DIFFERENTIAL FORMS ON STRATIFIED SPACES

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Abstract. First, we extend the notion of stratified spaces to diffeology. Then, we characterize the subspace of stratified differential forms, or zero-perverse forms in the sense of Goresky-MacPherson, which can be extended smoothly into differential forms on the whole space. For that we introduce an index which outlines the behavior of the perverse forms on the neighborhood of the singular strata.

Introduction

Stratified spaces have been introduced in the founding papers [Whi47, Tho62, Mat70]. Then, the theory has been developed by many authors in various ways, and had resulted in an important literature on the subject.

From a pure differential point of view, the standard definition of stratified spaces is unsatisfactory. The simple juxtaposition of the topological structure of the global space, with the relatively independent smooth structures of the strata that constitute the space, is very perturbing. Especially when it comes to Cartan calculus on stratified spaces, for which a global smooth structure is obviously needed.

As differential geometers we expect a unique smooth structure on the whole object that captures at the same time its global smooth structure, even stratified, and the individual structure of each stratum. Diffeology is a good candidate for a such framework [PIZ13], that can mix a global singular smooth structure with individual characteristics of the strata. And since a whole Cartan Calculus is well developed in diffeology, it will apply straightforwardly on stratified spaces.

Note that Diffeology has been already used to solve a few questions mixing singularities and smooth structures. It is the case, for example, for dense foliations [DI83] [IL90], or for orbifolds [IKZ10] [IZL17], and mixed with differential forms, to integrate general closed 1 and 2-forms, with any countable group of periods [PIZ13, §8.29, 8.42], or in symplectic reduction with singular orbits [PIZ16].

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We should cite also a few authors who have chosen the dual approach to the global smooth structure: the Differential Structure framework, for example M. Pflaum in [Pfl01] or J. Śniatycki in [Sni13].

As we shall see in the first section, it is not difficult to adapt the ordinary definition of stratified spaces to diffeology. It would be basically a diffeological space, given with a stratification for the D-topology (art. 1) (art. 3). That definition leaves a large degree of freedom on the adequation between the diffeology and the stratification, since there can be more than one diffeology on a space that give the same D-topology. That leads us to single out a subcategory of diffeological spaces for which the stratification is defined by its geometry, that is, by the action of its pseudo-group of local diffeomorphisms (art. 4). In this case, the stratification is completely encrypted in the diffeology: the strata are the connected components of the orbits of the pseudo-group of local diffeomorphisms. That defines a subcategory of {Diffeology} we can call {Stratified Diffeology}. Manifolds with corners are simple examples of such stratified diffeology, see [GIZ17]. But we left the study of this subcategory of *geometric stratified spaces* for a later time.

Coming back on the question of Cartan Calculus on stratified spaces, since diffeological space have a well defined De Rham complex, stratified diffeological spaces inherit this complex immediately. But in the literature on stratified spaces, there exists already a notion of "stratified (differential) form", these are the form of (general) perversity p, according to M. Goresky and R. MacPherson [MGRM77, MG81] and defined precisely by J.-L. Brylinski in [Bry86]. The aim of these authors is to establish a pairing between a complex of singular intersection chains and the complex of stratified forms with perversity p. These complexes of perverse forms are also involved in the computation of the equivariant intersection cohomology, originally by J.-L. Brylinski as cited above, but also by M. Brion in [Bri96] for example.

Hence, a natural question is to compare theses two classes of objects: differential forms versus stratified forms, beginning with perversity zero. That is the case we treat in this paper.

We should immediately precise that stratified forms are defined only on the regular part of a stratified space, with some conditions on the neighborhood of the strata, while differential forms are defined on the whole space [PIZ13, §6.28]. *The question is then to characterize the stratified forms that are restriction of differential forms*.

For that purpose we introduce an index that counts the number of different differential forms defined on each strata (Precisely, on the universal coverings of the strata). by a given stratified form (art. 5). Hence we show that if its index is equal to 1 for any stratum, then the stratified form extend into a differential form for the diffeology involved, (art. 6).

Conversely, we consider only the subcategory of stratified diffeological spaces that satisfy the condition that, for any two points in the regular subset there is always

a smooth path connecting them, that cuts the singular locus into a finite number of points. Because the fibers of the local tube systems are stratified spaces, that condition apply to them too. Then, in this context, we show that the restriction of a zero-perverse differential form has its index constant and equal to one for each stratum (art. 6). The condition we introduce here does not seem to be optimal, but it is satisfied on most common stratified spaces, for example on semi-algebraic sets. It is still not clear for us if this result is true generally.

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Stratified Spaces

In this section we recall the standard definition of stratified spaces, in the style of Kloeckner's survey [Klo07]. Then, we recall the notion of diffeology and the smooth category associated. That leads to a natural version of stratified spaces in diffeology.

1. Basic Stratified Spaces. A stratification on a diffeological space X is a partition $\mathscr{S}$ of X into *strata*,

$$X = \bigcup_{S \in \mathscr{S}} S \quad \text{with} \quad S \neq S' \Rightarrow S \cap S' = \varnothing,$$

that satisfy the following *boundary condition*

$$S \cap \bar{S}' \neq \varnothing \Rightarrow S \subset \bar{S}',$$

where $\bar{S}'$ represents the closure of S' for the D-topology of X [PIZ13, §2.8].

