^2\right\}\delta u\, du \\ &\quad - \int_0^{2\pi} \delta u\, du\,.\end{aligned}$$

The second term of the right hand side of this equality is a constant momentum of $\mathrm{Diff}_+(\mathrm{S}^1)$, so it can be forgottent. Thus, every moment map, up to a constant, is equal to this moment μ.

Souriau cocycle. The Souriau cocycle associated with the immersion x_0 is defined by $\theta(g) = \psi(x_0, \bar{g}(x_0))$; see (art. 9.10). We replace then, in the expression of ψ above,

x by $\bar{g}(x_0) = x_0 \circ g^{-1}$, that is, $x = x_0 \circ \gamma$, and $\theta(g)(r \mapsto \varphi)_r(\delta r) = \psi(x_0, x_0 \circ \gamma)$. Let us note next that

$$(x_0 \circ \gamma)'(u) = x_0'(\gamma(u))\gamma'(u) \text{ and } (x_0 \circ \gamma)''(u) = x_0''(\gamma(u))\gamma'(u)^2 + x_0'(u)\gamma''(u),$$

and let us recall that $\|x_0'\| = \|x_0''\| = 1$, and that $\langle x_0' \mid x_0'' \rangle = 0$. We get,

$$\|x'(u)\|^2 = \gamma'(u)^2 \text{ and } \|x''(u)\|^2 = \gamma'(u)^4 + \gamma''(u)^2,$$

which gives

$$\begin{aligned} \frac{\|x''(u)\|^2}{\|x'(u)\|^2} &= \gamma'(u)^2 + \frac{\gamma''(u)^2}{\gamma'(u)^2}, \\ \frac{d^2}{du^2} \log \|x'(u)\|^2 &= 2\frac{\gamma'''(u)\gamma'(u) - \gamma''(u)^2}{\gamma''(u)^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \theta(g)(r \mapsto \varphi)_r(\delta r) &= \int_0^{2\pi} \frac{3\gamma''(u)^2 - 2\gamma'''(u)\gamma'(u)}{\gamma'(u)^2}\, \delta u\, du \\ &+ \int_0^{2\pi} \gamma'(u)^2\, \delta u\, du - \int_0^{2\pi} \delta u\, du\,. \end{aligned}$$

But, after a change of variable $u \mapsto v = \gamma(u)$, we get

$$\int_0^{2\pi} \gamma'(u)^2\, \delta u\, du = \int_0^{2\pi} (\delta u \gamma'(u))\, \gamma'(u)\, du = \int_0^{2\pi} \delta v\ dv.$$

Then, the two last terms cancel each other, and we get the value claimed in the proposition for the Souriau cocycle θ.

Bott's cocycle. The real function $\mathrm{F}(g \circ h) - \mathrm{F}(g) \circ \bar{h} - \mathrm{F}(h)$ is constant since X is connected, and its differential is equal to $(\bar{g} \circ \bar{h})^*(\alpha) - \bar{h}^*(\bar{g}^*(\alpha))$, that is, 0. Now, to make $\beta(g, g') = \mathrm{F}(g) \circ \bar{g}' + \mathrm{F}(g') - \beta(g, g') - \mathrm{F}(g \circ g')$ explicit, it is sufficient to compute the right hand member on the standard immersion x_0, for which the speed norm is equal to 1, and thus $\log \|x'(t)\| = 0$ for all t. We get then

$$\begin{aligned} \beta(g,h) &= \mathrm{F}(g)(x_0 \circ h^{-1}) - \mathrm{F}(h)(x_0) - \mathrm{F}(g \circ h)(x_0) \\ &= +\int_0^{2\pi} \log \|(x_0 \circ h^{-1})'(t)\|\ d \log g'(t) \\ &= +\int_0^{2\pi} \log (h^{-1})'(t)\ d \log g'(t) \\ &= -\int_0^{2\pi} \log h'(h^{-1}(t))\ d \log g'(t) \\ &= -\int_0^{2\pi} \log h'(s)\ d \log g'(h(s)) \end{aligned}$$

