Dimension in diffeology

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Abstract

We define the *dimension function* for diffeological spaces, a simple but new invariant. We show then how it can be applied to prove that, for two different integers m and n the quotient spaces $\mathbf{R}^m/\mathbf{O}(m)$ and $\mathbf{R}^n/\mathbf{O}(n)$ are not diffeomorphic, and not diffeomorphic to the half-line $[0,\infty[\subset \mathbf{R}$.

Introduction

The notion of *dimension* in diffeology, which we introduce in Section 3, gives a quick and easy answer to the question: *For two different integers n and m, are the diffeological spaces $\Delta_n = \mathbf{R}^n/\mathrm{O}(n)$ and $\Delta_m = \mathbf{R}^m/\mathrm{O}(m)$ diffeomorphic?* In section 4, we show that since $\dim(\Delta_n) = n$ and since the dimension is a diffeological invariant, the answer is *No, they are not*. This method simplifies a partial result, obtained in a more complicated way in [Igl85], stating that Δ_1 and Δ_2 are not diffeomorphic. The half-line $\Delta_\infty = [0,\infty[\subset \mathbf{R}$ is a similar example for which $\dim(\Delta_\infty) = \infty$. Hence, Δ_m is not diffeomorphic to the half-line Δ_∞ for any integer m. Dimension appears to be a simple but a powerful diffeological invariant. Hopefully, the diffeological dimension coincides with the usual definition when the diffeology space is a manifold. That is, when the diffeology is generated by local diffeomorphisms with $\mathbf{R}^n$, for some integer n.

For more details, the reader who is not familiar with diffeology can look at [PIZ05].

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1 Diffeologies and diffeological spaces

1.1 Parametrizations of a set Let X be a set, we call *parametrization in* X any map defined on any open subset of any space $\mathbf{R}^n$ for any integer n, with values in X. The set of all the parametrizations in X will be denoted by Param(X). For any parametrization $\mathrm{P} : \mathrm{U} \longrightarrow \mathrm{X}$, the *numerical domain* U is called the *domain* of P and is denoted by dom(P). If U is a subset of $\mathbf{R}^n$ we say that P is a n-*parametrization*, the integer n will be called the *dimension of the parametrization* P, and we shall denote $\dim(\mathrm{P}) = n$.

1.2 Diffeology and diffeological spaces Let X be a set. A *diffeology on* X is a set $\mathcal{D}$ of parametrizations in X, that is $\mathcal{D} \subset \mathrm{Param}(\mathrm{X})$, such that

D1. *Covering* Every point of X is contained in the range of some $\mathrm{P} \in \mathcal{D}$.

D2. *Locality* If $\mathrm{P} \in \mathrm{Param}(\mathrm{X})$ and there exist $\mathrm{P}_i \in \mathcal{D}$, $i \in \mathcal{I}$, such that the $\mathrm{dom}(\mathrm{P}_i)$, $i \in \mathcal{I}$ form an open covering of dom(P) and $\mathrm{P}_i = \mathrm{P}_j$ on $\mathrm{dom}(\mathrm{P}_i) \cap \mathrm{dom}(\mathrm{P}_j)$ for every $i, j \in \mathcal{I}$, then $\mathrm{P} \in \mathcal{D}$.

D3. *Smooth compatibility* If $\mathrm{P} \in \mathcal{D}$ and F is a C^∞ mapping from a open subset V of R^m to dom(P), then $\mathrm{P} \circ \mathrm{F} \in \mathcal{D}$.

The first axiom can be replaced by the original "constant parametrizations belong to $\mathcal{D}$". The second axiom clearly means that to be an element of $\mathcal{D}$ is a local condition. Note that the third axiom implies in particular that the restriction of any element of $\mathcal{D}$ to an open subset of its domain still belongs to $\mathcal{D}$.
Equipped with a diffeology $\mathcal{D}$, X is a *diffeological space*[1]. Note that the definition of a diffeology does not assume any pre-existing structure on the underlying set.

1.3 Smooth maps and diffeomorphisms Let X and X′ be two sets equipped with the diffeologies $\mathcal{D}$ and $\mathcal{D}'$ respectively. A map $\mathrm{F} : \mathrm{X} \longrightarrow \mathrm{Y}$ is said to

[1] If X denotes a diffeological space, the elements of its diffeology are usually called the *plots* of X.

be *smooth* if for each $P \in \mathcal{D}$ we have $F \circ P \in \mathcal{D}'$. The set of smooth maps from X to Y is denoted by $\mathcal{C}^\infty(X, Y)$. A bijective map $F : X \longrightarrow Y$ is said to be a *diffeomorphism* if both F and F^{-1} are smooth. The set of diffeomorphisms of X is a group denoted by Diff(X). Diffeological spaces are the objects of the category {Diffeology} whose morphisms are *smooth maps*, and isomorphisms are *diffeomorphisms*. This category is stable by set theoretic operations. In particular, let $\sim$ be an equivalence relation on X, let $Q = X/\sim$ and $\pi : X \longrightarrow Q$ be the projection. There exists a natural *quotient diffeology* on Q, for which π is smooth, defined by the parametrizations which can be lifted locally along π by elements of $\mathcal{D}$. As well, there exists on every subset $A \subset X$ a natural *subset diffeology*, for which the inclusion is smooth, defined by the elements of $\mathcal{D}$ which take their values in A.