The boundary condition can be formulted as follows: the closure of a stratum is a union of strata. In the usual case, where strata are manifolds, the strata are organised by dimension and define a filtration:

$$X_0 \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k = X.$$

The *subsets of the filtration* X_j are the unions of all the strata with dimension less or equal to a given dimension, let us say n_j. The subset $X_j - X_{j-1}$, is open in X_j and its components are the strata of dimension n_j. The subset $X_k - X_{k-1}$ is called the *regular part* of X and is denoted by X_{reg}. It is the union of maximal dimension strata. The subset X_{k-1} is called the *singular part* and is denoted by X_{sing}.

For all $j > 0$, the subset $X' = X_j$ of the filtration is itself a stratified space with regular part $X'_{\text{reg}} = X_j - X_{j-1}$. The subset X_0 is the union of minimal dimension strata, it is a stratified space without singular part.

In the following we shall assume this:

- The space X is connected, Hausdorff and metrizable.

- The regular part is an open dense subset.
- Equipped with the subset diffeology, the strata are locally closed manifolds.
- The number of strata is finite

In a future work it could possible to ease these conditions.

2. Diffeology And Diffeological Spaces. A *diffeology* on a set X is the choice of a set $\mathscr{D}$ of parametrizations in X which satisfies the following axioms.

(1) Covering: $\mathscr{D}$ contains the constant parametrizations.
(2) Locality: Let P be a parametrization in X. If for all $r \in \mathrm{dom}(P)$ there is an open neighbourhood V of r such that $P \restriction V \in \mathscr{D}$, then $P \in \mathscr{D}$.
(3) Smooth compatibility: For all $P \in \mathscr{D}$, for all $F \in C^\infty(V, \mathrm{dom}(P))$, where V is a Euclidean domain, $P \circ F \in \mathscr{D}$.

We recall that a parametrization is a map defined on an open subset of an Euclidean space. A set X equipped with a diffeology is a *diffeological space*. The elements of $\mathscr{D}$ are then called plots of the diffeological space.

Smooth Maps. — *A map $f : X \to X'$ is said to be* smooth *if for any plot* P *in* X, $f \circ P$ *is a plot in* X'. *If f is smooth, bijective, and if its inverse f^{-1} is smooth, then f is said to be a* diffeomorphism.

Diffeological spaces and smooth maps constitute the category {Diffeology} whose isomorphisms are diffeomorphisms.

Subset Diffeology, Subspaces. — *Let* A *be a subset of a diffeological space* X. *The plots in* X *which take their values in* A *are a diffeology called* subset diffeology. *Equipped with this diffeology,* A *is said to be a* subspace *of* X.

Local Smooth Maps. — *The finest topology on* X *such that the plots are continuous is called D-Topology. A map $f : A \to X'$, where* A *is a subset of* X, *is said to be* local smooth *if* A *is a D-open subset of* X, *and f is smooth for the subset diffeology. We denoted by* $C^\infty_{\mathrm{loc}}(X, X')$ *the set of local smooth maps from* X *to* X'.

Actually f is local smooth if and only if, for all plot P in X, the composite $f \circ P : P^{-1}(A) \to X'$ is a plot. That implies in particular that A is D-open, by definition of the D-topology.

Local Diffeomorphisms. — *We say that $f : A \to X'$ is a local diffeomorphism if it is a local smooth injective map, as well as its inverse $f^{-1} : f(A) \to X$. We denoted by* $\mathrm{Diff}_{\mathrm{loc}}(X, X')$ *the set of local diffeomorphisms from* X *to* X'.

Diffeological Fibration. — *We say that a smooth projection $\pi : T \to B$ between diffeological spaces is a diffeological fibration, or* T *is a diffeological fiber bundle over* B, *if for all plot* $P : U \to B$, *the pullback* $\mathrm{pr}_1 : P^*(T) \to U$ *is locally trivial, see* [Igl85] *and* [PIZ13, §8.8, 8.9].

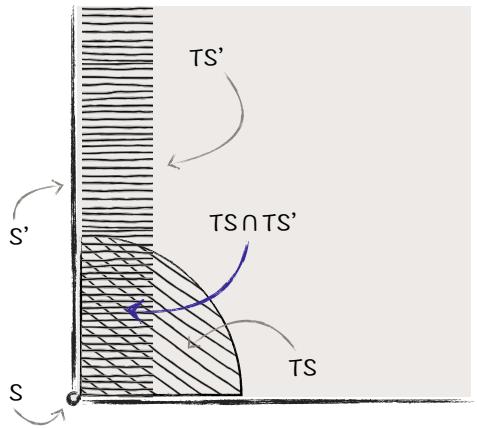

FIGURE 1. — Tubes Around Strata

3. Locally Fibered Stratified Spaces. Consider a diffeological space X equipped with a stratification $\mathscr{S}$. We shall say that the stratification is *locally fibered* if there exists a *tube system* $\{\pi_S\colon TS \to S)\}_{S\in\mathscr{S}}$ such that:

(1) TS is an open neighborhood of S, called a *tube* over S.
(2) The map $\pi_S\colon TS \to S$ is a smooth retraction which is a diffeological fibration, with fibers stratified spaces.
(3) For all $x \in TS \cap TS' \cap \pi_{S'}^{-1}(TS)$, one has $\pi_S(\pi_{S'}(x)) = \pi_S(x)$.

