Dimension in diffeology

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Communicated by Prof. J.J. Duistermaat at the meeting of September 24, 2007

ABSTRACT

We define the \textit{dimension function} for diffeological spaces, a simple but new invariant. We show then how it can be applied to prove that, for two different integers \( m \) and \( n \) the quotient spaces \( \mathbb{R}^m/O(m) \) and \( \mathbb{R}^n/O(n) \) are not diffeomorphic, and not diffeomorphic to the half-line \([0, \infty[ \subset \mathbb{R}\). 

INTRODUCTION

The notion of \textit{dimension} in diffeology, which we introduce in Section 3, gives a quick and easy answer to the question: \textit{For two different integers \( n \) and \( m \), are the diffeological spaces \( \Delta_n = \mathbb{R}^n/O(n) \) and \( \Delta_m = \mathbb{R}^m/O(m) \) diffeomorphic?} In Section 4, we show that since \( \dim(\Delta_n) = n \) and since the dimension is a diffeological invariant, the answer is \textit{No, they are not.} This method simplifies a partial result, obtained in a more complicated way in [4], stating that \( \Delta_1 \) and \( \Delta_2 \) are not diffeomorphic. The half-line \( \Delta_\infty = [0, \infty[ \subset \mathbb{R} \) is a similar example for which \( \dim(\Delta_\infty) = \infty \). Hence, \( \Delta_m \) is not diffeomorphic to the half-line \( \Delta_\infty \) for any integer \( m \). Dimension appears to be a simple but a powerful diffeological invariant. Fortunately, the diffeological dimension coincides with the usual definition when the diffeology space is a manifold. That is, when the diffeology is generated by local diffeomorphisms with \( \mathbb{R}^n \), for some integer \( n \).

For more details, the reader who is not familiar with diffeology can look at [10].

\textsuperscript{1} http://math.huji.ac.il/~piz/
1. DIFFEOLOGIES AND DIFFEOLOGICAL SPACES

1.1. Parametrizations of a set

Let X be a set, we call parametrization in X any map defined on any open subset of any space $\mathbb{R}^n$ for any integer $n$, with values in X. The set of all the parametrizations in X will be denoted by $\text{Param}(X)$. For any parametrization $P : U \to X$, the numerical domain $U$ is called the domain of $P$ and is denoted by $\text{dom}(P)$. If $U$ is a subset of $\mathbb{R}^n$ we say that $P$ is an $n$-parametrization, the integer $n$ will be called the dimension of the parametrization $P$, and we shall denote $\text{dim}(P) = n$.

1.2. Diffeology and diffeological spaces

Let X be a set. A diffeology on X is a set $\mathcal{D}$ of parametrizations in X, that is $\mathcal{D} \subset \text{Param}(X)$, such that

D1. Covering Every point of X is contained in the range of some $P \in \mathcal{D}$.

D2. Locality If $P \in \text{Param}(X)$ and there exist $P_i \in \mathcal{D}$, $i \in \mathcal{I}$, such that the $\text{dom}(P_i)$, $i \in \mathcal{I}$ form an open covering of $\text{dom}(P)$ and $P_i = P_j$ on $\text{dom}(P_i) \cap \text{dom}(P_j)$ for every $i, j \in \mathcal{I}$, then $P \in \mathcal{D}$.

D3. Smooth compatibility If $P \in \mathcal{D}$ and $F$ is a $C^\infty$ mapping from a open subset $V$ of $\mathbb{R}^m$ to $\text{dom}(P)$, then $P \circ F \in \mathcal{D}$.

The first axiom can be replaced by the original “constant parametrizations belong to $\mathcal{D}$”. The second axiom clearly means that to be an element of $\mathcal{D}$ is a local condition. Note that the third axiom implies in particular that the restriction of any element of $\mathcal{D}$ to an open subset of its domain still belongs to $\mathcal{D}$.

Equipped with a diffeology $\mathcal{D}$, X is a diffeological space. Note that the definition of a diffeology does not assume any pre-existing structure on the underlying set.