1.4 GENERATING FAMILIES Let X be a set, and let $\mathcal{F} \subset \mathrm{Param}(X)$. There exists a smallest diffeology $\mathcal{D}$ containing $\mathcal{F}$. We call it the *diffeology generated* by $\mathcal{F}$ and we denote $\mathcal{D} = \langle \mathcal{F} \rangle$. A parametrization $P : U \longrightarrow X$ belongs to $\langle \mathcal{F} \rangle$ if and only if for any point r of U there exists an open subset $V \subset U$ containing r such that either $P \restriction V$ is a constant parametrization, or there exists $F : W \longrightarrow X$ element of $\mathcal{F}$, and a smooth mapping $Q \in \mathcal{C}^\infty(V, W)$ such that $P \restriction V = F \circ Q$. Note that, for example, the empty family generates the discrete diffeology.

1.5 STANDARD DIFFEOLOGY ON NUMERICAL DOMAINS This is the very basic example of diffeological space. Any open set U of any $\mathbf{R}^n$ has a natural *smooth diffeology* defined as the set of all smooth parametrizations of U. Now, for any numerical domain U equipped with the smooth diffeology, for any set X equipped with a diffeology $\mathcal{D}$ we have $\mathcal{C}^\infty(U, X) = \{P \in \mathcal{D} \mid \mathrm{dom}(P) = U\}$. And obviously, the singleton $\{\mathbf{1}_U\}$ is a generating family of U.

2 Locality and diffeology

2.1 LOCAL SMOOTH MAPS AND LOCAL DIFFEOMORPHISMS Let X and X′ be two sets equipped with the diffeologies $\mathcal{D}$ and $\mathcal{D}'$ respectively. Let f be a map defined on a subset $A \subset X$ to X′. We say that f is a *local smooth map* if for every $P \in \mathcal{D}$ we have $f \circ P \in \mathcal{D}'$. This implies in particular that $P^{-1}(A)$ is a numerical domain[2]. The composite of local smooth maps is still a local smooth map. Now,

[2]This condition means that A is open for the D-topology of X [PIZ05].

f is said to be a *local diffeomorphism* if f is an injective local smooth map as well as its inverse f^{-1}, defined on $f(\mathrm{A}) \subset \mathrm{X}'$. In particular, manifolds are diffeological spaces generated by local diffeomorphisms with $\mathbf{R}^n$, for some integer n.

2.2 Generating the diffeology locally Let X be a set equipped with a diffeology $\mathcal{D}$. We shall say that a family $\mathcal{F} \subset \mathcal{D}$ *generates* $\mathcal{D}$ *locally at the point* $x \in \mathrm{X}$ if

1. For every $\mathrm{F} \in \mathcal{F}$ the point x belongs to the range of F.
2. If $\mathrm{P} \in \mathcal{D}$ and x belongs to the range of P, then there is an open subset U of dom(P), an element $\mathrm{F} \in \mathcal{F}$, and a $\mathcal{C}^\infty$ mapping Q from U to dom(F) such that $x \in \mathrm{P}(\mathrm{U})$ and $\mathrm{P} \restriction \mathrm{U} = \mathrm{F} \circ \mathrm{Q}$.

By analogy with (art. 1.4), we will denote this property by $\mathcal{D}_x = \langle \mathcal{F} \rangle$.

