THE BEGINNING OF DIFFEOLOGICAL SPACES

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This is the story, as I remember it, of the beginning of diffeologies and diffeological spaces as “groupes et espaces différentiels"[1] by Souriau and his team back in the 1980s.

The word "difféologies" appears for the first time in Souriau's article *Groupes différentiels*, the text of a talk he presented at a conference in Salamanca in 1979 and published in 1980 [Sou80]. In this paper a "difféologie" denotes an abstract structure used to define exclusively what Souriau called "groupes différentiels". A “difféologie" is there defined by five axioms, the last two axioms relate specifically to the group structure. Even if Souriau could have pulled out the two last axioms of this list, and thus should have defined the general concept of "espaces différentiels", we notice that he did not do that in this paper and considered his “difféologies" only in the framework of groups.

The general concept of “espace différentiel", as it will emerge later, is indeed absent from this 1980 paper. We shall see that it will take three years, and some unexpected developments, for this notion to appear independently.

Actually, Souriau focusing on groups is somewhat understandable when we know that he was looking at this time for a renewal of his method of quantization through the Gelfand-Naimark-Segal construction[2], that is, using positive-definite functions on a given

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[1] The facsimiles of the preprints cited in this note are available on line, look at their URL in the references below.

[2] Not to mention Souriau's profound bias for groups vis-à-vis general spaces.

group of symmetries to build a Hilbert space and a unitary representation. What he suggested then was a mean to conditionally connect these representations, depending on some positive function, with an appropriate coadjoint orbit, and thus achieve the Dirac quantization program. But because the Dirac program requires to represent the whole group of symplectomorphisms[3], Souriau had to leave the category he used to work in, that is, the category of finite dimensional Lie groups, to deal with the huge infinite dimensional group of all the symplectomorphisms of a symplectic manifold. A group about which Kostant told him once: “It is too big".

Souriau knew very well that dealing with infinite dimensional groups is a serious challenge, especially if you don't want to just be heuristic but mathematically correct. He knew also that he didn't need to involve a too sophisticated structure on the group of symplectomorphisms to get what he was looking for. Just a few "differentiable" properties would be enough. Then, instead of paying tribute to the thick theory of topological groups, he preferred to build his own light category of groups that he called "groupes différentiels", by retaining just the minimal properties he needed for his purpose. And that gave his paper on "Groupes différentiels", with its five axioms, we are talking about [Sou80].

Then, during a couple of years Souriau continued to work on this approach and tried to figure out a way to fulfill Dirac program of quantization in this context.

At that time, Paul Donato, a member of our team, came around this new concept of "groupe différentiel". He wrote an essay on the characterization of the fundamental group of a manifold through its group of diffeomorphisms, published in 1981 [Don81], and continued to investigate the fundamental group of groups of diffeomorphisms of manifolds and more generally the fundamental group, and coverings, of cosets of "groupes différentiels". Eventually, this made the subject of his dissertation that he defended later in 1984 [Don84]. On my side, I was working on something very different, the classification of SO(3) symplectic manifolds, and these "groupes différentiels" did not speak to me that much.

[3]Originally infinitesimally.

The next step in the theory of "groupes différentiels" happened in June 1983 during a conference in Lyon[4] where Souriau was invited to talk and Paul and I were attending. Souriau talked again about his new theory of "groupes différentiels" applied to the quantization program, he exposed his new developments. At the same conference we heard also a series of talks concerning what we called later "irrational tori", that is, the quotients of the 2-torus by an irrational line. This question was related to the study of quasi-periodic potentials in quantum physics. Alain Connes had already introduced his "non-commutative geometry" to study these objects that escapes the traditional differential geometry.

At that moment Paul had already developed some tools to compute the fundamental group of a coset of a "groupe différentiel" and even to built its universal covering. We decided then to investigate the "irrational torus" from the "groupe différentiel" point of view. We computed its fundamental group and universal covering, for each irrational number. We were happy to find non trivial results, but we were really amazed when we found that two irrational numbers gave two diffeomorphic tori if and only if they were conjugate modulo $GL(2, \mathbf{Z})$, we couldn't have hoped for better, but we could get less. We were even more surprised by the computation of the connected components of the group of diffeomorphisms of irrational tori that distinguish between the quadratic and non-quadratic irrational numbers. We published these results in a preprint titled "Exemple de groupes différentiels : flots irrationnels sur le tore", published in July 1983 [PDPI83]. As far as I know it is in this preprint that, for the first time, the expression "espace différentiel" is used. It appears between quotes, at the second page of the preprint (page 112 of the CPT preprints page numbering), in the sentence:

> Les applications différentiables de $\mathcal{T}_\alpha$ dans un "espace différentiel" E sont les applications $\varphi : \mathcal{T}_\alpha \to E$ telles que...

On the other hand, the formal (axiomatic) definition of "espace différentiel" appeared a couple of months later, precisely in October 1983, in the Souriau preprint of the Lyon conference talk

[4] *Feuilletages et Quantification Géométrique*, Journées Lyonnaises de la SMF, 14-17 juin 1983.

[Sou83]. Actually I don't remember Souriau introducing this definition during his talk in Lyon in June and I don't believe that, as Souriau's students, we should have omitted to refer to Souriau's talk if he did. Moreover, I remember that during this period we had a few heated discussions about the need of such a general notion, in particular during the summer 1983 in the cafeteria of the Luminy campus, where the whole team was around the table for a coffee-break. Paul and I insisted to have a formal definition of "espace différentiel", and Souriau offered some resistance because, at this time, he thought that such a generality was useless or at least not pertinent[5]. We could not agree because, for his doctorate thesis, Paul was in need of a formal framework of "espaces différentiels" at least to introduce coherently his "espaces différentiels homogènes" [Don84]. On my side, following our paper with Paul, I began to think on a generalization of our results through general homotopy and fiber bundles, and for that I needed a general theory of "espaces différentiels" as fundation. Indeed, homotopy and fiber bundles in diffeology became my doctorate thesis I defended in 1985, where "espaces différentiels" became "espaces difféologiques" [Igl85][6]. Under this pressure, Souriau finally changed his mind and included this definition a few weeks later, it was in his October preprint.

In conclusion, whatever how the events precisely happened, we can wonder why it took so long to pull off two axioms from five.

References

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http://math.huji.ac.il/~piz/documents/EDGDFISLT.pdf

[5] Indeed, if we have only in mind the needs of Geometric Quantization, involving just homogeneous spaces.

[6] Fixing that way a language incoherence highlighted by A.E. Van Est.

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