$$= \quad + \int_0^{2\pi} \log(g \circ h)'(t)\, d\log h'(t).$$

And this is the standard expression of Bott's cocycle. □

Exercise

✎ EXERCISE 147 (The moment of imprimitivity). Let X be a diffeological space. Let $\Omega^1(\mathrm{X})$ be the vector space of 1-forms of X, equipped with the functional diffeology (art. 6.45). Let Taut be the tautological 1-form defined on $\mathrm{X} \times \Omega^1(\mathrm{X})$ and let Liouv be the Liouville 1-form defined on the cotangent bundle $\mathrm{T}^*\mathrm{X}$; see (art. 6.48) and (art. 6.49). Let us consider then the additive diffeological group of smooth functions $\mathscr{C}^\infty(\mathrm{X}, \mathbf{R})$, acting smoothly on $\mathrm{X} \times \Omega^1(\mathrm{X})$ by *right action*,

$$\bar{f} : (x, \alpha) \mapsto (x, \alpha - df)$$

for all $f \in \mathscr{C}^\infty(\mathrm{X}, \mathbf{R})$ and $(x, \alpha) \in \mathrm{X} \times \Omega^1(\mathrm{X})$. This action has a natural projection on the cotangent $\mathrm{T}^*\mathrm{X}$, and this action will be denoted the same way,

$$\bar{f} : (x, a) \mapsto (x, a - df(x))$$

for all $f \in \mathscr{C}^\infty(\mathrm{X}, \mathbf{R})$ and $(x, a) \in \mathrm{T}^*\mathrm{X}$.

1) Show that, for all $f \in \mathscr{C}^\infty(\mathrm{X}, \mathrm{R})$, the variance of the tautological form and the Liouville form are given by

$$\bar{f}^*(\mathrm{Taut}) = \mathrm{Taut} - \mathrm{pr}_1^*(df) \text{ and } \bar{f}^*(\mathrm{Liouv}) = \mathrm{Liouv} - \pi^*(df).$$

Observe that the exterior derivatives $d\mathrm{Taut}$ and $\omega = d\mathrm{Liouv}$ are invariant by the action of $\mathscr{C}^\infty(\mathrm{X}, \mathbf{R})$.

2) Let p be a path in $\mathrm{T}^*\mathrm{X}$, connecting $(x_0, a_0) = p(0)$ to $(x_1, a_1) = p(1)$. Show that the paths moment map Ψ and the 2-points moment map ψ, with respect to the 2-form $\omega = d\mathrm{Liouv}$, are given by

$$\Psi(p) = \psi((x_0, a_0), (x_1, a_1)) = d[f \mapsto f(x_1)] - d[f \mapsto f(x_0)].$$

3) Check that, for all $x \in \mathrm{X}$, the real function $[f \mapsto f(x)]$ is smooth. We call it the *Dirac function* of the point x, and we denote it by δ_x.

$$\delta_x = [f \mapsto f(x)] \in \mathscr{C}^\infty(\mathscr{C}^\infty(\mathrm{X}, \mathbf{R}), \mathbf{R}).$$

Show that the differential $d\delta_x = d[f \mapsto f(x)]$ is an invariant 1-form[14] of the additive group $\mathscr{C}^\infty(\mathrm{X}, \mathbf{R})$. Show that every moment map of the action of $\mathscr{C}^\infty(\mathrm{X}, \mathrm{R})$ on $\mathrm{T}^*\mathrm{X}$ is, up to a constant, equal to the invariant moment map

$$\mu : (x, a) \mapsto d\delta_x = d[f \mapsto f(x)].$$

[14]This differential has nothing to do with the derivative of the Dirac distributions in the sense of De Rham's currents.

Note that the moment μ is constant on the fibers $T_x^*X = \pi^{-1}(x)$, and if the real smooth functions separate[15] the points of X, then the image of the moment map μ is the space X, identified with the space of Dirac's functions.

4) Show that the action of $\mathscr{C}^\infty(X, \mathbf{R})$ on (T^*X, ω) is Hamiltonian and exact, that is, $\Gamma = \{0\}$ and $\sigma = 0$.

This example, in the case of differential manifolds, appears informally in Ziegler's construction of a symplectic analogue for *systems of imprimitivity* in representation theory [Zie96]. It is why the moment map μ may be called the *moment of imprimitivity*. The diffeological framework then gives to it a full formal status and even extends it.