It is worth noticing here that the case of the *locally cone-like* stratified spaces enter in this category. And then, following Larry Siebenmann [Sie72], essentially all kinds of stratified spaces. Note also that in diffeology this wording is ambiguous. Indeed not all kinds of diffeological spaces that look like cones are equivalent, even if they share the same D-topology, as shows the following example. Consider the cone

$$\mathscr{C} = \left\{(x,y,z) \in \mathbf{R}^3 \mid z = \sqrt{x^2+y^2}\right\}.$$

We can equip $\mathscr{C}$ with the subset diffeology and also with the diffeology of the *cone over the circle* S^1. That is, the pushforward of the diffeology of the cylinder, by

$$\pi\colon S^1 \times [0,\infty[\to \mathscr{C} \quad \text{with} \quad \pi(u,t) = (tu,t).$$

Then, the parametrization defined on $\mathbf{R}$ by

$$\gamma\colon s \mapsto e^{-1/s^2}\begin{pmatrix}\cos 1/s^2\\ \sin 1/s^2\\ 1\end{pmatrix} \quad \text{if} \quad s \neq 0 \quad \text{and} \quad \gamma(0) = 0.$$

is a plot for $\mathscr{C}$ embedded in $\mathbf{R}^3$ but not for the cone over the circle, because the parametrization $s \mapsto (\cos 1/s^2, \sin 1/s^2)$ does not converge for $s = 0$.

4. Formal and Geometric Stratifications. The general definition of a diffeological space leave some space between topological and smooth condition. The same

partition of a space can be a stratification for different diffeologies, since different diffeology can have the same D-topology. We shall see here an example of this situation.

Example 1. — Consider the real line $\mathbf{R}$, equipped with the standard diffeology. We define the strata

$$\mathrm{S}_- =]-\infty, 0[, \quad \mathrm{S}_0 = \{0\} \quad \text{and} \quad \mathrm{S}_+ =]0, +\infty[.$$

And the tube system:

$$\mathrm{TS}_\pm = \mathrm{S}_\pm, \quad \pi_{\mathrm{S}_\pm} : x \mapsto x; \quad \mathrm{TS}_0 = \mathbf{R}, \quad \pi_{\mathrm{S}_0} : x \mapsto 0.$$

One can check that that describes a locally fibered stratified space.

Example 2. — Consider the positive right angle L in $\mathbf{R}^2$ made of point (x, y) such that: $(x, y \geq 0)$ and $(x = 0$ or $y = 0)$.

$$\mathsf{L} = \{(0, y) \mid y \geq 0\} \cup \{(x, 0) \mid x \geq 0\}.$$

We equip L with the subset diffeology of $\mathbf{R}^2$. Let us define the strata

$$\mathrm{S}_- = \{(0, y) \mid y > 0\}, \quad \mathrm{S}_+ = \{(x, 0) \mid x > 0\} \quad \text{and} \quad \mathrm{S}_0 = \{(0, 0)\}.$$

We define the tube system:

$$\mathrm{TS}_\pm = \mathrm{S}_\pm, \text{ with } \pi_{\mathrm{S}_\pm} : (x, y) \mapsto (x, y); \text{ and } \mathrm{TS}_0 = \mathsf{L}, \text{ with } \pi_{\mathrm{S}_0} : (x, y) \mapsto (0, 0).$$

One can check that that describes a locally fibered stratified space.

Let $f : \mathsf{L} \to \mathbf{R}$ be the bijection defined by $f(x, y) = x - y$. Then, equip $\mathbf{R}$ with the pushforward of the diffeology of L by f. Denote it by $\mathbf{R}_{\mathsf{L}}$. The map f send the stratification of L onto the stratification of $\mathbf{R}$ from Example 1. Obviously, the diffeology of $\mathbf{R}_{\mathsf{L}}$ does not coincide with the standard diffeology, it is strictly finer, but it induces the same D-topology. We have then two identical smooth stratifications on the same set, but equipped with different diffeologies.

The main difference between the two previous examples is the adequation between the stratification and the action of the pseudogroup of local diffeomorphisms. In the second example the stratification is given by the action of the pseudogroup of diffeomorphisms: the strata are the connected components of its orbits [GIZ16]. *A contrario*, in the first example the pseudogroup of local diffeomorphisms is transitive, the stratification is transparent for the local diffeomorphisms, it has no structural geometric frame. This suggests a specification for *geometric stratified spaces* in diffeology:

Every diffeological space X admits a natural partition $\mathscr{S}$ in connected components of the orbits of the pseudogroup $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{X})$ of local diffeomorphisms. They are called the Klein strata [PIZ13, 1.48]. We can single out the spaces X for which this partition is a stratification, or more precisely a local fibered stratification. We shall talk in this case of *geometric stratification*, when the stratification is given by the action of the pseudogroup of local diffeomorphisms, and of *formal stratification* in the opposite case.

