DIFFERENTIAL FORMS ON MANIFOLDS WITH BOUNDARY AND CORNERS

SERAP GÜRER AND PATRICK IGLESIAS-ZEMMOUR

Abstract. We identify the category of manifolds with boundary and corners with a subcategory of the category {Diffeology}. Then, we show that any differential form on a manifold with boundary or corners, embedded in a smooth manifold, extends to a differential form on an open neighbourhood. As an example of application, we discuss the structure of $\mathrm{SO}(2)^n$ invariant closed 2-forms on $\mathbf{R}^{2n}$.

Introduction

The n-corner $\mathrm{K}^n \subset \mathbf{R}^n$ is defined by $x_i \geq 0$, $i = 1, \ldots, n$, it is equipped with the subset diffeology. The definition of manifolds with corners follows naturally:

Definition. — *A n-manifold with corners is a diffeological space which is everywhere locally diffeomorphic to the corner* K^n.

The introduction of manifolds with corners goes back to J. Cerf and A. Douady in [Cer61, Dou62], and has been since adapted or refined by many authors, for example [ADLH73, GP74, Lee06, Joy10] and more.

In (§3) we show that the classical approach and the diffeological definition lead to the same category, which was mandatory. This result extends the special case of manifolds with boundary addressed in [PIZ13, §4.16].

Considering next differential forms defined on corners, we prove in (§5) the following extension lemma:

Lemma. *Every differential form on the corner* K^n *extends to a smooth form on an open neighbourhood of* K^n *in* $\mathbf{R}^n$.

From which we deduce the main theorem:

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THEOREM. *Every differential form on a manifold with corners, embedded in a smooth manifold, extends smoothly to an open neighbourhood.*

We could have said that every differential form defined on a manifold with corners extends smoothly to an open neighbourhood, but only in the sense that a manifold with corners can always be embedded in itself as a «pièce à coins»[1]; according to Douady et al. in *Arrondissement des Variétés à Coins* [ADLH73, Proposition 3.1]. Let us notice also that the theorem above applies in particular to differential forms on manifolds with boundary, since they are a particular case of manifolds with corners.

In (§6) we give a variation of this theorem for other corners: powers of various *half-lines* $\Delta_k = \mathbf{R}^k/\mathrm{O}(k)$. And, in (§7) we give an application of this analysis to the decomposition of $\mathrm{SO}(2)^n$ invariant closed 2-forms on $\mathbf{R}^{2n}$.

NOTE. — For more details on diffeology, the reader is referred to the textbook *Diffeology* [PIZ13].

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DIFFEOLOGY OF MANIFOLDS WITH CORNERS

We first recall some definitions in diffeology that we shall use in the following.

1. DIFFEOLOGY AND DIFFEOLOGICAL SPACES. A *diffeology* on a set X is the choice of a set $\mathcal{D}$ of parametrizations in X which satisfies the following axioms.

1. COVERING : $\mathcal{D}$ contains the constant parametrizations.
2. LOCALITY : Let P be a parametrization in X. If for all $r \in \mathrm{dom}(\mathrm{P})$ there is an open neighbourhood V of r such that $\mathrm{P} \restriction \mathrm{V} \in \mathcal{D}$, then $\mathrm{P} \in \mathcal{D}$.
3. SMOOTH COMPATIBILITY : For all $\mathrm{P} \in \mathcal{D}$, for all $\mathrm{F} \in \mathrm{C}^\infty(\mathrm{V}, \mathrm{dom}(\mathrm{P}))$, where V is a Euclidean domain, $\mathrm{P} \circ \mathrm{F} \in \mathcal{D}$.

We recall that a parametrization is a map defined on an open subset of an Euclidean space. A set X equipped with a diffeology is a *diffeological space*. The elements of $\mathcal{D}$ are then called plots of the diffeological space.

SMOOTH MAPS. — *A map $f : \mathrm{X} \to \mathrm{X}'$ is said to be* smooth *if for any plot* P *in* X, $f \circ \mathrm{P}$ *is a plot in* X'. *If f is smooth, bijective, and if its inverse f^{-1} is smooth, then f is said to be a* diffeomorphism.

Diffeological spaces and smooth maps constitute the category {Diffeology} whose diffeomorphisms are isomorphisms.

[1]Definition according to Douady and Herault : «Let V be a manifold (without boundary) and let X be a closed set in V. We say that X is a "piece with corners" (*pièce à coins*) of V if, for all $x_0 \in \mathrm{X}$, there exists a neighbourhood U of x_0 in V and functions $u_1, \ldots, u_k$ of class C^∞ on U such that $d_{x_0}u_1, \ldots, d_{x_0}u_k$ are linearly independent and that $\mathrm{X} \cap \mathrm{U}$ is the set of $x \in \mathrm{U}$ such that $u_1(x) \geq 0, \ldots, u_k(x) \geq 0$.»

Subset Diffeology, Subspaces. — *Let* A *be a subset of a diffeological space* X. *The plots in* X *which take their values in* A *are a diffeology called* subset diffeology. *Equipped with this diffeology,* A *is said to be a* subspace *of* X.

Local Smooth Maps. — *The finest topology on* X *such that the plots are continuous is called* D-Topology. *A map* $f : \mathrm{A} \to \mathrm{X}'$, *where* A *is a subset of* X, *is said to be* local smooth *if* A *is a D-open subset of* X, *and* f *is smooth for the subset diffeology. We denote by* $\mathrm{C}^\infty_{\mathrm{loc}}(\mathrm{X}, \mathrm{X}')$ *the set of local smooth maps from* X *to* X′.

