p-FORMS ON HALF-SPACES

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ref. http://math.huji.ac.il/~piz/documents/DBlog-Rmk-kFOHS.pdf

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We prove that every differential p-form on the half-space $\mathrm{H}^n = [0,\infty[\times \mathbf{R}^{n-1}$ is the restriction of a smooth p-form on $\mathbf{R}^n$.

This is the natural extension of the previous blog post on 1-forms on half-spaces [SGPIZ16].

Proposition — Let $\mathrm{H}^n = [0,\infty[\times \mathbf{R}^{n-1}$ be the half n-space, equipped with the subset diffeology. Let $\omega \in \Omega^p(\mathrm{H}^n)$ be a differential p-form on H^n. Then, there exists a smooth p-form $\bar{\omega}$ defined on some neighborhood of $\mathrm{H}^n \subset \mathbf{R}^n$ such that $\omega = \bar{\omega} \upharpoonright \mathrm{H}^n$.

Proof. Since $\mathring{\mathrm{H}}^n =]0,\infty[\times \mathbf{R}^{n-1} \subset \mathrm{H}^n = [0,\infty[\times \mathbf{R}^{n-1}$ inherits the usual smooth diffeology,

$$\begin{aligned} \omega \upharpoonright \mathring{\mathrm{H}}^n &= \sum_{1<j<\cdots<k} a_{1j\ldots k}(x,y)dx \wedge dy_j \wedge \cdots \wedge dy_k \\ &+ \sum_{i<j<\cdots<k} b_{ij\ldots k}(x,y)dy_i \wedge dy_j \wedge \cdots \wedge dy_k, \end{aligned}$$

where $(x,y) \in \mathring{\mathrm{H}}^n$ and $a_{1j\ldots k}, b_{ij\ldots k} \in \mathcal{C}^\infty(\mathring{\mathrm{H}}^n, \mathbf{R})$.

Now, let

$$\mathrm{sq}_1 : (t,y) \mapsto (t^2, y),$$

then let

$$\begin{aligned} \mathrm{sq}_1^*(\omega) &= \omega((t,y) \mapsto (t^2,y)) \\ &= \sum_{1<j<\cdots<k} \mathrm{A}_{1j\ldots k}(t,y)dt \wedge dy_j \wedge \cdots \wedge dy_k \\ &+ \sum_{i<j<\cdots<k} \mathrm{B}_{ij\ldots k}(t,y)dy_i \wedge dy_j \wedge \ldots dy_k, \end{aligned}$$

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with $A_{1j\ldots k}, B_{ij\ldots k} \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R})$. And for all $t \neq 0$,

$$A_{1j\ldots k}(t,y) = 2t\, a_{1j\ldots k}(t^2,y) \text{ and } B_{ij\ldots k}(t,y) = b_{ij\ldots k}(t^2,y).$$

Next, $\mathrm{sq}_1^*(\omega)$ is invariant by $\varepsilon : (t,y) \mapsto (-t,y)$, thus

$$\begin{aligned} \mathrm{sq}_1^*(\omega) &= \varepsilon^*(\mathrm{sq}_1^*(\omega)) \\ &= \sum_{1<j<\cdots<k} A_{1j\ldots k}(-t,y)(-dt \wedge dy_i \wedge \cdots \wedge dy_k) \\ &+ \sum_{i<j<\cdots<k} B_{ij\ldots k}(-t,y) dy_i \wedge dy_j \wedge \cdots \wedge dy_k. \end{aligned}$$

Thus, $-A_{1j\ldots k(-t,y)} = A_{1j\ldots k}(t,y)$ and $B_{ij\ldots k}(t,y) = B_{ij\ldots k}(-t,y)$. In particular, $A_{1j\ldots k}(0,y) = 0$. Hence, there exists a smooth function $\underline{A}_{1j\ldots k} \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R})$ such that $A_{1j\ldots k}(t,y) = 2t\underline{A}_{1j\ldots k}(t,y)$, for all $t \in \mathbf{R}$. Thus, for all $t \neq 0$, $a_{1j\ldots k}(t^2,y) = \underline{A}_{1j\ldots k}(t,y)$. Now, $\underline{A}_{1j\ldots k}$ is even in t, as well as the $B_{ij\ldots k}$. We can then apply the Hassler Whitney Theorem [Whi43, Theorem 1 and final remark], stated as follows:

Theorem [H. Whitney] *If a smooth function $f(t,x)$ is even in t, $f(t,x) = f(-t,x)$, then there exists a smooth function $g(t,x)$ such that $f(t,x) = g(t^2,x)$.*