Diffeological spaces that admit a geometric stratification form naturally a full subcategory in {Diffeology}, we could call it {Stratified Diffeology}. Obviously, diffeomorphisms between diffeologcal spaces respect their natural stratifications. Manifolds with corners are the first example of such geometric stratified diffeological spaces [GIZ17].

Differential Forms

In this section we give a necessary and sufficient condition for a stratified differential form, defined on the regular part of some locally fibered stratified space X, to be the restriction of a differential form in the sense of diffeology, defined on the whole space. Let us recall the two definitions in play here.

Stratified Differential Form. We call *stratified differential form* on a stratified space, any differential form with perversity zero, according to M. Goresky and R. MacPherson [MGRM77, MG81] and defined by J.-L. Brylinski in [Bry86]. Perecisely:

Definition. — *A* stratified k-form, *on a locally fibered stratified space* X *is a differential k-form* α, *defined on the regular part* $X_{reg} \subset X$, *such that: for every stratum* $S \in \mathscr{S}$, *for all points* $x \in TS \cap X_{reg}$, *for all vectors* $\xi \in \ker D(\pi_S)_x$,

$$\alpha_x(\xi) = 0 \quad \textit{and} \quad d\alpha_x(\xi) = 0, \tag{$\clubsuit$}$$

where $d\alpha_x(\xi)$ *denotes the contraction of* $d\alpha_x$ *with* ξ, *denoted equivalently by* $i_\xi(d\alpha_x)$ *or sometimes by* $\xi \bullet d\alpha_x$.

The space of stratified differential k-forms α is denoted by $\Omega^k_0[X]$. Note that α belongs *a priori* to $\Omega^k(X_{reg})$.

Differential forms in diffeology. Differential forms in diffeology have been introduced by Jean-Marie Souriau in [Sou84], and then developed in [PIZ13].

Definition. — *A* differential k-form α *on a diffeological space* X *is a mapping that associates with every plot* $P\colon U \to X$, *a smooth k-form* $\alpha(P)$ *on* U *such that: for any smooth parametrization* F *in* U,

$$\alpha(P \circ F) = F^*(\alpha(P)).$$

5. Strata Forms Associated With A Form Of Perversity Zero. Let X be a diffeological space equipped with a stratification $\mathscr{S}$ and a tube system $\{\pi_S\colon TS \to S\}_{S\in\mathscr{S}}$, as described above. Let $\alpha \in \Omega^k_0[X]$, $S \in \mathscr{S}$ and $\mathrm{pr}\colon \tilde{S} \to S$ be its universal covering. Consider the restriction

$$\pi_S \restriction TS \cap X_{reg}\colon TS \cap X_{reg} \to S.$$

This is a fiber bundle over S, let us denote by F its fiber. With α, we shall now associate a set $\mathscr{A}_S$ of differential k-forms on $\tilde{S}$, indexed by the connected components of F,

$$\mathscr{A}_S = \{\bar{\alpha}_a\}_{a\in\pi_0(F)} \quad \text{with} \quad \bar{\alpha}_a \in \Omega^k(\tilde{S}), \tag{$\heartsuit$}$$

in the way described below.

(Step 1.) — Let $\mathrm{pr}\colon \tilde{S} \to S$ be the universal covering of S. The strata are always assumed to be connected. Let $\mathrm{pr}^*(TS\cap X_{reg})$ be the pullback of $\pi_S \restriction TS\cap X_{reg}$ by pr. That is,

$$\mathrm{pr}^*(TS\cap X_{reg}) = \{(\tilde{x}, y) \in \tilde{S}\times TS\cap X_{reg} \mid \mathrm{pr}(\tilde{x}) = \pi_S(y)\}.$$

Let $\mathrm{pr}_1\colon \mathrm{pr}^*(TS\cap X_{reg}) \to \tilde{S}$ and $\mathrm{pr}_2\colon \mathrm{pr}^*(TS\cap X_{reg}) \to TS\cap X_{reg}$ be the first and second projections:

$$\begin{tikzcd}
\mathrm{pr}^*(TS\cap X_{reg}) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & TS\cap X_{reg} \arrow[d, "\pi_S"] \\
\tilde{S} \arrow[r, "\mathrm{pr}"'] & S
\end{tikzcd}$$

The exact homotopy sequence of pr_1 gives

$$\pi_1(\tilde{S}) = \{0\} \longrightarrow \pi_0(F) \longrightarrow \pi_0(\mathrm{pr}^*(TS\cap X_{reg})) \longrightarrow \pi_0(\tilde{S}) = \{\tilde{S}\}.$$

Thus,

$$\pi_0(F) \simeq \pi_0(\mathrm{pr}^*(TS\cap X_{reg})).$$

Now, since pr_1 is a fibration with fiber F, each connected component a of F defines a connected component of $\mathrm{pr}^*(TS\cap X_{reg})$ over $\tilde{S}$. The total space $\mathrm{pr}^*(TS\cap X_{reg})$ is then diffeomorphic to the sum of these connected components over $\tilde{S}$. What we could denote by

$$\mathrm{pr}^*(TS\cap X_{reg}) = \coprod_{a\in\pi_0(F)} \left\{\mathrm{pr}^*(TS\cap X_{reg})\right\}_a.$$

The restriction of pr_1 to the component $\left\{\mathrm{pr}^*(TS\cap X_{reg})\right\}_a$ is a fiber bundle with fiber $a \in \pi_0(F)$.