Actually f is local smooth if and only if, for all plots P in X, the composite $f \circ \mathrm{P} \colon \mathrm{P}^{-1}(\mathrm{A}) \to \mathrm{X}'$ is a plot. That implies in particular that A is D-open, by definition of the D-topology.

Local Diffeomorphisms. — *We say that* $f : \mathrm{A} \to \mathrm{X}'$ *is a* local diffeomorphism *if it is an injective local smooth map and* f^{-1}, *defined on* $f(\mathrm{A})$, *is also a local smooth map. We denote by* $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{X}, \mathrm{X}')$ *the set of local diffeomorphisms from* X *to* X′. *A smooth map or a local smooth map is said to be* étale *if it is a local diffeomorphism at each point.*

2. Corners as Diffeological Spaces. We denote by K^n the *corner*

$$\mathrm{K}^n = \{(x_i)_{i=1}^n \in \mathbf{R}^n \mid x_i \geq 0,\ i = 1, \ldots, n\},$$

and we equip it with the subset diffeology[2]. A plot in K^n is just a smooth parametrization to $\mathbf{R}^n$ but taking its values in K^n.

The corner K^n is *embedded* in $\mathbf{R}^n$, and closed. That is, the D-topology of the induction $\mathrm{K}^n \subset \mathbf{R}^n$ coincides with the induced topology[3] of $\mathbf{R}^n$.

The natural filtration of K^n, $\mathrm{X}_0 = \{0\} \subset \mathrm{X}_1 \subset \cdots \subset \mathrm{X}_n = \mathrm{K}^n$, is defined by

$$\mathrm{X}_j = \{(x_i)_{i=1}^n \in \mathrm{K}^n \mid \text{there exist } i_1 < \cdots < i_{n-j} \text{ such that } x_{i_\ell} = 0\}.$$

The subset $\mathrm{S}_j = \mathrm{X}_j - \mathrm{X}_{j-1}$ is the set of points in $\mathbf{R}^n$ which have j coordinates positive, and the others null:

$$\mathrm{S}_j = \left\{ (x_i)_{i=1}^n \in \mathbf{R}^n \;\middle|\; \begin{array}{l} \text{There exist } i_1 < \cdots < i_j \text{ such that } x_{i_\ell} > 0, \\ \text{and } x_m = 0 \text{ for all } m \notin \{i_1, \ldots, i_j\} \end{array} \right\}.$$

The *layer* S_j is D-open in X_j, for $j \geq 1$. It is the sum of $\binom{n}{j}$ connected components called *strata*, indexed by a string of j ones and $n - j$ zeros.

Theorem 1. — *A map* $f : \mathrm{K}^n \to \mathbf{R}^k$ *is smooth for the subset diffeology if and only if it is the restriction of a smooth map on some open neighbourhood of* K^n.

This statement means technically that, if for all smooth parametrizations $\mathrm{P} \colon \mathrm{U} \to \mathbf{R}^n$ such that $\mathrm{P}(\mathrm{U}) \subset \mathrm{K}^n$, $f \circ \mathrm{P} \in \mathrm{C}^\infty(\mathrm{U}, \mathbf{R}^k)$, then there exists an open neighbourhood $\mathscr{O}$ of K^n and $\mathrm{F} \in \mathrm{C}^\infty(\mathscr{O}, \mathbf{R}^k)$ such that $f = \mathrm{F} \restriction \mathrm{K}^n$.

[2]The corner K^n is the diffeological n-power of the half-line $\mathrm{K} = [0, \infty[\subset \mathbf{R}$, equipped with the subset diffeology.

[3]The standard topology of $\mathbf{R}^n$ is the D-topology of its standard smooth structure.

THEOREM 2. — *Let* $f \in \mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n)$*. Then,* f *respects the natural stratification, i.e. if* $x \in \mathrm{S}_j$*, then* $f(x) \in \mathrm{S}_j$*. Moreover,* f *is the restriction of an étale map defined on an open neighbourhood of its domain of definition.*

Proof. Theorem 1 was already proven in [PIZ13, §4.16]. For Theorem 2, let P be a parametrization in an Euclidean domain. We denote by $\mathrm{rk}(\mathrm{P})_x$ the rank of P at the point x, that is, the dimension of the image of the tangent linear map $\mathrm{D}(\mathrm{P})(x)$.

LEMMA. — *Let* $\mathrm{P}\colon \mathrm{U} \to \mathrm{K}^n$ *be a plot. If* $\mathrm{P}(r) \in \mathrm{S}_j$*, then* $\mathrm{rk}(\mathrm{P})_r \leq j$.