Hence, there exists smooth functions $\underline{a}_{1j\ldots k}(t,y)$ and $\underline{b}_{ij\ldots k}(t,y)$, such that $\underline{A}_{1j\ldots k}(t,y) = \underline{a}_{1j\ldots k}(t^2,y)$ and $B_{ij\ldots k}(t,y) = \underline{b}_{ij\ldots k}(t^2,y)$. Then, for all $t > 0$, $a_{1j\ldots k}(t,y) = \underline{a}_{1j\ldots k}(t,y)$ and $b_{ij\ldots k}(t,y) = \underline{b}_{1j\ldots k}(t,y)$.

Let us then define $\bar{\omega}$ on $\mathbf{R}^n$,

$$\begin{aligned} \bar{\omega} &= \sum_{1<j<\cdots<k} \underline{a}_{1j\ldots k}(x,y) dx \wedge dy_j \wedge \cdots \wedge dy_k \\ &+ \sum_{i<j<\cdots<k} \underline{b}_{ij\ldots k}(t,y) dy_i \wedge dy_j \wedge \cdots \wedge dy_k. \end{aligned}$$

The form $\bar{\omega}$ is a smooth p-form defined on an open neighborhood of H^n, and $\omega \restriction \mathring{H}^n = \bar{\omega} \restriction \mathring{H}^n$. Let us prove now that ω and $\bar{\omega}$ coincide on the whole H^n. Since ω and $\bar{\omega} \restriction H^n$ are two differential p-forms on H^n, it is enough to show that they take the same value on any smooth p-path.

Let σ be any p-path in H^n. Let $\mathcal{O} = \sigma^{-1}(\mathring{H}^n)$, $\mathcal{O} \subset \mathbf{R}^p$ is open, and on this open subset $\bar{\omega}(\sigma) = \omega(\sigma)$. Hence, by continuity $\bar{\omega}(\sigma) = \omega(\sigma)$ on the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ (since $\bar{\omega}(\sigma)$ and $\omega(\sigma)$ are smooth). But on the open subset $\mathbf{R}^p - \overline{\mathcal{O}}$, σ takes its values in $\partial H^n = \{0\} \times \mathbf{R}^{n-1}$; $\bar{\sigma} = \sigma \restriction \mathbf{R}^p - \overline{\mathcal{O}}$ is a plot of the boundary ∂H^n. Let $i : \mathbf{R}^{n-1} \to \partial H^n$,

$i(y) = (0, y)$. Then, $i^*(\omega)$ and $i^*(\bar\omega)$ are both p-forms on $\mathbf{R}^{n-1}$. Let us prove that they coincide. On the one hand

$$i^*(\bar\omega) = \sum_{i<j<\cdots<k} \underline{b}_{ij\ldots k}(0, y) dy_i \wedge dy_j \wedge \cdots \wedge dy_k.$$

On the other hand, let us notice that

$$i = \mathrm{sq}_1 \circ i : y \mapsto (0, y) \mapsto (0^2, y).$$

Thus, $i^*(\omega) = i^*(\mathrm{sq}_1^*(\omega))$ and then

$$i^*(\omega)_y(\delta_1 y, \ldots, \delta_{p-1} y) = \mathrm{sq}_1^*(\omega)_{(0,y)}(0, \delta_1 y, \ldots, \delta_{p-1} y).$$

Hence,

$$\begin{aligned} \mathrm{sq}_1^*(\omega) &= \sum_{i<j<\cdots<k} \mathrm{B}_{ij\ldots k}(0, y) dy_i \wedge dy_j \wedge \cdots \wedge dy_k \\ &= \sum_{i=1}^{n-1} \underline{b}_{ij\ldots k}(0, y) dy_i \wedge dy_j \wedge \cdots \wedge dy_k, \end{aligned}$$

since $\mathrm{A}(0, y) = 0$ and $\mathrm{B}_{ij\ldots k}(t, y) = \underline{b}_{ij\ldots k}(t^2, y)$. Hence ω and $\bar\omega$ coincide on $\partial \mathrm{H}^n$ and then $\bar\omega(\sigma)$ and $\omega(\sigma)$ coincide everywhere. Therefore, since $\bar\omega$ and ω coincide on the p-plots, they coincide as p-forms [PIZ13, §6.37], and then, $\omega = \bar\omega \restriction \mathrm{H}^n$. ☐

References

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