(Step 2.) — Consider the k-forms

$$\tilde{\alpha} = \mathrm{pr}_2^*(\alpha) \in \Omega^k(\mathrm{pr}^*(TS\cap X_{reg})) \quad \text{and the} \quad \tilde{\alpha}_a = \tilde{\alpha} \restriction \left\{\mathrm{pr}^*(TS\cap X_{reg})\right\}_a,$$

with $a \in \pi_0(F)$, restrictions of $\tilde{\alpha}$ to the connected components of $\mathrm{pr}^*(TS\cap X_{reg})$.

The forms $\tilde{\alpha}_a$ defined on $\left\{\mathrm{pr}^*(TS\cap X_{reg})\right\}_a$ satisfy the condition ($\clubsuit$) of perversity zero. Indeed, $D(\mathrm{pr}_2)$ maps $\ker(D(\mathrm{pr}_1))$ to $\ker(D(\pi_S))$. Now, since $\left\{\mathrm{pr}^*(TS\cap X_{reg})\right\}_a$ is connected, the condition ($\clubsuit$) means exactly that $\tilde{\alpha}_a$ is basic, that is, there exists a k-form $\bar{\alpha}_a$ on the covering $\tilde{S}$ such that:

$$\tilde{\alpha}_a = \mathrm{pr}_1^*(\bar{\alpha}_a) \quad \text{and then,} \quad \mathscr{A}_S = \{\bar{\alpha}_a\}_{a\in\pi_0(F)}.$$

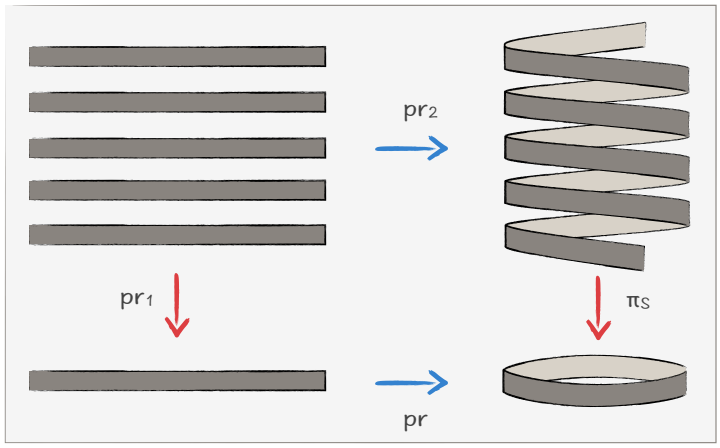

FIGURE 2. — Pullback of $\pi_S \restriction TS \cap X_{reg}$ by pr.

In other words,

$$pr_2^*(\alpha) \restriction \left\{ pr^*(TS \cap X_{reg}) \right\}_a = pr_1^*(\bar{\alpha}_a) \qquad (\spadesuit)$$

Then,

THEOREM. — *There exists a differential k-form α_S on the stratum S such that $\alpha \restriction TS = \pi_S^*(\alpha_S)$ if and only if, for all $a, b \in \pi_0(F)$, $\bar{\alpha}_a = \bar{\alpha}_b$. In this case, for all $a \in \pi_0(F)$, $\bar{\alpha}_a = pr^*(\alpha_S)$.*

Introducing the *index* of the form $\alpha \in \Omega_0^*[X]$ at the stratum S, as the number

$$\nu_S(\alpha) = \mathrm{card}(\mathscr{A}_S),$$

we can paraphrase the proposition as follows: *There exists a differential k-form α_S on the stratum S such that $\alpha \restriction TS = \pi_S^*(\alpha_S)$ if and only if $\nu_S(\alpha) = 1$.*

Proof. Let us assume that there exists α_S such that $\alpha \restriction TS = \pi_S^*(\alpha_S)$. Then, let

$$pr_{1,a} = pr_1 \restriction \left\{ pr^*(TS \cap X_{reg}) \right\}_a \quad \text{and} \quad pr_{2,a} = pr_2 \restriction \{pr^*(TS \cap X_{reg})\}_a.$$

Hence, $\pi_S \circ pr_{2,a} = pr \circ pr_{1,a}$. Now,

$$pr_{2,a}^*(\alpha) = pr_{2,a}^*(\pi_S^*(\alpha_S)) = pr_{1,a}^*(pr^*(\alpha_S)).$$

But we have

$$pr_{2,a}^*(\alpha) = pr_{1,a}^*(\bar{\alpha}_a).$$

Thus,

$$pr_{1,a}^*(\bar{\alpha}_a) = pr_{1,a}^*(pr^*(\alpha_S)).$$

But since $pr_{1,a}$ is a fibration, $\bar{\alpha}_a = pr^*(\alpha_S)$. That is, $\bar{\alpha}_a = \bar{\alpha}_b$ for all $a, b \in \pi_0(F)$.

