Diffeology of Manifolds with Boundary

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Abstract

We show that smooth manifolds with boundary are diffeological spaces modeled on half-spaces, equipped with the subset diffeology.

Introduction

First of all, we show that every real local smooth map, defined in a half-space $\mathbf{H}_n \subset \mathbf{R}^n$ equipped with the subset diffeology, is the restriction of an ordinary real smooth map. This gives a characterization of the local diffeomorphisms of $\mathbf{H}_n$ as restrictions of ordinary local étale maps of $\mathbf{R}^n$. We apply these lemma to show that any n-manifold M with boundary is locally diffeomorphic at each point to the half-space $\mathbf{H}_n$. And conversely, any diffeological space which is locally diffeomorphic at each point to $\mathbf{H}_n$ is naturally a n-manifold with boundary. In brief, manifolds with boundary are just diffeological spaces modeled on half-spaces, equipped with the subset diffeology.

We consider the elementary examples of the half-line $\Delta_\infty = [0, +\infty[$, the closed or semi-closed intervals, and we compare Δ_∞ with the family of half-lines

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$\Delta_n = \mathbf{R}^n/\mathrm{O}(n)$, identified with the segment $[0, +\infty[$ thanks to the square of the norm. We show that, first of all $\mathcal{C}^\infty(\Delta_n, \mathbf{R}^p) = \mathcal{C}^\infty(\Delta_\infty, \mathbf{R}^p)$, and secondly that $\mathrm{Diff}(\Delta_\mathrm{n}) = \mathrm{Diff}(\Delta_\infty)$. This result, together with non-equivalence of these various half-lines [Piz07], confirms — if it was necessary — that the diffeology of quotients, even simple ones, is richer than their « contravariant » differential structure, defined by their real smooth maps.

In the first section we remember the basic construction in diffeology used here. For more details see [Piz05].

Thanks — I am grateful to professor Hans Duistermaat who drew my attention to the property stated in Subsection 4.4.

1 Diffeology

1.1 Subset diffeology Let A be any subset of a diffeological space X, the *subset diffeology* of A, induced (or inherited) by X, is made of all the plots of X whose take their values in A.

1.2 D-Topology Let X be a diffeological space. The finest topology on X such that the plots are continuous is called *D-Topology*. A subset A of X is *D-open*, that is open for the D-topology, if and only if for every n, for every n-plot P of X, the pullback $\mathrm{P}^{-1}(\mathrm{A})$ is open in $\mathbf{R}^n$.

1.3 Local smooth maps Let X and X′ be two diffeological spaces, and let $\mathrm{A} \subset \mathrm{X}$. We say that $f : \mathrm{A} \longrightarrow \mathrm{X}'$ is a *local smooth map from* X *to* X′*, defined on* A, if for every plot P of X, $f \circ \mathrm{P}$ is a plot of X′. This condition implies in particular that A is open for the D-topology.

1.4 Local diffeomorphisms Let X and X′ be two diffeological spaces, and let $\mathrm{A} \subset \mathrm{X}$. We say that $f : \mathrm{A} \longrightarrow \mathrm{X}'$ is a *local diffeomorphism from* X *to* X′ *defined on* A, if f is injective, and if f is a local smooth map as well as $f^{-1} : f(\mathrm{A}) \longrightarrow \mathrm{X}$. This is equivalent to say that A and $f(\mathrm{A})$ are D-open and that $f \restriction \mathrm{A}$ is a diffeomorphism from A to $f(\mathrm{A})$, where A and $f(\mathrm{A})$ are equipped with the subset diffeology.

1.5 Modeling spaces Let $\mathcal{M}$ be a diffeological space, we say that a space X is *modeled on* $\mathcal{M}$ if for every point x of X there exists a local diffeomorphism $\mathrm{F} : \mathrm{A} \longrightarrow \mathrm{X}$, where $\mathrm{A} \subset \mathcal{M}$ such that $x \in \mathrm{F}(\mathrm{A})$. So, the diffeology of X is generated by a family of plots writing $\mathrm{F} \circ \mathrm{P}$, where F is a local diffeomorphism from $\mathcal{M}$ to X, and P is a plot of $\mathcal{M}$.