1.3. Smooth maps and diffeomorphisms

Let $X$ and $X'$ be two sets equipped with the diffeologies $\mathcal{D}$ and $\mathcal{D}'$ respectively. A map $F : X \to Y$ is said to be smooth if for each $P \in \mathcal{D}$ we have $F \circ P \in \mathcal{D}'$. The set of smooth maps from X to Y is denoted by $C^\infty(X, Y)$. A bijective map $F : X \to Y$ is said to be a diffeomorphism if both $F$ and $F^{-1}$ are smooth. The set of diffeomorphisms of X is a group denoted by $\text{Diff}(X)$. Diffeological spaces are the objects of the category $\{\text{Diffeology}\}$ whose morphisms are smooth maps, and isomorphisms are diffeomorphisms. This category is stable by set theoretic operations. In particular, let $\sim$ be an equivalence relation on X, let $Q = X/\sim$ and $\pi : X \to Q$ be the projection. There exists a natural quotient diffeology on Q, for which $\pi$ is smooth, defined by the parametrizations which can be lifted locally along $\pi$ by elements of $\mathcal{D}$. As well, there exists on every subset $A \subset X$ a natural subset diffeology, for which the inclusion is smooth, defined by the elements of $\mathcal{D}$ which take their values in A.

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2 If X denotes a diffeological space, the elements of its diffeology are usually called the plots of X.
1.4. Generating families

Let $X$ be a set, and let $\mathcal{F} \subset \text{Param}(X)$. There exists a smallest diffeology $\mathcal{D}$ containing $\mathcal{F}$. We call it the \textit{diffeology generated} by $\mathcal{F}$ and we denote $\mathcal{D} = \langle \mathcal{F} \rangle$. A parametrization $P : U \to X$ belongs to $\langle \mathcal{F} \rangle$ if and only if for any point $r$ of $U$ there exists an open subset $V \subset U$ containing $r$ such that either $P \upharpoonright V$ is a constant parametrization, or there exists $F : W \to X$ element of $\mathcal{F}$, and a smooth mapping $Q \in C^\infty(V, W)$ such that $P \upharpoonright V = F \circ Q$. Note that, for example, the empty family generates the discrete diffeology.

1.5. Standard diffeology on numerical domains

This is the very basic example of diffeological space. Any open set $U$ of any $\mathbb{R}^n$ has a natural smooth diffeology defined as the set of all smooth parametrizations of $U$. Now, for any numerical domain $U$ equipped with the smooth diffeology, for any set $X$ equipped with a diffeology $\mathcal{D}$ we have $C^\infty(U, X) = \{ P \in \mathcal{D} \mid \text{dom}(P) = U \}$. And obviously, the singleton $\{1_U\}$ is a generating family of $U$.

2. Locality and diffeology

2.1. Local smooth maps and local diffeomorphisms

Let $X$ and $X'$ be two sets equipped with the diffeologies $\mathcal{D}$ and $\mathcal{D}'$ respectively. Let $f$ be a map defined on a subset $A \subset X$ to $X'$. We say that $f$ is a local smooth map if for every $P \in \mathcal{D}$ we have $f \circ P \in \mathcal{D}'$. This implies in particular that $P^{-1}(A)$ is a numerical domain.\(^3\) The composite of local smooth maps is still a local smooth map. Now, $f$ is said to be a local diffeomorphism if $f$ is an injective local smooth map as well as its inverse $f^{-1}$, defined on $f(A) \subset X'$. In particular, manifolds are diffeological spaces generated by local diffeomorphisms with $\mathbb{R}^n$, for some integer $n$.

2.2. Generating the diffeology locally

Let $X$ be a set equipped with a diffeology $\mathcal{D}$. We shall say that a family $\mathcal{F} \subset \mathcal{D}$ generates $\mathcal{D}$ locally at the point $x \in X$ if

1. For every $F \in \mathcal{F}$ the point $x$ belongs to the range of $F$.
2. If $P \in \mathcal{D}$ and $x$ belongs to the range of $P$, then there is an open subset $U$ of $\text{dom}(P)$, an element $F \in \mathcal{F}$, and a $C^\infty$ mapping $Q$ from $U$ to $\text{dom}(F)$ such that $x \in P(U)$ and $P \upharpoonright U = F \circ Q$.

By analogy with Subsection 1.4, we will denote this property by $\mathcal{D}_x = \langle \mathcal{F} \rangle$.