Discussion on Symplectic Diffeology

Symplectic diffeology is more a program than a complete theory. The last decades of theoretical research in mechanics — in completely integrable systems, quantum mechanics, or quantum field theory etc. — have shown a special interest in structures which seem to be symplectic even if they do not live on manifolds but on spaces, generally infinite dimensional, where the formal constructions of symplectic geometry do not apply as is. Building a formal framework for these symplectic-like structures is not just a desire of formalism, but a need to embed these heuristic constructions in a well delimited and workable mathematical construction. There are different ways to approach these problems, such as functional analysis, infinite dimensional manifolds à la Banach, or maybe others. A diffeological approach is one of them, but has the virtue of involving a very light apparatus of mathematical tools. Axiomatics, reduced to only three axioms, cannot be simpler and the great stability of the category, under set theoretic operations, is a gift for this kind of problem, where infinite dimensions and what is admitted to be considered as singularities are a burden for classical geometry. On the other hand, the light structure of diffeology does not seem to be a weakness. The construction of the moment maps, associated with a closed 2-form on any diffeological space, shows the whole — and deals correctly with the — complexity of the various situations without exaggerating technicalities. This is clearly shown in the few examples given in the previous sections of this chapter. So, even if it seems clear that diffeology is adequate for such a generalization of symplectic (or presymplectic) geometry, we still need a good definition, or at least a serious discussion, about *what is a symplectic diffeological space*? Or *what does it mean to be symplectic in diffeology*?

A relatively good answer has been given at least in the case of homogeneous spaces, that is, in the case of a closed 2-form ω defined on a space X homogeneous

[15] which means, $f(x) = f(x')$ for all smooth real functions f if and only if $x = x'$.

for its group of automorphisms $\mathrm{Diff}(X, \omega)$; see (art. 9.18) and (art. 9.19). In this case the situation is clear: a universal moment map distinguishes between being *presymplectic* or *symplectic*. The homogeneous space (X, ω) is symplectic if a universal moment map μ_ω is a covering onto its image, or, which is equivalent, if the preimages of its values are diffeologically discrete. Otherwise it is presymplectic, and the characteristics of μ_ω can be regarded as the characteristics of the 2-form ω, by analogy with what happens in the special case of homogeneous manifolds (art. 9.26). The symplectic homogeneous case has been illustrated, in particular, by the Hilbert space $\mathscr{H}$ or the infinite dimensional projective space $\mathscr{P}$ (art. 9.33), even by the singular irrational tori (art. 9.30). The fact that these spaces are symplectic is now a well defined property, according to the definition given in (art. 9.19).

Why is the homogeneous case so important? Because for a closed 2-form ω defined on a manifold M, being symplectic implies to be homogeneous under $\mathrm{Diff}(M, \omega)$, with moreover the universal moment maps μ_ω injective (art. 9.23). The homogeneity of M under the action of $\mathrm{Diff}(M, \omega)$ is a strong consequence of two things: firstly ω is invertible, has no kernel; and secondly, every symplectic manifold is locally flat, that is, looks locally like some $(\mathbf{R}^{2n}, \omega_{\mathrm{st}})$. There exists only one local model of symplectic manifold in each dimension, this is the famous Darboux theorem.

In diffeology, we have not a unique model to propose for all the covered cases: from singular tori to infinite projective space. But we can replace advantageously the local model of symplectic manifolds by the strong requirement of homogeneity. We remark that we could replace the global homogeneity by a local homogeneity, but this would lead to unwanted subtleties, for now. Our question is then: *Do we want to preserve the fundamental property of homogeneity for symplectic diffeological spaces?* The answer to this question is not that obvious.

If we want to preserve the property of homogeneity, the problem is just solved by (art. 9.19). But the example of the cone orbifold $\mathscr{C}_m$ (art. 9.32) is a real cultural or sociological obstacle. Many mathematicians want to consider the cone orbifold, equipped with the 2-form described in (art. 9.32), as symplectic, because it is the quotient of the symplectic space $(\mathbf{R}^2, \omega_{\mathrm{st}})$ by a finite group preserving the standard symplectic form ω_{st}. Unfortunately the cone orbifold is not homogeneous, the origin is fixed by every diffeomorphism. However, even according to diffeology, regarding $\mathscr{C}_m$ as a symplectic diffeological space makes sense: the cone orbifold is generated by *symplectic plots* — in this case the only plot $\pi_m : \mathbf{R}^2 \to \mathscr{C}_m$. Let us remark that the corner orbifold (art. 9.31) does not satisfy this property, and cannot be symplectic even according to this definition. Yael Karshon suggested, in private discussions, to define symplectic diffeological spaces as diffeological spaces generated by symplectic plots,[16] that is, plots P such that $\omega(P)$ is symplectic.