2 The diffeology of half-spaces

2.1 Half-spaces We denote by $\mathbf{H}_n$ the *standard half-space* of $\mathbf{R}^n$, that is the set of points $x = (r, t) \in \mathbf{R}^n$ such that $r \in \mathbf{R}^{n-1}$ and $t \in [0, +\infty[$, and by $\partial\mathbf{H}_n$ its boundary $\mathbf{R}^{n-1} \times \{0\}$. The subset diffeology of $\mathbf{H}_n$, inherited from $\mathbf{R}^n$, is made of all the smooth parametrizations $\mathrm{P} : \mathrm{U} \longrightarrow \mathbf{R}^n$ such that $\mathrm{P}_n(r) \geq 0$ for all $r \in \mathrm{U}$, $\mathrm{P}_n(r)$ being the n-th coordinate of $\mathrm{P}(r)$. The D-topology of $\mathbf{H}_n$ is the usual topology defined by its inclusion into $\mathbf{R}^n$.

2.2 Smooth real maps from half-spaces A map $f : \mathbf{H}_n \longrightarrow \mathbf{R}^p$ is smooth for the subset diffeology of $\mathbf{H}_n$ if and only if there exists an ordinary smooth map F, defined on an open superset of $\mathbf{H}_n$, such that $f = \mathrm{F} \restriction \mathbf{H}_n$. Actually, there exists such an F defined on the whole $\mathbf{R}^n$.

Remark — As an immediate corollary, any map f defined on $\mathcal{C} \times [0, \varepsilon[$ to $\mathbf{R}^p$, were $\mathcal{C}$ is an open cube of $\partial\mathbf{H}_n$ centered at some point $(r, 0)$, smooth for the subset diffeology, is the restriction of a smooth map $\mathrm{F} : \mathcal{C} \times]-\varepsilon, +\varepsilon[\longrightarrow \mathbf{R}^p$.

Proof – First of all, if f is the restriction of a smooth map $\mathrm{F} : \mathbf{R}^n \longrightarrow \mathbf{R}^p$, it is obvious that for every smooth parametrization $\mathrm{P} : \mathrm{U} \longrightarrow \mathbf{H}_n$, $f \circ \mathrm{P} = \mathrm{F} \circ \mathrm{P}$ is smooth. Conversely, let f_i be a coordinate of f. Let $x = (r, t)$ denote a point of $\mathbf{R}^n$, where $r \in \mathbf{R}^{n-1}$ and $t \in \mathbf{R}$. If f_i is smooth for the subset diffeology then $\phi_i : (r, t) \mapsto f_i(r, t^2)$, defined on $\mathbf{R}^n$, is smooth. Now, ϕ_i is even in the variable t, $\phi_i(r, t) = \phi_i(r, -t)$, according to Hassler Whitney [Whi43, Theorem 1 & final remark] there exists a smooth map $\mathrm{F}_i : \mathbf{R}^n \longrightarrow \mathbf{R}$ such that $\phi_i(r, t) = \mathrm{F}_i(r, t^2)$. And thus, $f_i(r, t) = \mathrm{F}_i(r, t)$ for every $r \in \mathbf{R}^{n-1}$ an every $t \in [0, +\infty[$. ☐

2.3 Local diffeomorphisms of half-spaces A map $f : \mathrm{A} \longrightarrow \mathbf{H}_n$, with $\mathrm{A} \subset \mathbf{H}_n$, is a local diffeomorphism for the subset diffeology of $\mathbf{R}^n$ if and only if: A is open in $\mathbf{H}_n$, f is injective, $f(\mathrm{A} \cap \partial \mathbf{H}_n) \subset \partial \mathbf{H}_n$, and for all $x \in \mathrm{A}$ there exists an open ball $\mathcal{B} \subset \mathbf{R}^n$ centered at x and a local diffeomorphism $\mathrm{F} : \mathrm{B} \longrightarrow \mathbf{R}^n$ such that f and F coincide on $\mathcal{B} \cap \mathbf{H}_n$.

