USE OF THE MOMENT MAP IN GEODESIC CALCULUS

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ref.
http://math.huji.ac.il/~piz/documents/DBlog-Rmk-UOTMMIGC.pdf

A fact or two about geodesics, proved thanks to the Moment Map.

The questions on geodesics we treat here came around in a session of our working group with Jean-Paul Mohsen.

1. On Geodesics

We consider the unparametrized geodesics of a Riemannian manifold (M, g). They are defined as the characteristics of the presymplectic 2-form $d\lambda$ on the Unit Tangent Bundle:

$$\mathrm{UM} = \{(x, u) \in \mathrm{T}(\mathrm{M}) \mid x \in \mathrm{M}, u \in \mathrm{T}_x(\mathrm{M}) \text{ and } u \cdot u = 1\},$$

where $\lambda \in \Omega^1(\mathrm{UM})$ is defined by its evaluation on a tangent vecteur $\delta(x, u) \in \mathrm{T}_{(x,u)}(\mathrm{UM})$

$$\lambda_{(x,u)}(\delta(x, u)) = u \cdot \delta x.$$

We note $v \cdot w$ for $g_x(v, w)$, and $\|v\|^2 = v \cdot v$, where $v, w \in \mathrm{T}_x(\mathrm{M})$. The characteristics of $d\lambda$ satisfies the differential equations

$$\delta(x, u) \in \ker(d\lambda_{(x,u)}) \quad \text{iff} \quad \delta x \propto u \quad \text{and} \quad \hat{\delta} u = 0,$$

where $\hat{\delta}$ stands for the covariant differentiation. In a chart:

$$\hat{\delta} u^\mu = \delta u^\mu + \Gamma^\mu_{\nu\rho} u^\nu \delta x^\rho,$$

where the $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols of the Levi-Civita connexion. The map $\delta(x, v) \mapsto (\delta x, \hat{\delta} v)$ is a isomorphism from $\mathrm{T}_{(x,v)}(\mathrm{T}(\mathrm{M}))$ to $\mathrm{T}_x(\mathrm{M}) \times \mathrm{T}_x(\mathrm{M})$. What we have to know now is this:

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Proposition — The integral curves of the distribution

$$(x, u) \mapsto \ker(d\lambda_{(x,u)})$$

projects on M on the geodesics trajectories (or unparametrized geodesics) of the Riemannian metric g.

Actually, for every integral curve there exists always a parametrization $t \mapsto x(t)$, defined on some interval, such that $u(t) = dx(t)/dt$, with $(x(t), u(t)) \in \mathrm{UM}$, and $[t \mapsto (x(t), u(t))]$ is a characteristic of $d\lambda$. And that is the way one usually understands the wording "unparametrized geodesics".

Now, if a Lie-group H acts on M preserving the metric g, that is, $h_{\mathrm{M}}^*(g) = g$ for all $h \in \mathrm{H}$, then its natural action on $\mathrm{T(M)}$ preserves UM,

$$h_{\mathrm{UM}}(x, u) = (h_{\mathrm{M}}(x), h_{\mathrm{M}*}(u) = \mathrm{D}(h_{\mathrm{M}})(x)(u)),$$

and also λ on UM,

$$h_{\mathrm{UM}}^*(\lambda) = \lambda.$$

The Moment Map of this action is the pullback by the orbit map of the 1-form λ [PIZ10], that is,

$$\mu(x, u) = [h \mapsto h_{\mathrm{M}}(x, u)]^*(\lambda).$$

Applied on a vector Z in the Lie algebra $\mathcal{H}$ of H, that gives

$$\mu(x, u) \cdot \mathrm{Z} = u \cdot \mathrm{Z}_{\mathrm{M}}(x),$$

where $\mathrm{Z}_{\mathrm{M}}(x)$ denotes the action of Z on M, that is, the infinitesimal action of the 1-parameter group generated by Z.

The **Noether Theorem** states that the Moment Map is constant on the characteristics. That means that if $[t \mapsto x(t)]$ is a geodesic with $u(t) = dx(t)/dt$ unitary, then for all t in the domain of the curve, for all $\mathrm{Z} \in \mathcal{H}$,

$$u(t) \cdot \mathrm{Z}_{\mathrm{M}}(x(t)) = u_0 \cdot \mathrm{Z}_{\mathrm{M}}(x_0),$$

where $(x_0, u_0) = (x(t_0), u(t_0))$, for an arbitrary t_0 in the domain of the curve.