◀ *Proof.* Let $\mathrm{P}\colon \mathrm{U} \to \mathrm{K}^n$ be a plot and assume that $\mathrm{P}(r) \in \mathrm{S}_j$. Then, $\mathrm{P}(r) = (\mathrm{P}_1(r), \mathrm{P}_2(r), \ldots, \mathrm{P}_n(r))$ where there exist exactly $i_1 < \cdots < i_{n-j}$ indices such that $\mathrm{P}_{i_k}(r) = 0$. Since $\mathrm{P}_{i_k}(r') \geq 0$ for all $r' \in \mathrm{U}$ and $\mathrm{P}_{i_k}(r) = 0$, then $\mathrm{D}(\mathrm{P}_{i_k})(r) = 0$. That is, $\mathrm{rk}(\mathrm{P})_r \leq j$. ▶

Now, let us come back to the local diffeomorphism f, and assume first that f is defined on all K^n. Let $x \in \mathrm{S}_j$ and $x' = f(x) \in \mathrm{S}_k$ and $k \neq j$. We can choose $k > j$. There exists a smooth map F defined on an open neighbourhood $\mathscr{O} \supset \mathrm{K}^n$, such that f and F coincide on K^n, $f = \mathrm{F} \restriction \mathrm{K}^n$. And also, there exists a smooth map G defined on an open neighbourhood $\mathscr{O}' \supset \mathrm{K}^n$, such that f^{-1} and G coincide on K^n, $f^{-1} = \mathrm{G} \restriction \mathrm{K}^n$. The restriction of G on S_k is a plot in K^n, and $\mathrm{G} \restriction \mathrm{S}_k : x' \mapsto x \in \mathrm{S}_j$. By the lemma, $\mathrm{rk}(\mathrm{G} \restriction \mathrm{S}_k)_{x'} \leq j$. But $\mathrm{G} \restriction \mathrm{S}_k = \mathrm{G} \circ j_k$, where $j_k : \mathrm{S}_k \hookrightarrow \mathrm{K}^n$ is identified with a plot. And we know that $(\mathrm{F} \circ \mathrm{G} \restriction \mathrm{S}_k)(t) = \mathrm{F} \circ \mathrm{G} \circ j_k(t) = \mathrm{F} \circ \mathrm{G}(j_k(t))$. But j_k takes values in $\partial \mathrm{K}^n$ (the boundary of K^n). Now, since f is a homeomorphism of K^n for the D-topology, it maps the boundary into the boundary, and G and f^{-1} coincide on the boundary. So we have $\mathrm{F} \circ \mathrm{G}(j_k(t)) = \mathrm{F} \circ f^{-1}(j_k(t))$. As well, F and f coincide on the boundary, and $\mathrm{F} \circ \mathrm{G}(j_k(t)) = f \circ f^{-1}(j_k(t)) = j_k(t)$. Thus, $\mathrm{rk}(\mathrm{F} \circ \mathrm{G} \restriction \mathrm{S}_k)_{x'} = \mathrm{rk}(j_k)_{x'} = k \leq \mathrm{rk}(\mathrm{G} \restriction \mathrm{S}_k)_{x'} \leq j$. But, we assumed that $k > j$ which is a contradiction, and $k = j$.

Now, consider the smooth parametrization $\mathrm{G} \circ \mathrm{F}\colon \mathscr{U} \to \mathbf{R}^n$, with $\mathscr{U} = \mathrm{F}^{-1}(\mathscr{O}')$. Then, $\mathscr{U} \supset \mathrm{K}^n$ and $\mathrm{G} \circ \mathrm{F} \restriction \mathrm{K}^n = \mathbf{1}_{\mathrm{K}^n}$. Hence, for all $x \in \mathring{\mathrm{K}}^n$, $\mathrm{D}(\mathrm{G} \circ \mathrm{F})(x) = \mathrm{D}(f^{-1} \circ f)(x) = \mathrm{D}(\mathbf{1}_{\mathring{\mathrm{K}}^n}) = \mathbf{1}_{\mathbf{R}^n}$, and by continuity, for all $x \in \mathrm{K}^n$, $\mathrm{D}(\mathrm{G} \circ \mathrm{F})(x) = \mathbf{1}_{\mathbf{R}^n}$. Therefore, for all $x \in \mathrm{K}^n$, $\mathrm{rk}(\mathrm{F})_x$ is maximum and equal to n. Hence, F is étale on K^n. And obviously, the same for G. The situation, where f is only local, is similar. □

3. MANIFOLDS WITH CORNERS AS DIFFEOLOGICAL SPACES. The concept of manifolds with corners goes back to Cerf [Cer61, Chap. 1 §1.2], and Douady [Dou62, §4] (as *variétés à bords anguleux*). Over time the various descriptions of manifolds with boundary or corners evolved to a commonly accepted definition, see for example Lee in [Lee06, pp. 251-252] from which we extract the following definition[4].

[4]See also Joyce in [Joy10, Chap. 2].

CLASSICAL DEFINITION. — *Let* M *be a paracompact Hausdorff topological space. A n-chart with corners for* M *is a pair* (U, φ), *where* U *is an open subset of* K^n, *and* φ *is a homeomorphism from* U *to an open subset of* M. *Two charts with corners* (U, φ) *and* (V, ψ) *are said to be smoothly compatible if the composite map* $\psi^{-1} \circ \varphi \colon \varphi^{-1}(\psi(\mathrm{V})) \to \psi^{-1}(\varphi(\mathrm{U}))$ *is a diffeomorphism in the sense that it admits a smooth extension to an open set in* $\mathbf{R}^n$. *An n-atlas with corners for* M *is a pairwise compatible family of n-charts with corners covering* M. *A maximal atlas is an atlas which is not a proper subset of any other atlas. An n-manifold with corners is a paracompact Hausdorff topological space* M *equipped with a maximal n-atlas with corners.*

Then, from the diffeology point of view:

DIFFEOLOGICAL DEFINITION. — *A n-manifold with corners is a diffeological space* X *which is everywhere locally diffeomorphic to the corner* K^n.