NOTE — This implies in particular, that there exists an open superset $\mathcal{U}$ of A and an étale application $g : \mathcal{U} \longrightarrow \mathbf{R}^n$ such that f and g coincide on A.

PROOF – Let us assume that f is a local diffeomorphism for the subset diffeology. Since f is a local diffeomorphisms, for the D-topology f is a local homeomorphism, and A is open in $\mathbf{H}_n$. In particular, f maps the boundary $\partial \mathrm{A} = \mathrm{A} \cap \partial \mathbf{H}_n$ into $\partial \mathbf{H}_n$. As well, f maps the complementary $\mathrm{A} - \partial \mathrm{A}$ into $\mathbf{H}_n - \partial \mathbf{H}_n$. Now, since A is open in $\mathbf{H}_n$, $\mathrm{A} - \partial \mathrm{A}$ is open in $\mathbf{H}_n - \partial \mathbf{H}_n$, thus the restriction $f \restriction \mathrm{A} - \partial \mathrm{A}$ is a local diffeomorphism to $\mathbf{H}_n - \partial \mathbf{H}_n$. Therefore, for every $x \in \mathrm{A} - \partial \mathrm{A}$ there exists an open ball $\mathcal{B} \subset \mathrm{A} - \partial \mathrm{A}$, centered at x, such that $\mathrm{F} = f \restriction \mathcal{B}$ is a local diffeomorphism in $\mathbf{R}^n$.

Now, let $(r, 0) \in \mathrm{A}$. Since A is open in $\mathbf{H}_n$, $\partial \mathrm{A}$ is open in $\partial \mathbf{H}_n$. Therefore, there exists an open cube $\mathcal{C} \subset \partial \mathrm{A}$ centered at $(r, 0)$, and $\varepsilon > 0$ such that $\mathcal{C} \times [0, +\varepsilon[\subset \mathrm{A}$. The restriction of f to $\mathcal{C} \times [0, +\varepsilon[$ is a local diffeomorphism, for the subset diffeology, to $\mathbf{H}_n$. Thanks to Subsection 2.2, there exists a smooth map F defined on $\mathcal{C} \times]-\varepsilon, +\varepsilon[$ to $\mathbf{R}^n$, such that f and F coincide on $\mathcal{C} \times [0, +\varepsilon[$. Since f is a diffeomorphism, f maps $\mathcal{C} \times [0, +\varepsilon[$ to some open set $\mathrm{A}' \subset \mathbf{H}_n$ and $\mathcal{C}$ to some open subset of $\partial \mathbf{H}_n$. So, we have $(r', 0) = f(r, 0) \in \partial \mathrm{A}' = \mathrm{A}' \cap \partial \mathbf{H}_n$. Considering now f^{-1}, for the same reason there exist an open cube $\mathcal{C}' \subset \partial \mathrm{A}'$, centered at $(r', 0)$, there exists $\varepsilon' > 0$ such that $\mathcal{C}' \times [0, +\varepsilon'[\subset \mathrm{A}'$ and a smooth map G defined on $\mathcal{C}' \times]-\varepsilon', +\varepsilon'[$ to $\mathbf{R}^n$ such that f^{-1} and G coincide on $\mathcal{C}' \times [0, +\varepsilon'[$. Now, let $\mathcal{O} = \mathrm{F}^{-1}(\mathcal{C}' \times]-\varepsilon', +\varepsilon'[)$, and $\mathcal{O}' = \mathrm{G}^{-1}(\mathcal{C} \times]-\varepsilon, +\varepsilon[)$, $\mathcal{O}$ and $\mathcal{O}'$ are open subset of $\mathbf{R}^n$, with $(r, 0) \in \mathcal{O}$ and $(r', 0) \in \mathcal{O}'$. For every $t \geq 0$ such that $(r, t) \in \mathcal{O}$ we have $\mathrm{D}(\mathrm{G} \circ \mathrm{F})(r, t) = \mathrm{D}(f^{-1} \circ f)(r, t) = \mathbf{1}_{n+1}$. Thus, since F and G are smooth parametrizations, we have on one hand: $\lim_{t \mapsto 0^+} \mathrm{D}(\mathrm{G} \circ \mathrm{F})(r, t) = \mathbf{1}_{n+1}$, and on the other hand: $\lim_{t \mapsto 0^+} \mathrm{D}(\mathrm{G} \circ \mathrm{F})(r, t) = \mathrm{D}(\mathrm{G})(r', 0) \circ \mathrm{D}(\mathrm{F})(r, 0)$. So, $\mathrm{D}(\mathrm{G})(r', 0) \circ \mathrm{D}(\mathrm{F})(r, 0) = \mathbf{1}_{n+1}$, and thus $\mathrm{D}(\mathrm{F})(r, 0)$ is nondegenerate. Therefore, thanks to the implicit function theorem, there exists an open ball $\mathcal{B}$ centered at $x = (r, 0)$ such that $\mathrm{F} \restriction \mathcal{B}$ is a local diffeomorphism to $\mathbf{R}^n$, and such that f and F coincide on $\mathcal{B} \cap \mathbf{H}_n$.