2. Special Metrics on Principal Bundles

Consider now a principal bundle $\pi : M \to B$ with structure group H, equipped with an H-invariant metric g. Consider a connexion on M defined by orthogonal projectors: for all $x \in M$ let $Q_x : T_x(M) \to T_x(H_M(x))$ be the vertical projector, and $P_x = \mathbf{1}_x - Q_x$ be the horizontal one. In other words, the horizontal subspace is orthogonal to the fibers. Assume also that the metric g is "calibrated vertically", we mean that there exists a left-invariant metric ε on G such that

$$\hat{x}^*(g) = \varepsilon,$$

where $\hat{x}$ is the orbit map $\hat{x}(h) = h_M(x)$, for all $x \in M$ and $h \in H$. Now, we have two interesting consequences of the conservation of the Moment Map along the geodesics:

Proposition 1 — If a geodesic is horizontal at some point, it is horizontal at everypoint.

Proof. To be horizontal in one point means that the unit tangent vector u_0 is orthogonal to the fiber at x_0. Next, being orthogonal to a fiber at t_0 means: $u_0 \cdot Z_M(x_0) = 0$ for all $Z \in \mathcal{H}$, since the vertical tangent space is spaned by the infinitesimal action of H. Then, the moment map being constant along the geodesic , $u(t) \cdot Z_M(x(t)) = 0$ for all t. The whole geodesic is horizontal. □

Actually this is a particular case of a more general property:

Proposition 1' — Let H be a group of isometries of (M, g). If a geodesic is orthogonal to an orbit of H at one point, it is orthogonal to every orbit it meets.

In particular, if the group has a fix point, every geodesic emerging from this point is orthogonal to every orbit it meets. Consider for example the 2-sphere, and the rotations around an axe. The poles are fixed and the geodesics passing through the poles are the meridians, orthogonal to the parallels, orbits of the rotations.

Proposition 2 — The fibers are totally geodesic.

Proof. Let us recall that being totally geodesic means that a geodesic which is tangent to a fiber at some point is tangent to that fiber at every point.

Let $\{Z_i\}_{i=1}^k$ be an orthonormal basis of $\mathcal{H}$, for the metric ε. Then, according to the hypothesis $\hat{x}^*(g) = \varepsilon$, the vectors $Z_{iM}(x)$ form an orthonormal basis of $\mathrm{T}_x(\mathrm{H_M}(x))$, for all $x \in \mathrm{M}$. Let $t \mapsto x(t)$ be a geodesic with unitary velocity $u(t)$, and let $u(t) = u_{\mathrm{Q}}(t) + u_{\mathrm{P}}(t)$, where u_{Q} is the vertical part and u_{P} the horizontal. Then, $\|u_{\mathrm{Q}}(t)\|^2 = \sum_{i=1}^{k} u_i(t)^2$, with $u_i(t) = u(t) \cdot \mathrm{Z}_{i\mathrm{M}}(x(t))$. Thus, since the $u_i(t)$ are constant on the geodesic, $\|u_{\mathrm{Q}}(t)\|^2$ is also constant on the geodesic. Thus, if $u_{\mathrm{P}}(x_0) = 0$, then $\|u_{\mathrm{Q}}(t)\|^2 = \|u_{\mathrm{Q}}(0)\|^2 = 1$ for all t, and then $\|u_{\mathrm{P}}(t)\|^2 = 0$. Therefore, if the geodesic is tangent to the fiber somewhere it is tangent everywhere. $\square$

Remark — On $\mathbf{R}^2$ the standard metric is $\mathrm{SO}(2, \mathbf{R})$-invariant. And if we forget 0, the action of $\mathrm{SO}(2, \mathbf{R})$ is principal. The geodesics are the lines and the orbits, the circles centered at the origin. A line which is tangent to a circle somewhere is not tangent to any other circle. Thus, the orbits are not geodesics. So? Did we miss something? It happens that the metric is not calibrated with respect to the action of $\mathrm{SO}(2, \mathbf{R})$. Precisely $\hat{x}^*(g) = \|x\|^2\varepsilon$. Hence, we can see that for the proposition 2, the hypothesis for the metric to be calibrated is crucial.

References

[PIZ10] Patrick Iglesias-Zemmour. *The Moment Maps in Diffeology*. Memoir of the AMS 972, R.I. USA, 2010.

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