3 Dimension of diffeological spaces

3.1 The dimension function of a diffeological space Let X be a set. We define the *dimension* of any family $\mathcal{F}$ of parametrizations of X as

$$\dim(\mathcal{F}) = \sup\{\dim(\mathrm{F}) \mid \mathrm{F} \in \mathcal{F}\},$$

where the dimension of a parametrization has been defined in Subsection 1.1. If for any $n \in \mathbf{N}$ there exists $\mathrm{F} \in \mathcal{F}$ such that $\dim(\mathrm{F}) = n$ then $\dim(\mathcal{F}) = \infty$. Let X be a diffeological space, and let $\mathcal{D}$ be its diffeology. Let $x \in \mathrm{X}$, we define the *dimension of the diffeological space* X *at the point* x as the infimum of the dimensions of the families of parametrizations generating the diffeology $\mathcal{D}$ at the point x. In other words,

$$\dim_x(\mathrm{X}) = \inf\{\dim(\mathcal{F}) \mid \langle \mathcal{F} \rangle = \mathcal{D}_x\}.$$

The *dimension function* $x \mapsto \dim_x(\mathrm{X})$, of the diffeological space X, takes its values in $\mathbf{N} \cup \{\infty\}$. The *global dimension* of X can be defined as the supremum of the dimension map of X, and we have

$$\dim(\mathrm{X}) = \sup_{x \in \mathrm{X}} \{\dim_x(\mathrm{X})\} = \inf\{\dim(\mathcal{F}) \mid \langle \mathcal{F} \rangle = \mathcal{D}\}.$$

See [PIZ05] for the proof of the second equality.

3.2 THE DIMENSION MAP IS A LOCAL INVARIANT Let X and X′ be two diffeological spaces. If $x \in \mathrm{X}$ and $x' \in \mathrm{X}'$ are two points related by a local (a fortiori global) diffeomorphism then $\dim_x(\mathrm{X}) = \dim_{x'}(\mathrm{X}')$.

3.3 DIMENSIONS OF NUMERICAL DOMAINS Let $\mathrm{U} \subset \mathbf{R}^n$ be an open set equipped with the smooth diffeology defined in Subsection 1.5. We have, $\dim(\mathrm{U}) = n$. And, thanks to the proposition 3.2, dimension for diffeological spaces coincides with the usual notion in the case of manifolds.

4 Examples of the half-lines

4.1 THE HALF-LINES Δ_n Let $\Delta_n = \mathbf{R}^n/\mathrm{O}(n, \mathbf{R})$ equipped with the quotient diffeology, $n \in \mathbf{N}$. So, $\dim_0(\Delta_n) = n$, and $\dim_x(\Delta_n) = 1$ if $x \neq 0$. Thus, $\dim(\Delta_n) = n$ and for $n \neq m$ the half-lines Δ_n and Δ_m are not diffeomorphic.

PROOF – The case $n = 0$ is trivial. Let us assume $n > 0$, and let us denote by $\pi_n : \mathbf{R}^n \longrightarrow \Delta_n$ the projection from $\mathbf{R}^n$ onto its quotient. There is a natural bijection $f : \Delta_n \longrightarrow [0, \infty[$ such that $f \circ \pi_n = \nu_n$, where $\nu_n(x) = \|x\|^2$. Now, thanks to the uniqueness of quotients [PIZ05], we use f to identify Δ_n with $[0, \infty[$ equipped with the diffeology $\mathcal{D}_n$ generated by ν_n. The elements of $\mathcal{D}_n$ consist of the parametrizations which locally can be lifted along ν_n by smooth parametrizations of $\mathbf{R}^n$. So, since $\dim(\nu_n) = n$, we get $\dim(\Delta_n) \leq n$. Let us prove now that $\dim(\Delta_n) \geq n$. Let us assume that ν_n, which is an element of $\mathcal{D}_n$, can be lifted locally at the point 0_n, along $\mathrm{P} \in \mathcal{D}_n$ with $\dim(\mathrm{P}) = p < n$. So, there exists a smooth parametrization $\phi : \mathrm{V} \longrightarrow \mathrm{dom}(\mathrm{P})$ such that $\mathrm{P} \circ \phi = \nu_n \restriction \mathrm{V}$. We can assume without loss of generality that $0_p \in \mathrm{dom}(\mathrm{P})$, $\mathrm{P}(0_p) = 0$ and $\phi(0_n) = 0_p$. Now, since P is an element of $\mathcal{D}_n$ there exists a smooth parametrization $\psi : \mathrm{W} \longrightarrow \mathbf{R}^n$ such that $0_p \in \mathrm{W}$ and $\nu_n \circ \psi = \mathrm{P} \restriction \mathrm{W}$. Let $\mathrm{V}' = \phi^{-1}(\mathrm{W})$, and $\mathrm{F} = \psi \circ \phi \restriction \mathrm{V}'$, we get $\nu_n \restriction \mathrm{V}' = \nu_n \circ \mathrm{F}$, with $\mathrm{F} \in \mathcal{C}^\infty(\mathrm{V}', \mathbf{R}^n)$, $0_n \in \mathrm{V}'$ and $\mathrm{F}(0_n) = 0_n$, that is $\|x\|^2 = \|\mathrm{F}(x)\|^2$. The second derivative of this identity computed at the point 0_n gives $\mathbf{1}_n = \mathrm{M}^t\mathrm{M}$ with $\mathrm{M} = \mathrm{D}(\mathrm{F})(0_n)$. But $\mathrm{M} = \mathrm{AB}$ with $\mathrm{A} = \mathrm{D}(\psi)(0_p)$ and $\mathrm{B} = \mathrm{D}(\phi)(0_n)$. So, $\mathbf{1}_n = \mathrm{B}^t\mathrm{A}^t\mathrm{AB}$ which is impossible because $\mathrm{rank}(\mathrm{B}) \leq p < n$. Therefore, $\dim(\Delta_n) = n$. And, since the dimension is a diffeological invariant, Δ_n is not diffeomorphic to Δ_m for $n \neq m$. □