The {Manifolds with Corners} form a subcategory of {Diffeology}. The smooth maps between manifolds with corners are just the smooth maps between diffeological spaces. Theorems 1 and 2 of the previous subsections ensure that morphisms between manifolds with corners preserve their natural stratifications.

THEOREM. — *Let* $(\mathrm{M}, \mathscr{A})$ *be a n-manifold with corners according to the classical framework,* $\mathscr{A}$ *being the maximal atlas of* M. *The diffeology* $\mathscr{D}$ *on* M *generated*[5] *by the charts* $\mathrm{F} \in \mathscr{A}$ *is a diffeology of manifold with corners for which* $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M}) = \mathscr{A}$. *The D-topology of* $(\mathrm{M}, \mathscr{D})$ *coincides with the given topology of* $(\mathrm{M}, \mathscr{A})$. *We shall denote* $\Phi \colon (\mathrm{M}, \mathscr{A}) \mapsto (\mathrm{M}, \mathscr{D})$ *this association. Conversely, let* $(\mathrm{M}, \mathscr{D})$ *be a diffeological n-manifold with corners. Equip* M *with its D-topology, then* $\mathscr{A} = \mathrm{Diff}_{\mathrm{loc}}(\mathrm{K}^n, \mathrm{M})$ *is a maximal atlas equipping* M *with a usual structure of manifold with corners. Let* $\Psi \colon (\mathrm{M}, \mathscr{D}) \mapsto (\mathrm{M}, \mathscr{A})$ *be this association. Then* Φ *and* Ψ *are inverse of each other.*

Hence, the classical category of manifolds with corners, defined using the heuristic of what smoothness means for maps on closed subsets of Euclidean domains, fits fully and faithfully in the category {Diffeology}. That shows, by the way, that it is unnecessary to adopt the axiomatics where plots are defined on convex subsets, as in Chen's approach [Che77], to include manifolds with corners in the new category. Specifying the local model to be a corner equipped with its subset diffeology is, indeed, sufficient.

NOTE 1. — As in the case of ordinary manifolds, the category {Manifolds with corners} is closed for products and sums but is not closed for the other usual set theoretical operations. As members of the category {Diffeology}, manifolds with corners inherit of all the diffeological constructions: fiber bundles, homotopy, differential calculus, homology, cohomology, etc.

NOTE 2. — Since, for a corner, coordinates are smooths, manifolds with corners are *reflexive diffeological spaces* [PIZ13, Exercise 79]. Also, the set of smooth real maps from a manifold with corner is a *differential structure* in the sense of Sikorski [Sik72].

[5]See [PIZ13, §1.66] for the definition of *generating family*.

Indeed, real smooth maps from K^n to $\mathbf{R}^m$ are still the restrictions of smooth maps on open neighbourhoods of K^n.

NOTE 3. — The diffeology framework gives a different perspective on the definition of strata of a n-manifold with corners M. We can regard it as the *active point of view*. Indeed, instead of defining the strata through local chart — the *passive point of view*— following Theorem 2 of (§2), one can define the different strata of M as the connected components of the orbits of the pseudogroup of local diffeomorphisms $\mathrm{Diff_{loc}}(M)$. That is,

$$\mathrm{Strat}(M) = \{\mathscr{O}_i \in \pi_0(\mathscr{O}) \mid \mathscr{O} \in M/\mathrm{Diff_{loc}}(M)\}.$$

Moreover, Strat(M) does not capture only, as a set, the decomposition of M in strata, but its quotient diffeology can be regarded as the *transversal smooth structure* of the stratification, see [PIZ13, §1.42]. Note also that the *regular part* of M, that is, the principal orbit of $\mathrm{Diff_{loc}}(M)$, is the union of strata of dimension n. It is a regular n-submanifold and an open dense subset of M as it must be. Actually, M has a (geometrical) structure of locally conelike stratified space [GIZ18].

Proof. Let us consider a manifold with corners $(M, \mathscr{A})$, according to the classical definition. The finest diffeology $\mathscr{D}$ making the charts $F \in \mathscr{A}$ smooth is the set of parametrizations $P\colon U \to M$ that satisfy the following: there is a covering of U by a family of open sets U_i, and for each index i a chart $F_i \in \mathscr{A}$ and a smooth maps $Q_i\colon U_i \to K^n$ such that $P \upharpoonright U_i = F_i \circ Q_i$. We write $P = \sup F_i \circ Q_i$.

Now, the charts $F \in \mathscr{A}$ are smooth, by construction, and injective. Their domains are open for the induced topology of K^n, which is also the D-topology of K^n, according to above.

Let us show now that the topology of M and its D-topology coincide. Let first $\mathscr{O} \subset M$ be an open subset of M. Let P be a plot of M, then $P = \sup_i F_i \circ Q_i$ for some family of indices, with the F_i in $\mathscr{A}$ and the Q_i smooth parametrizations in K^n. Then, $P^{-1}(\mathscr{O}) = (\sup F_i \circ Q_i)^{-1}(\mathscr{O}) = \cup_i Q_i^{-1}(F_i^{-1}(\mathscr{O}))$. And since the F_i and the Q_i are continuous, $P^{-1}(\mathscr{O})$ is open. Thus, $\mathscr{O}$ is open for the D-topology. Conversely, let $\mathscr{O}$ be open for the D-topology. For all $x \in \mathscr{O}$, there exists $F_x \in \mathscr{A}$ such that $x \in \mathrm{val}(F_x)$. Since F_x is a plot for $\mathscr{D}$, $F_x^{-1}(\mathscr{O})$ is open in K^n, and since F_x is a local homeomorphism from K^n to M, $F_x \upharpoonright F_x^{-1}(\mathscr{O})$ is still a local homeomophism. Then, $\mathrm{val}(F_x \upharpoonright F_x^{-1}(\mathscr{O}))$ is open in M. But $\mathrm{val}(F_x \upharpoonright F_x^{-1}(\mathscr{O})) = \mathscr{O} \cap \mathrm{val}(F_x)$, thus $\mathscr{O} = \cup_x \mathscr{O} \cap \mathrm{val}(F_x)$ is a union of open subsets, then open in M. Therefore the topologies coincide.

