VAGUE ADJUNCTION OF A POINT TO A SPACE

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ref.
http://math.huji.ac.il/~piz/documents/DBlog-Rmk-VAOAPTAS.pdf

We add a point to a diffeological space, counting for nothing, and we look at the consequences.

Consider a diffeological space X. Let $\bar{X} = X \cup \{\omega\}$, where ω is an arbitray point not contained in X. Define on $\bar{X}$ the following diffeology:

A parametrization $P : U \to \bar{X}$ is a plot if, for all r_0 such that $P(r_0) \in X$, there exists an open neighborhood V of r_0 such that $P \restriction V$ is a plot in X, and that's it. No condition is required when $P(r_0) = \omega$.

1. — $\bar{X}$ is connected. Precisely, let $P \colon \mathcal{B} \to \bar{X}$ be the parametrization defined by:

- $\mathcal{B}$ is an open ball.
- $P(0) = \omega$.
- $P(\mathcal{B} - \{0\}) = x$, where $x \in X$ is any point.

Then, P is a plot connecting any point x in X to ω.

Proof. Indeed $\mathcal{B} - \{0\}$ is open, and $P \restriction \mathcal{B} - \{0\}$ is constant then $P \restriction \mathcal{B} - \{0\}$ is a plot, and since it takes its value in X, P is a plot in $\bar{X}$. Now, since $\mathcal{B}$ is connected, x and ω are in the same connected component, for all $x \in X$. □

2. — The point ω is closed in $\bar{X}$

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Proof. Let $P\colon U \to \bar{X}$ be a plot, and let us check that $P^{-1}(X)$ is open. Let $r_0 \in U$, and $P(r_0) \in X$. Then, there exists an open neighborhood V of r_0 such that $P \restriction V$ is a plot in X. Thus, as a union of open sets $P^{-1}(X)$ is open. Therefore, $X \subset \bar{X}$ is D-open and $\omega \in \bar{X}$ is D-closed. ☐

3. — Every point x in X is in the neighborhood of ω. In other words, ω has only one neighborhood, the space $\bar{X}$ itself.

Proof. Let Ω be an open neighborhood of ω. That is, for every plot P in $\bar{X}$, $P^{-1}(\Omega)$ is open. Let $x \in X$ and let $P\colon \mathcal{B} \to \bar{X}$ be a special plot defined in the first article, that sends 0 to ω and the rest of the ball on x. Since $P^{-1}(\Omega)$ is open and 0 is sent to ω, there exists a small open ball $\mathcal{B}'$ centered at 0 such that its image by P is contained in Ω. By construction $P(\mathcal{B}' - \{0\}) = x \in \Omega$. Therefore Ω contains every point in X. That is, $\Omega = \bar{X}$. ☐

4 (Note). — This kind of situation is not proper to diffeology. Consider for example the quotient of $\mathbf{R}^n$ by the action of $]0,\infty[$, $(\alpha, x) \mapsto \alpha x$, where $(\alpha, x) \in]0,\infty[\times\mathbf{R}^n$. For the quotient topology of $\mathbf{R}^n/]0,\infty[$, any neighborhood of class(0) contains the whole quotient. Set theorethically, the quotient is the union of $S^{n-1} = (\mathbf{R}^n - \{0\})/]0,\infty[$ with class(0).

References

[DBook] Patrick Iglesias-Zemmour. *Diffeology,* Mathematical Surveys and Monographs, vol. 185. Am. Math. Soc., Providence, 2012. http://www.ams.org/bookstore-getitem/item=SURV-185

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