[16] I would be tempted to call these spaces, *quasi symplectic*.

		Moment map	
		injective	non injective
Automorphisms	transitive	• Irrational tori • Infinite projective space • Symplectic manifolds	• Infinite Hilbert sphere • Homogeneous presymplectic manifolds
	intransitive	• Cone orbifold • $\mathbf{R}^2$, $\omega = (x^2+y^2)dx\wedge dy$	• Corner orbifold • Almost everything

FIGURE 9.3. Distribution of examples.

Although this makes sense, it is a way we have not explored yet. In particular, the relationship between such a space and the moment maps needs to be clarified, as well as the relationship with the action of the group of automorphisms and the nature of its orbits. The idea behind this is to consider symplectic reductions as symplectic spaces independently of the regularity of the distribution of the moment map strata.

Another approach should consist of only requiring the universal moment maps to be injective or at least to have discrete preimages, since the preimages of a universal moment map seem to realize the characteristics of ω. Let us note that the cone orbifold satisfies this property, but not the corner orbifold, which is satisfactory. Unfortunately, the space $\mathbf{R}^2$ equipped with the 2-form $\omega = (x^2 + y^2)dx \wedge dy$ also satisfies this property, and it is clearly not symplectic. It seems that we have to either abandon this approach, or to go deeper into it.

So, what remains? If we want to include the cone orbifold in the category of symplectic diffeological spaces we have to give up homogeneity by the group of automorphisms. Or, if we do not want to give up homogeneity, we abandon the cone orbifold and certainly many other examples of diffeological spaces, generated by symplectic plots. For now, the set of examples and theorems in symplectic diffeology is not large enough to make a reasonable choice. It is why there is no general definition of symplectic diffeological spaces here. But, we can just deal with these spaces, equipped with closed 2-forms, by trying to get the maximum of information as we did in the examples above, sumarized in the Figure 9.3: the nature of the characteristics of the universal moment map, the injectivity or not of the universal moment map, the transitivity of the group of automorphisms or the nature of its orbits, the computation of the universal holonomy, and the Souriau class, etc. We have all the tools needed to treat the examples given by the literature or by the physicists, and maybe to define what the word symplectic means in diffeology is not that important after all…

A few more words. Why do we need the 2-form to be nondegenerate? From a physicist's point of view, we do not need that, actually it is the opposite. A pair (X, ω), with ω a closed 2-form, represents what we could call a *dynamical structure*. The *characteristics* of ω, that is, the connected components of the preimages of the universal moment map, describe the dynamics of the system, together with the partition into orbits by the group of automorphisms. This is what physicists are interested in: they want equations describing the evolution of their systems. The fact that, in classical mechanics, the space of characteristics is symplectic is fortuitous, a consequence of the presymplectic nature of the dynamical structures involved.[17] Then, the group of automorphisms of the space of characteristics is transitive and, by equivariance, the image of the moment map is one coadjoint orbit. In the general case of infinite dimension spaces, orbifolds, singular reductions etc., there is no reason for the automorphisms to be transitive, and the image of the universal moment map is just some union of coadjoint orbits, and then, maybe, not *symplectic*. The picture is clear, and that is what we have to deal with. Another critical question is of the *symplectic reduction* of a subspace $W \subset X$: the reduced space may be simply defined as the space of characteristics of the dynamical 2-form ω restricted to W, or its regularization, that is, the image of the universal moment map of the restriction. It is actually not different from the general case where the subspace coincides with the whole space. The task is then to describe this image and what, from the 2-form, passes to the quotient or to the image? For the structure of the space of characteristics we get two diffeologies: the first one is the natural quotient diffeology on the space of characteristics, the second one is the diffeology induced on the image of the universal moment map by the ambient space of momenta. They may coincide or not, but they both have their role to play.

[17]However, as Lagrange showed, the fact that the motion space is symplectic is important for the perturbations calculus in mechanics, see [Igl98].