Conversely, assume that $\bar{\alpha}_a = \bar{\alpha}_b$ for all $a, b \in \pi_0(F)$. Let us denote by $\bar{\alpha}$ this differential form on $\tilde{S}$. We want to prove that, for all $k \in \pi_1(S)$, $\underline{k}^*(\bar{\alpha}) = \bar{\alpha}$.

Let us remark first that, because $\mathrm{pr}(\underline{k}(\tilde{x})) = \mathrm{pr}(\tilde{x})$, $\pi_1(\mathrm{S})$ acts on $\mathrm{pr}^*(\mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}})$ by

$$\text{for all } k \in \pi_1(\mathrm{S}), \quad \underline{k}(\tilde{x}, y) = (\underline{k}(\tilde{x}), y),$$

where $\underline{k}$ denotes indifferently the two actions of k. Note that:

- The action of $\pi_1(\mathrm{S})$ on $\mathrm{pr}^*(\mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}})$ is free. Every element k in $\pi_1(\mathrm{S})$ acts by diffeomorphism. In particular it exchanges the connected components.
- The two actions of $\pi_1(\mathrm{S})$ intertwin pr_1, that is, $\mathrm{pr}_1 \circ \underline{k} = \underline{k} \circ \mathrm{pr}_1$.
- The projection pr_2 is invariant by $\pi_1(\mathrm{S})$, $\mathrm{pr}_2 \circ \underline{k} = \mathrm{pr}_2$.

Lemma. — *If $\underline{k}$ sends the component relative to $a \in \pi_0(\mathrm{F})$ onto the component relative to b, then $\bar{\alpha}_a = \underline{k}^*(\bar{\alpha}_b)$.*

◀ *Proof.* — First of all, $\tilde{\alpha}$ is invariant under the action of $\pi_1(\mathrm{S})$. Indeed, $\underline{k}^*(\mathrm{pr}_2^*(\alpha)) = \mathrm{pr}_2^*(\alpha)$, that is, $\underline{k}^*(\tilde{\alpha}) = \tilde{\alpha}$. Let $k \in \pi_1(\mathrm{S})$, and maps the components

$$\underline{k} \colon \left\{ \mathrm{pr}^*(\mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}}) \right\}_a \to \left\{ \mathrm{pr}^*(\mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}}) \right\}_b.$$

Let $j_i \colon \left\{ \mathrm{pr}^*(\mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}}) \right\}_i \to \mathrm{pr}^*(\mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}})$ be the inclusion of the component, for all $i \in \pi_0(\mathrm{F})$. Then, $\underline{k} \circ j_a = j_b \circ \underline{k}$. Hence, $(\underline{k} \circ j_a)^*(\tilde{\alpha}) = (j_b \circ \underline{k})^*(\tilde{\alpha})$, that is, $j_a^*(\underline{k}^*(\tilde{\alpha})) = \underline{k}^*(j_b^*(\tilde{\alpha}))$. But, $\underline{k}^*(\tilde{\alpha}) = \tilde{\alpha}$, then $j_a^*(\tilde{\alpha}) = \underline{k}^*(j_b^*(\tilde{\alpha}))$. That is, $\tilde{\alpha}_a = \underline{k}^*(\tilde{\alpha}_b)$, where $\tilde{\alpha}_i = \tilde{\alpha} \restriction \left\{ \mathrm{pr}^*(\mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}}) \right\}_i$. Hence, since $\tilde{\alpha}_i = \mathrm{pr}_1^*(\bar{\alpha}_i)$, $\mathrm{pr}_1^*(\bar{\alpha}_a) = \underline{k}^*(\mathrm{pr}_1^*(\bar{\alpha}_b))$. Now, $\underline{k} \circ \mathrm{pr}_1 = \mathrm{pr}_1 \circ \underline{k}$, then $\mathrm{pr}_1^*(\bar{\alpha}_a) = \mathrm{pr}_1^*(\underline{k}^*(\bar{\alpha}_b))$. And since pr_1 is a fibration, we get finally $\bar{\alpha}_a = \underline{k}^*(\bar{\alpha}_b)$. ▶

As a corollary, if $\bar{\alpha}_a = \bar{\alpha}_b$ (denoted by $\bar{\alpha}$), then $\bar{\alpha}$ is invariant by $\pi_1(\mathrm{S})$. Hence, $\bar{\alpha}$ is basic with respect to the group $\pi_1(\mathrm{S})$, then there exists $\alpha_{\mathrm{S}} \in \Omega^k(\mathrm{S})$ such that $\bar{\alpha} = \mathrm{pr}^*(\alpha_{\mathrm{S}})$. Next, on the one hand, by the commutative diagram $\pi_{\mathrm{S}} \circ \mathrm{pr}_2 = \mathrm{pr} \circ \mathrm{pr}_1$, we have $\mathrm{pr}_2^*(\pi_{\mathrm{S}}^*(\alpha_{\mathrm{S}})) = \mathrm{pr}_1^*(\mathrm{pr}^*(\alpha_{\mathrm{S}})) = \mathrm{pr}_1^*(\bar{\alpha})$. On the other hand, (♠) writes $\mathrm{pr}_{2,a}^*(\alpha) = \mathrm{pr}_1^*(\bar{\alpha}_a)$, and since $\bar{\alpha}_a = \bar{\alpha}_b = \bar{\alpha}$, $\mathrm{pr}_2^*(\alpha) = \mathrm{pr}_1^*(\bar{\alpha})$. Therefore, $\mathrm{pr}_2^*(\pi_{\mathrm{S}}^*(\alpha_{\mathrm{S}})) = \mathrm{pr}_2^*(\alpha)$, and because pr_2 is a fibration, we get $\alpha \restriction \mathrm{TS} = \pi_{\mathrm{S}}^*(\alpha_{\mathrm{S}})$. That completes the proof. □