Let us prove now that $\mathrm{Diff_{loc}}(K^n, M) = \mathscr{A}$. Let $\Phi \in \mathrm{Diff_{loc}}(K^n, M)$. Since the two topologies coincide, we know already that Φ is a local homeomorphism from K^n to M. Now let $F \in \mathscr{A}$, thus $F^{-1} \circ \Phi = F^{-1} \circ (\sup F_i \circ Q_i)$, where $\Phi = \sup F_i \circ Q_i$, as previously. Hence, $F^{-1} \circ \Phi = \sup(F^{-1} \circ F_i) \circ Q_i$. But the $F^{-1} \circ F_i$ and the Q_i are smooth, and moreover local diffeomorphisms, thus $F^{-1} \circ \Phi$ is a local diffeomorphism, and then also $\Phi^{-1} \circ F$. Hence, since $\mathscr{A}$ is maximal, $\mathrm{Diff_{loc}}(K^n, M) \subset \mathscr{A}$. Next let $F \in \mathscr{A}$. We know already that F is smooth, and a local homeomorphism for both topologies. Let us show that $F^{-1}\colon \mathrm{val}(F) \to K^n$ is smooth. Let P be a plot in $\mathrm{val}(F) \subset M$, then $P = \sup F_i \circ Q_i$. Hence, $F^{-1} \circ P = \sup(F^{-1} \circ F_i) \circ Q_i$. Thus, F^{-1} is smooth and F is a local diffeomorphism. Therefore, $\mathscr{A} \subset \mathrm{Diff}_{\mathrm{loc}}(K^n, M)$, and then $(M, \mathscr{D})$ is a diffeological manifold with corners such that $\mathrm{Diff}_{\mathrm{loc}}(K^n, M) = \mathscr{A}$.

The proof of the converse, from $(M, \mathscr{D})$ to $(M, \mathscr{A})$ with $\mathscr{A} = \mathrm{Diff}_{\mathrm{loc}}(K^n, M)$, is of the same vein. We leave it as an exercise for the reader. □

Extension of Differential Forms on Manifolds with Corners

In this section, we prove that any differential form on a manifold with corners is the restriction of a differential form on an open neighbourhood, after having been pushed inside itself as a submanifold with corner, thanks to a construction due to Douady *et al.* described in [ADLH73, Proposition 3.1].

4. The Square Function Lemma. Let $\mathrm{sq}\colon \mathbf{R}^n \to K^n$ be the smooth parametrization:

$$\mathrm{sq}(x_1, \ldots, x_n) = (x_1^2, \ldots, x_n^2).$$

Then $\mathrm{sq}^*\colon \Omega^k(K^n) \to \Omega^k(\mathbf{R}^n)$ is injective. That is, for all $\alpha \in \Omega^k(K^n)$, if $\mathrm{sq}^*(\alpha) = 0$, then $\alpha = 0$.

Proof. Note that each component of $X_j - X_{j-1}$ is diffeomorphic to $\mathbf{R}^j$. Thus, if $\mathrm{sq}^*(\alpha) = 0$, then $\alpha \upharpoonright (X_j - X_{j-1}) = 0$, because $\mathrm{sq} \upharpoonright \mathrm{sq}^{-1}(X_j - X_{j-1})$ is a 2-fold covering over $X_j - X_{j-1}$. Hence, for all plots Q in $X_j - X_{j-1}$, $\alpha(Q) = 0$. Let then, for some $j \geq 1$, $P_j\colon U_j \to X_j$ be a plot. According to (§2), the subset $\mathscr{O}_j = P_j^{-1}(X_j - X_{j-1})$ is open, and $\alpha(P_j \upharpoonright \mathscr{O}_j) = \alpha(P_j) \upharpoonright \mathscr{O}_j = 0$. By continuity, $\alpha(P_j) \upharpoonright \overline{\mathscr{O}}_j = 0$, where $\overline{\mathscr{O}}_j$ is the closure of $\mathscr{O}_j$. Let then $U_{j-1} = U_j - \overline{\mathscr{O}}_j$ and $P_{j-1} = P_j \upharpoonright U_{j-1}$. Then, U_{j-1} is open and $P_{j-1}\colon U_{j-1} \to X_{j-1}$ is a plot. This construction gives a descending recursion, starting with any plot $P\colon U \to K^n$, by initializing $P_n = P$, $U_n = U$ and $X_n = K^n$. One has $P_j = P \upharpoonright U_j$, $U_{j-1} \subset U_j$, the recursion ends with a plot P_0 with values in $X_0 = \{0\}$, and $\alpha(P_0) = 0$ since P_0 is constant. Therefore $\alpha = 0$. □

5. Differential Forms On Corners. The extension of smooth real functions on corners (§2), is a particular case of the following lemma on differential forms of any degree.