Conversely, let us assume that A is open in $\mathbf{H}_n$, $f : \mathrm{A} \longrightarrow \mathbf{H}_n$ is injective, and for each $x \in \mathrm{A}$ there exists an open ball $\mathcal{B}$ of $\mathbf{R}^n$, centered at x, and a local diffeomorphism $\mathrm{F} : \mathcal{B} \longrightarrow \mathbf{R}^n$ such that f and F coincide on $\mathcal{B}$. Let us prove that f is a local smooth map for the subset diffeology. Let $\mathrm{P} : \mathrm{U} \longrightarrow \mathbf{R}^n$ be a smooth parametrization taking its values in $\mathbf{H}_n$. Since A is open in $\mathbf{H}_n$ and P is continuous, for the D-topology, $\mathrm{P}^{-1}(\mathrm{A})$ is open. Now let $r \in \mathrm{P}^{-1}(\mathrm{A})$ and $x = \mathrm{P}(r)$. Since P is continuous, $\mathrm{P}^{-1}(\mathcal{B})$ is opens, and since F is a local diffeomorphism from $\mathcal{B}$ to $\mathbf{R}^n$, $f \circ \mathrm{P} \restriction \mathrm{P}^{-1}(\mathcal{B}) = \mathrm{F} \circ \mathrm{P} \restriction \mathrm{P}^{-1}(\mathcal{B})$ is a smooth parametrization. Thus $f \circ \mathrm{P}$ is locally smooth at every point point, thus P is smooth, and therefore f is smooth. For the smoothness of f^{-1}, we just need to check that $f(\mathrm{A})$ is open, and the rest will follow the same way as for f. So, let $x \in \mathrm{A}$. If $x \in \mathrm{A} - \partial\mathrm{A}$, the ball $\mathcal{B}$ can be choose small enough to fit into $\mathrm{A} - \partial\mathrm{A}$. Now, by hypothesis $\partial\mathrm{A}$ is mapped into $\partial\mathbf{H}_n$ and f is injective, thus f maps the complementary $\mathrm{A} - \partial\mathrm{A}$ into $\mathbf{H}_n - \partial\mathbf{H}_n$. And since $f \restriction \mathcal{B}$ is a local diffeomorphism to $\mathbf{R}^n$, $f(\mathcal{B}) \subset f(\mathrm{A})$ is an open subset of $\mathbf{H}_n - \partial\mathbf{H}_n$. Now, let $x \in \partial\mathrm{A}$. So $f(x) \in \partial\mathbf{H}_n$ and $f(x) \in f(\mathcal{B} \cap \mathbf{H}_n) = \mathrm{F}(\mathcal{B}) \cap \mathbf{H}_n \subset f(\mathrm{A})$. Since F is a local diffeomorphism from $\mathcal{B}$ to $\mathbf{R}^n$, $\mathrm{F}(\mathcal{B})$ is open in $\mathbf{R}^n$ and $\mathrm{F}(\mathcal{B}) \cap \mathbf{H}_n$ is an open subset of $\mathbf{H}_n$. Finally, for every $x \in \mathrm{A}$, $f(x)$ is contained in some open subset $\mathcal{O}$ of $\mathbf{H}_n$ with $\mathcal{O} \subset f(\mathrm{A})$. Therefore $f(\mathrm{A})$ is a union of open subset of $\mathbf{H}_n$, that is an open subset of $\mathbf{H}_n$. $\square$