6. Stratified Differential Forms as Differential Forms. Let X be a diffeological space equipped with a locally fibered stratification, as described above. Considering the index function ν defined in (art. 5), we have:

Theorem. — *Let $\alpha \in \Omega^k_{\mathrm{o}}[\mathrm{X}]$. If $\nu_{\mathrm{S}}(\alpha) = 1$ for all $\mathrm{S} \in \mathscr{S}$, then then there exists a (unique) differential form $\underline{\alpha} \in \Omega^k(\mathrm{X})$ such that $\alpha = \underline{\alpha} \restriction \mathrm{X}_{\mathrm{reg}}$. Conversely, we consider the subcategory of stratified diffeological spaces, such that any two points in the regular part can be connected by a smooth path that passes through the singular locus in only a finite number of points. Let $\underline{\alpha} \in \Omega^k(\mathrm{X})$ such that $\alpha = \underline{\alpha} \restriction \mathrm{X}_{\mathrm{reg}} \in \Omega^k_{\mathrm{o}}[\mathrm{X}]$. Then, $\nu_{\mathrm{S}}(\alpha) = 1$ for all $\mathrm{S} \in \mathscr{S}$.*

Remark 1. — The perversity condition applying on the entire tube around the strata is clearly too strong. Indeed, in the case of the simple example treated in (art. 4), for 0-forms, that is real functions, one gets only restrictions of constant functions. That is clearly insufficient. We should get instead all the smooth functions locally

constant on the neighborood of the origin, logically. We can obtain that result by weakening the condition of perversity, and considering the germs of the tubes around the strata. That is, given a system of tubes, we should say that a form is 0-perverse if there exists a sub-system of tubes, made of restrictions of the original system around each stratum, for which the form is 0-perverse. In this condition the property of perversity becomes semi-local, which is more appropriate. This has been adopted by a few authors, for example [Pfl01, p. 23] [Pol05, p.83]. In the examples above we get then, as stratified real functions, the restrictions of any smooth function locally constant on the neighborhood of the origin. Considering 1-forms, we would obtain differential forms that vanish locally around the origin.

Note then that smooth functions for the diffeology are obviously every functions on $\mathbf{R}$, in the first case, and contains at least all the restrictions of any smooth function to L in the second case. We treated the general case of forms on corners in [GIZ17].

Remark 2. — In the second part of the theorem, the condition is only technical to ensure a converse property. It is possible that there is no need of this hypothesis but we could not find a general proof until now.

Proof. Let $\alpha \in \Omega^k_0[X]$ such that $\nu_S(\alpha) = 1$, for all $S \in \mathscr{S}$. Thanks to (art. 5), for all $S \in \mathscr{S}$, there exists $\alpha_S \in \Omega^k(S)$ such that $\alpha \restriction TS \cap X_{reg} = \pi_S^*(\alpha_S)$. Let S and S′ be two strata such that $TS \cap TS' \neq \varnothing$. On $TS \cap TS' \cap X_{reg}$, which is open, $\pi_S^*(\alpha_S) = \pi_{S'}^*(\alpha_{S'}) = \alpha$. Then, consider $x \in TS \cap TS'$ but $x \notin X_{reg}$, that is, $x \in TS \cap TS' \cap X_{sing}$, that is, $x \in S'' \cap TS \cap TS'$ with $S'' \subset X_{sing}$. Thus $S'' \cap TS \neq \varnothing$, that implies in particular $S \subset \bar{S}''$ and $\pi_S \circ \pi_{S''} = \pi_S$ on $TS \cap TS'' \cap \pi_{S''}^{-1}(TS)$, according to the definition of locally fibered stratified spaces. Hence, $(\pi_S \circ \pi_{S''})^*(\alpha_S) = \pi_S^*(\alpha_S)$, that is, $\pi_{S''}^*(\pi_S^*(\alpha_S)) = \pi_S^*(\alpha_S)$. On the regular part $TS \cap TS'' \cap \pi_{S''}^{-1}(TS) \cap X_{reg}$, $\pi_S^*(\alpha_S) = \alpha = \pi_{S''}^*(\alpha_{S''})$, thus, $\pi_{S''}^*(\pi_S^*(\alpha_S)) = \pi_{S''}^*(\alpha_{S''})$ on $TS \cap TS'' \cap \pi_{S''}^{-1}(TS) \cap X_{reg}$, that is,

$$\pi_{S''}^*(\pi_S^*(\alpha_S) - \alpha_{S''}) \restriction TS \cap TS'' \cap \pi_{S''}^{-1}(TS) \cap X_{reg} = 0.$$