Lemma. *Any differential k-form on the corner* K^n*, equipped with the subset diffeology of* $\mathbf{R}^n$*, is the restriction of a smooth differential k-form defined on some open neighbourhood of the corner.*

In other words, the pullback : $j^*\colon \Omega^k(\mathbf{R}^n) \to \Omega^k(K^n)$ is surjective, where j denotes the inclusion from K^n to $\mathbf{R}^n$.

As a corollary of the previous lemma we get:

THEOREM. *Let* M *be a Hausdorff paracompact submanifold with corners, embedded in a smooth manifold* M′. *Then, any form* $\omega \in \Omega^k(\mathrm{M})$ *can be smoothly extended to an open neighbourhood of* M *in* M′.

NOTE 1. Thanks to [ADLH73, Proposition 3.1], every manifold with corners can be embedded in itself. Thus, in the previous theorem, to be a submanifold with corners of a smooth manifold is not a restriction.

NOTE 2. The theorem applies obviously to manifold with boundary, since they are a particular case of manifolds with corners.

NOTE 3. There is a subtlety about the value of a form [PIZ13, §6.40] at the boundary of a manifold with corners. For example on the half-line $[0,\infty[\, \subset \mathbf{R}$, the restriction α of the constant 1-form $a\,dx$ is a differential form on the half line. Its value at 0 is zero, since it vanishes when precomposed by a plot and evaluated at zero, that is, $\alpha(\gamma)_0 = 0$ for all paths γ in $[0,\infty[$ such that $\gamma(0) = 0$. But, of course, the value of the extension $a\,dx$ of α on $\mathbf{R}$, at the point 0, is the number a, as usual. So, we have to be careful when using the notion of values when it comes to differential forms in diffeology.

Proof. Let us prove the lemma. Let $\omega \in \Omega^k(\mathrm{K}^n)$ and $\mathring{\mathrm{K}}^n = \{(x_i)_{i=1}^n \mid x_i > 0, i = 1,\ldots,n\}$. One has

$$\omega \restriction \mathring{\mathrm{K}}^n = \sum_{i_1<\cdots<i_k} a_{i_1\ldots i_k}(x_1,\ldots,x_n)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k},$$

with $i_j = 1,\ldots,n$ and $a_{i_1\ldots i_k} \in \mathrm{C}^\infty(\mathring{\mathrm{K}}^n,\mathbf{R})$. Recall that $\mathrm{sq}\colon (x_i)_{i=1}^n \mapsto (x_i^2)_{i=1}^n$, then

$$\mathrm{sq}^*(\omega) = \sum_{i_1<\cdots<i_k} \mathrm{A}_{i_1\ldots i_k}(x_1,\ldots,x_n)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k},$$

where $\mathrm{A}_{i_1\ldots i_k} \in \mathrm{C}^\infty(\mathbf{R}^n,\mathbf{R})$. Let $\varepsilon_j : (x_1,\ldots,x_j,\ldots,x_n) \mapsto (x_1,\ldots,-x_j,\ldots,x_n)$, then $\mathrm{sq}\circ\varepsilon_j = \mathrm{sq}$ and $(\mathrm{sq}\circ\varepsilon_j)^*(\omega) = \varepsilon_j^*(\mathrm{sq}^*(\omega))$, that is, $\mathrm{sq}^*(\omega) = \varepsilon_j^*(\mathrm{sq}^*(\omega))$. Hence,

$$\begin{aligned}\varepsilon_j^*(\mathrm{sq}^*(\omega)) &= \sum_{\substack{i_1<\cdots<i_k\\ i_\ell\neq j}} \mathrm{A}_{i_1\ldots i_k}(x_1,\ldots,-x_j,\ldots,x_n)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}\\ &\quad - \sum_{i_1<\cdots\leq j\leq\cdots<i_k} \mathrm{A}_{i_1\ldots j\ldots i_k}(x_1,\ldots,-x_j,\ldots,x_n)\, dx_{i_1}\wedge\cdots\wedge dx_j\wedge\cdots\wedge dx_{i_k}.\end{aligned}$$

Then,

$$\begin{aligned}\mathrm{A}_{\substack{i_1\ldots i_k\\ i_\ell\neq j}}(x_1,\ldots,-x_j,\ldots,x_n) &= \mathrm{A}_{i_1\ldots i_k}(x_1,\ldots,x_j,\ldots,x_n),\\ \mathrm{A}_{i_1\ldots j\ldots i_k}(x_1,\ldots,-x_j,\ldots,x_n) &= -\mathrm{A}_{i_1\ldots j\ldots i_k}(x_1,\ldots,x_j,\ldots,x_n).\end{aligned}$$

Hence,

$$\mathrm{A}_{i_1\ldots j\ldots i_k}(x_1,\ldots,x_j=0,\ldots,x_n) = 0.$$

Thus,

$$\mathrm{A}_{i_1\ldots j\ldots i_k}(x_1,\ldots,x_j,\ldots,x_n) = 2x_j\underline{\mathrm{A}}_{i_1\ldots j\ldots i_k}(x_1,\ldots,x_j,\ldots,x_n),$$

with $\underline{\mathrm{A}}_{i_1\ldots j\ldots i_k} \in \mathrm{C}^\infty(\mathbf{R}^n,\mathbf{R})$. By induction, there are real smooth functions $\hat{\mathrm{A}}_{i_1\ldots i_k}$ defined on $\mathbf{R}^n$ such that

$$\mathrm{A}_{i_1\ldots i_k}(x_1,\ldots,x_n) = 2^k x_{i_1}\ldots x_{i_k}\hat{\mathrm{A}}_{i_1\ldots i_k}(x_1,\ldots,x_n).$$