4.2 THE HALF-LINE Δ_∞ The dimension of a diffeological subspace $\mathrm{A} \subset \mathrm{X}$ can be less, equal or even greater than the dimension of X. The following example is a clear illustration of this phenomenon. Let $\Delta_\infty = [0, \infty[\subset \mathbf{R}$ equipped with the subset diffeology. So, $\dim_0(\Delta_\infty) = \infty$ and $\dim_x(\Delta_\infty) = 1$ if $x \neq 0$. Thus, $\dim(\Delta_\infty) = \infty$, and for any integer m, Δ_∞ is not diffeomorphic to Δ_m.

Proof – Let us assume that $\dim(\Delta_\infty) = \mathrm{N} < \infty$. For any integer n, the map $\nu_n : \mathbf{R}^n \longrightarrow \Delta_\infty$, defined by $\nu_n(x) = \|x\|^2$ belongs to $\mathcal{D}_\infty$, the subset diffeology on $[0, \infty[$. Hence, ν_n lifts locally at the point 0_n along some $\mathrm{P} \in \mathcal{D}_\infty$ where $\dim(\mathrm{P}) = p \leq \mathrm{N}$. Now, let us choose $n > \mathrm{N}$. So, P belongs to some $\mathcal{C}^\infty(\mathrm{U}, \mathbf{R})$ with $\mathrm{val}(\mathrm{P}) \subset [0, \infty[$, and there exists a smooth parametrization $\phi : \mathrm{V} \longrightarrow \mathrm{U}$ such that $\mathrm{P} \circ \phi = \nu_n \upharpoonright \mathrm{V}$. We can assume, without loss of generality, that $0_p \in \mathrm{U}$, $\phi(0_n) = 0_p$, and thus $\mathrm{P}(0_p) = 0$. Now, the first derivative of ν_n at a point $x \in \mathrm{V}' = \phi^{-1}(\mathrm{V})$ is given by $x = \mathrm{D}(\mathrm{P})(\phi(x)) \circ \mathrm{D}(\phi)(x)$. But, since P is smooth and positive, and since $\mathrm{P}(0) = 0$ we have $\mathrm{D}(\mathrm{P})(0_p) = 0$. So, the second derivative of ν_n computed at the point 0_n gives $\mathbf{1}_n = \mathrm{M}^t\mathrm{H}\mathrm{M}$, where $\mathrm{M} = \mathrm{D}(\phi)(0)$ and $\mathrm{H} = \mathrm{D}^2(\mathrm{P})(0)$. But since $\mathrm{rank}(\mathrm{M}) \leq p \leq \mathrm{N}$ and $n > \mathrm{N}$ this is impossible. Therefore $\dim(\Delta_\infty) = \infty$. □

5 Some other examples

5.1 Dimension zero spaces are discrete A diffeological space is said to be discrete if its diffeology is generated by the empty set. A diffeological space has dimension zero if and only if it is discrete.

5.2 Has the set $\{0, 1\}$ dimension 1? Let us consider the set $\{0, 1\}$ and $\pi : \mathbf{R} \longrightarrow \{0, 1\}$ be the parametrization defined by: $\pi(x) = 0$ if $x \in \mathbf{Q}$, and $\pi(x) = 1$ otherwise. Let $\{0, 1\}_\pi$ be the set $\{0, 1\}$ equipped with the diffeology generated by π. Thus $\dim(\{0, 1\}_\pi) = 1$. So, a diffeological space made of a finite number of points may have a non zero dimension.

5.3 Dimension of tori Let $\Gamma \subset \mathbf{R}$ be a strict subgroup of $(\mathbf{R}, +)$ and let T_Γ be the quotient $\mathbf{R}/\Gamma$. So, $\dim(\mathrm{T}_\Gamma) = 1$. This applies in particular to the circles $\mathbf{R}/a\mathbf{Z}$, with length $a > 0$, or to *irrational tori* [DI85] when Γ is generated by more than one generators, rationally independent.

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