3 Smooth manifolds with boundary

For definitions of smooth manifolds with boundary look at [GuPo74] or [Lee06]. We use here the second reference, except that charts have been inverted.

3.1 MANIFOLDS WITH BOUNDARY A *smooth n-manifold with boundary* is a topological space M, together with a family of local homeomorphisms F_i defined on some open sets U_i of the half-space $\mathbf{H}_n$ to M, such that: the values of the F_i cover M, for every two elements F_i and F_j of the family, the transition homeomorphism $\mathrm{F}_i^{-1} \circ \mathrm{F}_j$, defined on $\mathrm{F}_i^{-1}(\mathrm{F}_i(\mathrm{U}_i) \cap \mathrm{F}_j(\mathrm{U}_j))$ to $\mathrm{F}_j^{-1}(\mathrm{F}_i(\mathrm{U}_i) \cap \mathrm{F}_j(\mathrm{U}_j))$, is the restriction of some smooth map defined on an open superset of $\mathrm{F}_i^{-1}(\mathrm{F}_i(\mathrm{U}_i) \cap \mathrm{F}_j(\mathrm{U}_j))$. The boundary $\partial\mathrm{M}$ is the union of the $\mathrm{F}_i(\mathrm{U}_i \cap \partial\mathbf{H}_n)$. Such a family $\mathcal{F}$ of homeomorphisms is called an atlas of M, and its elements are called *charts*. There exists a *maximal atlas* $\mathcal{A}$

containing $\mathcal{F}$, made of all the local homeomorphisms from $\mathbf{H}_n$ to M such that the transition homeomorphisms with every element of $\mathcal{F}$ satisfy the condition given just above. We say that $\mathcal{A}$ gives to M its *structure of manifold with boundary.*

3.2 Diffeology of manifolds with boundary Let M be a smooth n-manifold with boundary. Let us remind that a parametrization $\mathrm{P} : \mathrm{U} \longrightarrow \mathrm{M}$ is smooth if for every $r \in \mathrm{U}$ there exists an open superset V of r, a chart $\mathrm{F} : \Omega \longrightarrow \mathrm{M}$ of M and a smooth parametrization $\mathrm{Q} : \mathrm{V} \longrightarrow \Omega$ such that $\mathrm{P} \restriction \mathrm{V} = \mathrm{F} \circ \mathrm{Q}$.

So, the set of smooth parametrizations of M is a natural diffeology. As a diffeological space M is modeled on $\mathbf{H}_n$, where $\mathbf{H}_n$ is equipped with the subset diffeology. Conversely, every diffeological space modeled on $\mathbf{H}_n$ is naturally a manifold with boundary. Moreover, the set of smooth maps from a manifold with boundary to another, for the category {Smooth Manifolds with Boundary} or for the category {Diffeology}, coincide.