Now,

$$\mathrm{sq}^*(\omega \restriction \mathring{\mathrm{K}}^n) = \mathrm{sq}^*(\omega) \restriction \{x_i \neq 0\}$$

implies

$$\begin{aligned}&\sum_{i_1<\cdots<i_k} 2^k x_{i_1}\ldots x_{i_k} a_{i_1\ldots i_k}(x_1^2,\ldots,x_n^2)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}\\ = &\sum_{i_1<\cdots<i_k} 2^k x_{i_1}\ldots x_{i_k}\hat{\mathrm{A}}_{i_1\ldots i_k}(x_1,\ldots,x_n)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}.\end{aligned}$$

Hence,

$$\hat{\mathrm{A}}_{i_1\ldots i_k}(x_1,\ldots,x_n) = a_{i_1\ldots i_k}(x_1^2,\ldots,x_n^2) \quad \text{for } x_i \neq 0, i = 1,\ldots,n.$$

Thus $(x_1,\ldots,x_n) \mapsto \hat{\mathrm{A}}_{i_1\ldots i_k}(x_1,\ldots,x_n)$, which belongs to $\mathrm{C}^\infty(\mathbf{R}^n,\mathbf{R})$, is even in each variable. Therefore, according to Whitney's theorem on even smooth functions [Whi43, Theorem 1 & Remark p.160], there exist

$$\underline{a}_{i_1\ldots i_k} \in \mathrm{C}^\infty(\mathbf{R}^n,\mathbf{R}),$$

such that

$$\hat{\mathrm{A}}_{i_1\ldots i_k}(x_1,\ldots,x_n) = \underline{a}_{i_1\ldots i_k}(x_1^2,\ldots,x_n^2).$$

One deduces:

$$\underline{a}_{i_1\ldots i_k}(x_1,\ldots,x_n) = a_{i_1\ldots i_k}(x_1,\ldots,x_n), \text{ for all } (x_1,\ldots,x_n)\in\mathring{\mathrm{K}}^n.$$

Then, defining the k-form $\underline{\omega}$ on $\mathbf{R}^n$ by

$$\underline{\omega} = \sum_{i_1<\cdots<i_k} \underline{a}_{i_1\ldots i_k}(x_1,\ldots,x_n)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k},$$

one has already

$$\underline{\omega} \restriction \mathring{\mathrm{K}}^n = \omega \restriction \mathring{\mathrm{K}}^n.$$

Let us show that $\underline{\omega} \restriction \mathrm{K}^n = \omega$. That is, let us check that for all plot $\mathrm{P}\colon \mathrm{U} \to \mathbf{R}^n$, $\mathrm{P}^*(\underline{\omega}) = \omega(\mathrm{P})$. Actually, one has

$$\mathrm{sq}^*(\omega) = \mathrm{sq}^*(\underline{\omega} \restriction \mathrm{K}^n).$$

Indeed:

$$\begin{aligned}
\mathrm{sq}^*(\omega) &= \sum_{i_1\ldots i_k} \mathrm{A}_{i_1\ldots i_k}(x_1,\ldots,x_n)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\
&= \sum_{i_1\ldots i_k} 2^k x_{i_1}\ldots x_{i_k}\hat{\mathrm{A}}_{i_1\ldots i_k}(x_1,\ldots,x_n)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\
&= \sum_{i_1\ldots i_k} 2^k x_{i_1}\ldots x_{i_k}\underline{a}_{i_1\ldots i_k}(x_1^2,\ldots,x_n^2)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}.
\end{aligned}$$

And, on the other hand:

$$\mathrm{sq}^*(\underline{\omega}\restriction \mathrm{K}^n) = \sum_{i_1\ldots i_k} 2^k x_{i_1}\ldots x_{i_k}\underline{a}_{i_1\ldots i_k}(x_1^2,\ldots,x_n^2)\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}.$$

Thus, $\mathrm{sq}^*(\omega - \underline{\omega}\restriction \mathrm{K}^n) = 0$. Therefore, according to the lemma (§4), $\omega - \underline{\omega}\restriction \mathrm{K}^n = 0$. And then, ω is the restriction on K^n of the smooth k-form $\underline{\omega}$ on $\mathbf{R}^n$.

For the theorem: thanks to the lemma, we consider a locally finite open cover $\{\mathrm{U}_i\}$ of the boundary of $\mathrm{M}\subset\mathrm{M}'$ where the form ω has been extended smoothly on each U_i by ω_i. Then, there exists a smooth partition of unity λ_j, subordinate to a subcover $\{\mathrm{V}_j\}$. Let $\omega_j = \omega_i \restriction \mathrm{V}_j$, for some index i such that $\mathrm{V}_j \subset \mathrm{U}_i$. The smooth form $\bar{\omega} = \sum_j \lambda_j\omega_j$, defined on the neighbourhood $\cup_j \mathrm{V}_j$ of the boundary of M, is an extension of ω on $(\cup_j \mathrm{V}_j)\cap \mathrm{M}$. It defines this way an extension of ω on an open neighbourhood of M in M′. □

6. Other Corners. The *half-line* $\Delta_k = \mathbf{R}^k/\mathrm{O}(k)$ is identified to the interval $[0,\infty[$, equipped with the pushforward of the smooth diffeology of $\mathbf{R}^k$ by the projection $\nu_k\colon \mathrm{X}\mapsto \|\mathrm{X}\|^2$, see [PIZ07]. Then, with each half-line we can associate a *n-corner* Δ_k^n. But note that, according to definition (§3), none of these corners is a manifold with corners. Then, let $\mathrm{J}_k^n : \Delta_k^n \to \mathbf{R}^n$ be the natural injection.