\(^3\) This condition means that $A$ is open for the $D$-topology of $X$ [10].
3. DIMENSION OF DIFELOGICAL SPACES

3.1. The dimension function of a diffeological space

Let \( X \) be a set. We define the dimension of any family \( \mathcal{F} \) of parametrizations of \( X \) as

\[
\dim(\mathcal{F}) = \sup\{\dim(F) \mid F \in \mathcal{F}\},
\]

where the dimension of a parametrization has been defined in Subsection 1.1. If for any \( n \in \mathbb{N} \) there exists \( F \in \mathcal{F} \) such that \( \dim(F) = n \) then \( \dim(\mathcal{F}) = \infty \). Let \( X \) be a diffeological space, and let \( \mathcal{D} \) be its diffeology. Let \( x \in X \), we define the dimension of the diffeological space \( X \) at the point \( x \) as the infimum of the dimensions of the families of parametrizations generating the diffeology \( \mathcal{D} \) at the point \( x \). In other words,

\[
\dim_x (X) = \inf\{\dim(U) \mid U = \mathcal{D}_x\}.
\]

The dimension function \( x \mapsto \dim_x (X) \), of the diffeological space \( X \), takes its values in \( \mathbb{N} \cup \{\infty\} \). The global dimension of \( X \) can be defined as the supremum of the dimension map of \( X \), and we have

\[
\dim(X) = \sup\{\dim_x (X) \mid x \in X\} = \inf\{\dim(U) \mid \langle \mathcal{U} \rangle = \mathcal{D}\}.
\]

See [10] for the proof of the second equality.

3.2. The dimension map is a local invariant

Let \( X \) and \( X' \) be two diffeological spaces. If \( x \in X \) and \( x' \in X' \) are two points related by a local (a fortiori global) diffeomorphism then \( \dim_x (X) = \dim_{x'} (X') \).

3.3. Dimensions of numerical domains

Let \( U \subset \mathbb{R}^n \) be an open set equipped with the smooth diffeology defined in Subsection 1.5. We have, \( \dim(U) = n \). And, thanks to the Proposition 3.2, dimension for diffeological spaces coincides with the usual notion in the case of manifolds.

4. EXAMPLES OF THE HALF-LINES

4.1. The half-lines \( \Delta_n \)

Let \( \Delta_n = \mathbb{R}^n / O(n, \mathbb{R}) \) equipped with the quotient diffeology, \( n \in \mathbb{N} \). So, \( \dim_0(\Delta_n) = n \), and \( \dim_x(\Delta_n) = 1 \) if \( x \neq 0 \). Thus, \( \dim(\Delta_n) = n \) and for \( n \neq m \) the half-lines \( \Delta_n \) and \( \Delta_m \) are not diffeomorphic.

**Proof.** The case \( n = 0 \) is trivial. Let us assume \( n > 0 \), and let us denote by \( \pi_n : \mathbb{R}^n \to \Delta_n \) the projection from \( \mathbb{R}^n \) onto its quotient. There is a natural bijection \( f : \Delta_n \to [0, \infty[ \) such that \( f \circ \pi_n = v_n \), where \( v_n(x) = \|x\|^2 \). Now, thanks to the uniqueness of quotients [10], we use \( f \) to identify \( \Delta_n \) with \( [0, \infty[ \) equipped with the
diffeology $\mathcal{D}_n$ generated by $\nu_n$. The elements of $\mathcal{D}_n$ consist of the parametrizations which locally can be lifted along $\nu_n$ by smooth parametrizations of $\mathbb{R}^n$. So, since $\dim(\nu_n) = n$, we get $\dim(\Delta_n) \leq n$. Let us prove now that $\dim(\Delta_n) \geq n$. Let us assume that $\nu_n$, which is an element of $\mathcal{D}_n$, can be lifted locally at the point $0_n$, along $P \in \mathcal{D}_n$ with $\dim(P) = p < n$. So, there exists a smooth parametrization $\phi : V \rightarrow \text{dom}(P)$ such that $P \circ \phi = \nu_n | V$. We can assume without loss of generality that $0_p \in \text{dom}(P)$, $P(0_p) = 0$ and $\phi(0_n) = 0_p$. Now, since $P$ is an element of $\mathcal{D}_n$ there exists a smooth parametrization $\psi : W \rightarrow \mathbb{R}^n$ such that $0_p \in W$ and $\nu_n \circ \psi = P | W$. Let $V' = \phi^{-1}(W)$, and $F = \psi \circ \phi | V'$, we get $\nu_n | V' = \nu_n \circ F$, with $F \in C^\infty(V', \mathbb{R}^n)$, $0_n \in V'$ and $F(0_n) = 0_n$, that is $\|x\|^2 = \|F(x)\|^2$. The second derivative of this identity computed at the point $0_n$ gives $I_n = M \cdot H \cdot M$ with $M = \text{D}(F)(0_n)$. But $M = AB$ with $A = \text{D}(\psi)(0_p)$ and $B = \text{D}(\phi)(0_n)$. So, $I_n = B' A' A B$ which is impossible because $\text{rank}(B) \leq p < n$. Therefore, $\dim(\Delta_n) = n$. And, since the dimension is a diffeological invariant, $\Delta_n$ is not diffeomorphic to $\Delta_m$ for $n \neq m$. 