Proof – We denote by $\mathcal{D}$ the set of smooth parametrizations of M. Now, let us prove that any chart $\mathrm{F} : \mathrm{U} \longrightarrow \mathrm{M}$ is a local diffeomorphism from $\mathbf{H}_n$ to M, where $\mathbf{H}_n$ is equipped with the subset diffeology, and M is equipped with $\mathcal{D}$. Since for any plot P of $\mathbf{H}_n$, $\mathrm{F} \circ \mathrm{P}$ — defined on $\mathrm{P}^{-1}(\mathrm{U})$ which is open — belongs obviously to $\mathcal{D}$, so F is smooth. Then, let us prove now that F^{-1} is smooth. Let $\mathrm{P} : \mathrm{U} \longrightarrow \mathrm{M}$ an element of $\mathcal{D}$, let $r \in \mathrm{U}$, let V, Q and F' as above, such that $\mathrm{P} \restriction \mathrm{V} = \mathrm{F}' \circ \mathrm{Q}$. So, $\mathrm{F}^{-1} \circ \mathrm{P} \restriction \mathrm{V} = \mathrm{F}^{-1} \circ \mathrm{F}' \circ \mathrm{Q}$. But since F and F' are charts of M, $\mathrm{F}^{-1} \circ \mathrm{F}'$ is the restriction of some ordinary smooth map defined on some open superset of $\mathrm{F}^{-1}(\mathrm{F}(\mathrm{U}) \cap \mathrm{F}'(\mathrm{U}'))$, therefore $\mathrm{F}^{-1} \circ \mathrm{F}' \circ \mathrm{Q}$ is a smooth parametrization of $\mathbf{H}_n$, that is $\mathrm{F}^{-1} \circ \mathrm{P} \restriction \mathrm{V}$. Now, since $\mathrm{P}^{-1}(\mathrm{F}(\mathrm{U}))$ is open, and since the parametrization $\mathrm{P} \circ \mathrm{F}^{-1}$ is locally smooth everywhere on $\mathrm{P}^{-1}(\mathrm{F}(\mathrm{U}))$, $\mathrm{F}^{-1} \circ \mathrm{P}$ is smooth. Thus, F^{-1} is a local smooth map. Therefore, F is a local diffeomorphism. This proves that, for the diffeology $\mathcal{D}$, M is modeled on $\mathbf{H}_n$. Conversely, let us assume that M is a diffeological space modeled on $\mathbf{H}_n$. Let $\mathcal{A}$ be the set of all local diffeomorphisms from $\mathbf{H}_n$ to M and let us equip M with the D-topology. So, the elements of $\mathcal{A}$ are local homeomorphisms. Let $\mathrm{F} : \mathrm{U} \longrightarrow \mathrm{M}$ and $\mathrm{F}' : \mathrm{U}' \longrightarrow \mathrm{M}$ be two elements of $\mathcal{A}$. Let us assume that $\mathrm{F}(\mathrm{U}) \cap \mathrm{F}'(\mathrm{U}')$ which is open, is not empty. Thus $\mathrm{F}^{-1}(\mathrm{F}(\mathrm{U}) \cap \mathrm{F}'(\mathrm{U}'))$ and $\mathrm{F}'^{-1}(\mathrm{F}(\mathrm{U}) \cap \mathrm{F}'(\mathrm{U}'))$ are open. But $\mathrm{F}'^{-1} \circ \mathrm{F} \restriction \mathrm{F}^{-1}(\mathrm{F}(\mathrm{U}) \cap \mathrm{F}'(\mathrm{U}'))$ and $\mathrm{F}^{-1} \circ \mathrm{F}' \restriction \mathrm{F}'^{-1}(\mathrm{F}(\mathrm{U}) \cap \mathrm{F}'(\mathrm{U}'))$

are local diffeomorphisms for the subset diffeology of $\mathbf{H}_n$, and according to Subsection 2.3 they are the restrictions of ordinary smooth maps. Therefore, the set $\mathcal{A}$ gives to M a structure of smooth manifold with boundary. It is clear that these two operations just described are inverse each other.