But $\pi_{S''} \restriction TS \cap TS'' \cap \pi_{S''}^{-1}(TS) \cap X_{reg}$ is a submersion on $S'' \cap TS$. Thus,

$$\alpha_{S''} \restriction S'' \cap TS = \pi_S^*(\alpha_S).$$

And therefore, on $TS'' \cap TS$:

$$\pi_{S''}^*(\alpha_{S''}) = \pi_{S''}^*(\pi_S^*(\alpha_S)) = (\pi_S \circ \pi_{S''})^*(\alpha_S) = \pi_S^*(\alpha_S).$$

As well, on $TS'' \cap TS'$, $\pi_{S''}^*(\alpha_{S''}) = \pi_{S'}^*(\alpha_{S'})$. Note that $TS'' \cap TS'$ and $TS'' \cap TS$ are two open neighborhood to x, Hence, $\pi_S^*(\alpha_S) = \pi_{S'}^*(\alpha_{S'})$ on an open neighborhood of x, which belongs in this case to $TS \cap TS' \cap X_{sing}$. Therefore, $\pi_S^*(\alpha_S) = \pi_{S'}^*(\alpha_{S'})$ on $TS \cap TS'$. And since differential forms on diffeological spaces are local [PIZ13, §6.36], there exists a differential form $\underline{\alpha}$, defined on X, such that, for all $S \in \mathscr{S}$, $\underline{\alpha} \restriction TS = \pi_S^*(\alpha_S)$, and in particular $\alpha = \underline{\alpha} \restriction X_{reg}$.

Now, let $\alpha = \underline{\alpha} \restriction X_{\text{reg}}$ with $\underline{\alpha} \in \Omega^k(X)$. The pullback $\text{pr}_1\colon \text{pr}^*(TS) \to \tilde{S}$ is a locally trivial fiber bundle. Let $\underline{F}$ be its fiber, and then $F = \underline{F}_{\text{reg}}$ be the regular part. The differential form $\text{pr}_2^*(\underline{\alpha})$ is defined on the whole pulback $\text{pr}^*(TS)$, and $\text{pr}_2^*(\alpha) = \text{pr}_2^*(\underline{\alpha}) \restriction \text{pr}^*(TS \cap X_{\text{reg}})$. Then, according to (♠), for each component $a \in \pi_0(F)$, $\text{pr}_2^*(\alpha) \restriction \{TS \cap X_{\text{reg}}\}_a = \text{pr}_1^*(\bar{\alpha}_a)$, that is, $\text{pr}_2^*(\underline{\alpha}) \restriction \{TS \cap X_{\text{reg}}\}_a = \text{pr}_1^*(\bar{\alpha}_a)$. Now, let $y \in F_a$ and $y' \in F_b$ be two points in F, belonging to two different connected components. There exists a smooth path $t \mapsto y_t$ in F that connects y to y' and which cut the singular subset $\underline{F}_{\text{sing}}$ in a finite number of points. The interval $]0,1[$ is then divided into a finite set of open intervals denoted by I_a where y_t belongs to the component $a \in \pi_0(F)$, separated by points belonging to $\underline{F}_{\text{sing}}$. Let $(r,y) \mapsto (\tilde{x}_r, y)$ be a local trivialisation of $\text{pr}_1\colon \text{pr}^*(TS) \to \tilde{S}$, then $(r,t) \mapsto (\tilde{x}_r, y_t)$ is a plot of $\text{pr}^*(TS)$. On the open subset of (r,t) such that $t \in I_a$ one has

$$\begin{aligned} \text{pr}_2^*(\underline{\alpha})((r,t) \mapsto (\tilde{x}_r, y_t))_{\binom{r}{t}} \begin{pmatrix} u_i \\ \varepsilon_i \end{pmatrix}_{i=1}^k &= \text{pr}_1^*(\bar{\alpha}_a)((r,t) \mapsto (\tilde{x}_r, y_t))_{\binom{r}{t}} \begin{pmatrix} u_i \\ \varepsilon_i \end{pmatrix}_{i=1}^k \\ &= \bar{\alpha}_a(r \mapsto \tilde{x}_r)_r(u_1)\dots(u_k), \end{aligned}$$

where $\underline{\alpha}$ is a k-form and the (u_i, ε_i) are tangents vectors at (t,r). But, for each r and $u_1 \dots u_k$, the map

$$t \mapsto \text{pr}_2^*(\underline{\alpha})((r,t) \mapsto (\tilde{x}_r, y_t))_{\binom{r}{t}} \begin{pmatrix} u_i \\ \varepsilon_i \end{pmatrix}_{i=1}^k,$$

is smooth but constant on each I_a, $a \in \pi_0(F)$ with value $\bar{\alpha}_a(r \mapsto \tilde{x}_r)_r(u_1)\dots(u_k)$. Thus, since $\underline{F}$ is connected and it is the closure of F, $\bar{\alpha}_a(r \mapsto \tilde{x}_r)_r(u_1)\dots(u_k) = \bar{\alpha}_b(r \mapsto \tilde{x}_r)_r(u_1)\dots(u_k)$ for all $r, u_1 \dots u_k$. □

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