Proposition. — *The pullback* $\mathrm{J}_k^{n*}\colon \Omega^*(\mathbf{R}^n)\to\Omega^*(\Delta_k^n)$ *is surjective. That is, every differential form* ε *on* Δ_k^n *is the pullback by* J_k^n *of some smooth form* $\alpha\in\Omega^*(\mathbf{R}^n)$.

Note. Let $j:\mathrm{K}^n\to\mathbf{R}^n$ be the natural injection, and let $j_k^n\colon\Delta_k^n\to\mathrm{K}^n$ be the injection such that $\mathrm{J}_k^n = j\circ j_k^n$. Then, thanks to the previous proposition, j_k^{n*} is a bijection.

Proof. First, we notice that the map $\mathrm{sq}\colon\mathbf{R}^n\to\Delta_k^n$ is still smooth. Then, we check that J_k^n is an embedding, that is, the pullback of the topology of $\mathbf{R}^n$ is the D-topology of Δ_k^n. We conclude that the lemma (§4) is still true, replacing K^n by Δ_k^n and $\mathring{\mathrm{K}}^n$ by $\mathring{\Delta}_k^n$. Next the proof of (§5) applies *mutatis mutandis* to Δ_k^n, considering $\mathrm{J}_k^{n*}(\omega)$ instead of $\omega\restriction\mathrm{K}^n$. □

7. An Application. We can apply the theorems above to describe the closed 2-forms on manifolds, invariant with respect to some action of a Lie group. Roughly speaking, a closed 2-form ω on a manifold M, invariant by the Hamiltonian action of a compact group[6] G, is characterized by its *moment map* $\mu\colon \mathrm{M} \to \mathscr{G}^*$, and for each given moment map, a closed 2-form $\varepsilon \in \mathrm{Z}^2(\mathrm{M}/\mathrm{G})$. Let us be more precise: the space of G-invariant closed 2-forms $\mathrm{Z}^2(\mathrm{M})^{\mathrm{G}}$ is a vector space, the space of G-equivariant maps from M to $\mathscr{G}^*$ is also a vector space, and the map associating its moment map[7] μ with each invariant closed 2-form ω is linear. What we claim is that the kernel of this map is exactly $\mathrm{Z}^2(\mathrm{M}/\mathrm{G})$, where M/G is equipped with the quotient diffeology. Denoting by $\mathscr{E}q_\bullet(\mathrm{M},\mathscr{G}^*) \subset \mathscr{E}q(\mathrm{M},\mathscr{G}^*)$ the space of moment maps of G-invariant closed 2-forms on M, as a subset of smooth equivariant maps, one has this exact sequence of smooth linear maps:

$$0 \longrightarrow \mathrm{Z}^2(\mathrm{M}/\mathrm{G}) \longrightarrow \mathrm{Z}^2(\mathrm{M})^{\mathrm{G}} \longrightarrow \mathscr{E}q_\bullet(\mathrm{M},\mathscr{G}^*) \longrightarrow 0.$$

An equivariant map is easy to conceive, but a differential form on the space of orbits, which is generally not a manifold, is more problematic. This is where the above theorem can help: M/G is a diffeological space which, sometimes, is not far from being a manifold with boundary or corners, as the following example shows.

Consider for example $\mathrm{M} = \mathbf{R}^{2n}$, and the space of closed 2-forms invariant by $\mathrm{SO}(2)^n$. The quotient space $\mathscr{Q}^n = \mathbf{R}^{2n}/\mathrm{SO}(2)^n$ is equivalent to the *other corner* Δ_2^n, with $\Delta_2 = \mathbf{R}^2/\mathrm{O}(2)$. Let ω and ω' be two closed 2-forms, sharing the same moment map $\mu\colon \mathbf{R}^{2n} \to \mathbf{R}^n$. Then, according to the previous sections, there exists a closed 2-form ε defined on an open neighbourhood of $\mathrm{K}^n \subset \mathbf{R}^n$ (which could be $\mathbf{R}^n$), such that $\omega' = \omega + \pi^*(\varepsilon)$, with $\pi\colon (z_1,\ldots,z_n) \mapsto (\|z_1\|^2,\ldots,\|z_n\|^2)$, $z_i \in \mathbf{R}^2$. Moreover, since K^n is contractible, $\varepsilon = d\alpha$, where α is a smooth extension of a 1-form on K^n. Therefore, $\omega' = \omega + d(\pi^*(\alpha))$.

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[6] There could a diffeological generalisation possible here to non compact group.

[7] The manifold M is supposed to be connected. To have a uniqueness of the moment maps we decide to fix their value to 0 at some base point $m_0 \in \mathrm{M}$, for example.

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Serap Gürer — Galatasaray University, Ortaköy, Çırağan Cd. No:36, 34349 Beşiktaş/İstanbul, Turkey.

E-mail address: `sgurer@gsu.edu.tr`

Patrick Iglesias-Zemmour — CNRS, France & The Hebrew University of Jerusalem, Israel.

E-mail address: `piz@math.huji.ac.il`