Now, thanks to Subsection 2.3 and Subsection 2.2, it is clear that to be a smooth map for the category of smooth manifolds with boundary or to be smooth for the natural diffeology associated is identical. $\square$

4 The diffeology of slices, half-lines and intervals

The statements given without proofs are simple corollaries of the previous Subsection 2.2 and Subsection 2.3.

4.1 Smooth real maps from slices A map $f : \mathbf{R}^{n-1} \times [a, b] \longrightarrow \mathbf{R}^p$ is a smooth map, for the subset diffeology if and only if there exists a smooth map $\mathrm{F} : \mathbf{R}^{n-1} \times]2a - b, 2b - a[$ to $\mathbf{R}^p$ such that $f = \mathrm{F} \restriction \mathbf{R}^{n-1} \times [a, b]$. So, the set $\mathcal{C}^\infty(\mathbf{R}^{n-1} \times [a, b], \mathbf{R}^p)$ is made of restrictions of smooth maps defined on an open superset of $\mathbf{R}^{n-1} \times [a, b]$. For $n = 1$ we get the case of closed intervals.

Proof – According to Subsection 2.2, $f \restriction \mathbf{R}^{n-1} \times [a, b[= \mathrm{F} \restriction \mathbf{R}^{n-1} \times [a, b[$ where $\mathrm{F} : \mathbf{R}^n \times]2a - b, b[\longrightarrow \mathbf{R}^p$ is smooth. As well, $f \restriction \mathbf{R}^n \times]a, b] = \bar{\mathrm{F}} \restriction \mathbf{R}^n \times]a, b]$ where $\bar{\mathrm{F}} : \mathbf{R}^n \times]a, 2b - a[\longrightarrow \mathbf{R}^p$ is smooth. Now, $\mathrm{F} \restriction \mathrm{dom}(\mathrm{F}) \cap \mathrm{dom}(\bar{\mathrm{F}}) = \bar{\mathrm{F}} \restriction \mathrm{dom}(\mathrm{F}) \cap \mathrm{dom}(\bar{\mathrm{F}}) = f \restriction \mathbf{R}^n \times]a, b[$, which is smooth. Therefore, the smallest common extension of F and $\bar{\mathrm{F}}$, defined on $\mathbf{R}^n \times]2a - b, 2b - a[$, is smooth and is an extension of f. $\square$

4.2 The diffeomorphisms of the half-line Every diffeomorphism f of the half-line $[0, +\infty[$ is the restriction of an increasing local diffeomorphism F preserving the origin. Conversely, any restriction of a local diffeomorphism defined on a superset of the half-line and preserving the origin is a diffeomorphism of the half-line, for the subset diffeology.

4.3 The diffeomorphisms of closed intervals Any diffeomorphisms f of a closed interval $[a, b]$, equipped with the subset diffeology, is the restriction of a local diffeomorphism F defined a superset of $[a, b]$ such that either F preserves the ends a and b, and F is increasing, or F exchanges the ends, and F is decreasing.