4.2. The half-line $\Delta_{\infty}$

The dimension of a diffeological subspace $A \subset X$ can be less, equal or even greater than the dimension of $X$. The following example is a clear illustration of this phenomenon. Let $\Delta_{\infty} = [0, \infty[ \subset \mathbb{R}$ equipped with the subset diffeology. So, $\dim_0(\Delta_{\infty}) = \infty$ and $\dim_X(\Delta_{\infty}) = 1$ if $x \neq 0$. Thus, $\dim(\Delta_{\infty}) = \infty$, and for any integer $m$, $\Delta_{\infty}$ is not diffeomorphic to $\Delta_m$.

**Proof.** Let us assume that $\dim(\Delta_{\infty}) = N < \infty$. For any integer $n$, the map $\nu_n : \mathbb{R}^n \rightarrow \Delta_{\infty}$, defined by $\nu_n(x) = \|x\|^2$ belongs to $\mathcal{D}_{\infty}$, the subset diffeology on $[0, \infty[$. Hence, $\nu_n$ lifts locally at the point $0_n$ along some $P \in \mathcal{D}_{\infty}$ where $\dim(P) = p \leq N$. Now, let us choose $n > N$. So, $P$ belongs to some $C^\infty(U, \mathbb{R})$ with $\text{val}(P) \subset [0, \infty[$, and there exists a smooth parametrization $\phi : V \rightarrow U$ such that $P \circ \phi = \nu_n | V$. We can assume, without loss of generality, that $0_p \in U$, $\phi(0_n) = 0_p$, and thus $P(0_p) = 0$. Now, the first derivative of $\nu_n$ at a point $x \in V' = \phi^{-1}(V)$ is given by $x = \text{D}(P)(\phi(x)) \circ \text{D}(\phi)(x)$. But, since $P$ is smooth and positive, and since $P(0) = 0$ we have $\text{D}(P)(0_p) = 0$. So, the second derivative of $\nu_n$ computed at the point $0_n$ gives $I_n = M'HM$, where $M = \text{D}(\phi)(0)$ and $H = \text{D}^2(P)(0)$. But since $\text{rank}(M) \leq p \leq N$ and $n > N$ this is impossible. Therefore $\dim(\Delta_{\infty}) = \infty$. 

5. SOME OTHER EXAMPLES

5.1. Dimension zero spaces are discrete

A diffeological space is said to be discrete if its diffeology is generated by the empty set. A diffeological space has dimension zero if and only if it is discrete.

5.2. Has the set $\{0, 1\}$ dimension 1?

Let us consider the set $\{0, 1\}$ and $\pi : \mathbb{R} \rightarrow \{0, 1\}$ be the parametrization defined by: $\pi(x) = 0$ if $x \in \mathbb{Q}$, and $\pi(x) = 1$ otherwise. Let $\{0, 1\}_\pi$ be the set $\{0, 1\}$ equipped
with the diffeology generated by \( \pi \). Thus \( \dim(\{0, 1\}) = 1 \). So, a diffeological space made of a finite number of points may have a non zero dimension.

### 5.3. Dimension of tori

Let \( \Gamma \subset \mathbb{R} \) be a strict subgroup of \((\mathbb{R}, +)\) and let \( T_\Gamma \) be the quotient \( \mathbb{R}/\Gamma \). So, \( \dim(T_\Gamma) = 1 \). This applies in particular to the circles \( \mathbb{R}/a\mathbb{Z} \), with length \( a > 0 \), or to **irrational tori** [3] when \( \Gamma \) is generated by more than one generators, rationally independent.

**ACKNOWLEDGEMENTS**

I am pleased to thank Prof. Hans Duistermaat for his advices concerning the writing of this paper. It is also a pleasure to thank the Hebrew University of Jerusalem Israel, for its warm hospitality.

**REFERENCES**


(Received September 2006)