4.4 The case of Δ_n half-lines Let us denote by Δ_n the quotient of $\mathbf{R}^n$ by the orthogonal group $\mathrm{O}(n)$, $n > 0$. The Δ_n are identified with the interval $[0, +\infty[$, equipped with the pushforward of the smooth diffeology of $\mathbf{R}^n$ by the projection $\nu_n : x \mapsto \|x\|^2$, see [Piz07]. We denote by Δ_∞ the interval $[0, \infty[\subset \mathbf{R}$ equipped with the subset diffeology. For every n the identity $j_n : \Delta_n \longrightarrow \Delta_\infty$ is smooth. Moreover, its pullback $j_n^* : \mathcal{C}^\infty(\Delta_\infty, \mathbf{R}^p) \longrightarrow \mathcal{C}^\infty(\Delta_n, \mathbf{R}^p)$ is the identity, for every n and every p. In other words, $\mathcal{C}^\infty(\Delta_n, \mathbf{R}^p) = \mathcal{C}^\infty(\Delta_\infty, \mathbf{R}^p)$, thus any smooth map from Δ_n to $\mathbf{R}^p$ is the restriction of a smooth map defined on $\mathbf{R}$. Moreover, for every integer n, $\mathrm{Diff}(\Delta_\mathrm{n}) = \mathrm{Diff}(\Delta_\infty)$. This result shows in particular that, the group of diffeomorphisms of these half-lines — as an abstract group — does not capture their diffeology.

Proof – First of all, let us remind that, by definition, the plots of Δ_n write locally $\mathrm{P} : r \mapsto \|\phi(r)\|^2$, where ϕ is a smooth parametrization of $\mathbf{R}^n$. Now, let $f \in \mathcal{C}^\infty(\Delta_\infty, \mathbf{R}^p)$ and let P be a plot of Δ_n. The parametrization $f \circ \mathrm{P}$ writes locally $r \mapsto f(\|\phi(r)\|^2)$ which, thanks to the definition of the subset diffeology, is smooth. So, $f \circ \mathrm{P}$ is locally smooth, that is smooth, therefore $f \in \mathcal{C}^\infty(\Delta_n, \mathbf{R}^p)$ and $\mathcal{C}^\infty(\Delta_\infty, \mathbf{R}^p) \subset \mathcal{C}^\infty(\Delta_n, \mathbf{R}^p)$. Conversely, let $f \in \mathcal{C}^\infty(\Delta_n, \mathbf{R}^p)$, and let $\gamma : t \mapsto (t, 0, ..., 0)$ from $\mathbf{R}$ to $\mathbf{R}^n$. By hypothesis, $\phi : t \mapsto f(t^2) = f(\|\gamma(t)\|^2)$ is smooth. So, ϕ is a smooth even function from $\mathbf{R}$ to $\mathbf{R}^p$. Thanks to Subsection 2.2, there exists a smooth real map F such that $f = \mathrm{F} \restriction [0, +\infty[$. So, $f \in \mathcal{C}^\infty(\Delta_\infty, \mathbf{R}^p)$, and therefore $\mathcal{C}^\infty(\Delta_n, \mathbf{R}^p) \subset \mathcal{C}^\infty(\Delta_\infty, \mathbf{R}^p)$. And finally, $\mathcal{C}^\infty(\Delta_\infty, \mathbf{R}^p) = \mathcal{C}^\infty(\Delta_n, \mathbf{R}^p)$, for every $n > 0$.

Now, let us consider the diffeomorphisms of Δ_n. Every diffeomorphism is an homeomorphism for the D-topology, which coincide with the standard topology of $[0, +\infty[$ [Piz05]. Since any homeomorphism of $[0, +\infty[$ preserves the origin[1], so does f. Then, considering the smooth parametrizations $\mathrm{F} : t \mapsto f(t^2) = f(\|\gamma(t)\|^2)$ and $\mathrm{G} : t \mapsto f^{-1}(t^2) = f^{-1}(\|\gamma(t)\|^2)$, the reproduction of the general argument of Subsection 4.2 leads to the result, $\mathrm{Diff}(\Delta_\mathrm{n}) = \mathrm{Diff}(\Delta_\infty)$. □

[1]For $n > 1$, we have $\dim_0 \Delta_n = n$ and $\dim_x \Delta_n = 1$ for $x \neq 0$ [Piz07], so $f(0) = 0$, without considering the D-topology. But this pure diffeological argument fails for $n = 1$.

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