SEIFERT ORBIFOLDS

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The quotient of a 3-manifold by an effective action of S^1, without fixed points, is a classical example of orbifold. But how does that fit within the diffeological frame? That is what we discuss here.

We'll talk about classical things and construction in differential geometry, but from the point of view of diffeology. We'll see how, in this example, the vocabulary and constructions of diffeology fit the needs of the problem. We shall consider in particular the diffeological definition of an orbifold, that is, a diffeological space locally diffeomorphic to a quotient $\mathbf{R}^n/\Gamma$, where Γ is a finite subgroup of the linear group $\mathrm{GL}(n, \mathbf{R})$, see [IKZ10] for the details.

Warning. One should not confuse what we call in this note a "Seifert orbifold" with what topologists call "Seifert fibered orbifold", as it appears in [BS85] for example. For us the orbifold is the space of Seifert fibers of a Seifert fibered manifold, and not an orbifold that would be the total space of a Seifert fibration.

A Little Bit of Smooth Lie Group Actions

Let M be a smooth manifold and consider a smooth action of a Lie group G on M.

A) — Diffeologically speaking, such an action is a smooth homomorphism $g \mapsto g_M$, from G to Diff(M), where Diff(M) is equipped with the functional diffeology [PIZ13, §7.4].

B) — Let $x \in M$ and $H = \mathrm{St}(x)$, the stabilizer of x. The orbit map $g \mapsto g_M(x)$ from G to M is strict, that means that the projection

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$\mathrm{class}(g) \mapsto g_M(x)$, defined on the coset G/H into M is an induction. In other words, the map class : $G/H \to \mathcal{O}_x$ is a diffeomorphism, where G/H is equipped with the quotient diffeology and $\mathcal{O}_x \subset M$ is equipped with the subset diffeology [IZK12, §1]: equipped with the subset diffeology, the orbit $\mathcal{O}_x$ is a manifold diffeomorphic to the coset G/H. Now, if G is compact, then this induction is an embedding and the orbit is an embedded submanifold.

C) — If G is compact, the type of stabilizers (or orbits) of a smooth action of G on a manifold M, is given by the *Theorem of Principal Orbits* [Bre72, Theorem 3.1 of Part IV].

Theorem (Principal Orbits). There exists a maximum orbit type G/H for G on M (i.e., H is conjugate to a subgroup of each isotropy group). The union $M_{(H)}$ of the orbits of type G/H is open and dense in M.

In particular, the stabilizers of any two points in $M_{(H)}$ are conjugate. The orbits of points in $M_{(H)}$ are called *principal orbits*, the other ones are called *singular orbits*. If a singular orbit has the same dimension than the principal orbits, the orbit is said to be *exceptional*.

D) — If G is compact, the Theorem 2.4 of part VI of *Compact Group Action on Manifolds* [Bre72] states that:

Theorem (Smooth Linear Tube). Let x be any point in M, and let H be the stabilizer of x. There exist a vector space V, an orthogonal action of H on V, and a local G invariant diffeomorphism $\varphi : G \times_H V \to M$, defined on $G \times_H V$.

The space $G \times_H V$ is the quotient of $G \times V$ by the diagonal action of H, $h_{G\times V} : (g, v) \mapsto (gh^{-1}, h_V(v))$, where $h \mapsto h_V$ denotes the action of H on V.

The image of the local diffeomorphism φ is an open invariant tube about the orbit $\mathcal{O}_x$, image by φ of the zero-section in $G \times_H V$.

Circle actions on manifolds

Consider a smooth action of the circle S^1 on a manifold M. According to the Theorem of Principal Orbits of compact groups actions on manifolds, there exists a S^1-invariant open dense subset of M such that all of its points have the same principal stabilizer H, a closed subgroup of S^1. And this principal stabilizer is contained in every stabilizer.

If the principal stabilizer is S^1 then all stabilizers are S^1 and the action is trivial.

If the action is not trivial then H is some cyclic group $\mathbf{Z}_m = \{\varepsilon \mid \varepsilon^m = 1\}$.

In the case of a non trivial action of S^1, it is always possible to consider the quotient group S^1/H, which is isomorphic to S^1, and the situation is reduced to an effective action of S^1, that is, an action with principal stabilizer $\{1\}$. This will be what we assume now.

If the action has no fixed points then the singular orbits are exceptional, and conversely.

Seifert Orbifolds

Now, we can present the object of our note. We refer to [Sco83] for the vocabulary and general context.

Definition — A *Seifert Fibered Space* (or Seifert fibration) is a 3-manifold M with an effective action of S^1 without fixed points.

Because all the stabilizers are cyclic, the *fibers* of the Seifert Fibered Space, that is, the orbits of the action of the circle, are diffeomorphic to the circle, when equipped with their subset diffeology [IZK12, §1]. What about the quotient space?

Theorem (Seifert Orbifolds). Let M be a 3-manifold with an effective action of S^1, without fixed points. Then, the quotient space $Q = M/S^1$ is a 2-manifold if the action is principal, or a 2-orbifold with isolated conic singularities otherwise.

Proof. Thanks to the Linear Tube Theorem, every orbit $\mathcal{O}_x$ has an equivariant open neighborhood diffeomorphic to a linear tube of type $S^1 \times_{\mathbf{Z}_m} \mathbf{C}$, where $\mathbf{Z}_m$ is the stabilizer of x. Then, about the orbit $\mathcal{O}_x \in Q$, the quotient space Q is locally diffeomorphic to $[S^1 \times_{\mathbf{Z}_m} \mathbf{C}]/S^1$, where the action of S^1 is given by $\tau\,\mathrm{class}(z, Z) = \mathrm{class}(\tau z, Z)$. Note in particular that $\mathrm{class}(z, Z) = z\,\mathrm{class}(1, Z)$.

Consider the map $J : \mathbf{C} \to S^1 \times_{\mathbf{Z}_m} \mathbf{C}$, defined by $J(Z) = \mathrm{class}(1, Z)$. The map J is an induction. Indeed, it is clearly injective. Moreover, let $r \mapsto \zeta(r)$ be a plot in $J(\mathbf{C}) \subset S^1 \times_{\mathbf{Z}_m} \mathbf{C}$. Since class :

$S^1 \times \mathbf{C} \to S^1 \times_{\mathbf{Z}_m} \mathbf{C}$ is a subduction, by construction, there exists always plots $r \mapsto (z(r), Z(r))$ in $S^1 \times \mathbf{C}$ such that, locally, $\zeta(r) = \mathrm{class}(z(r), Z(r))$. But since $\zeta(r) \in J(\mathbf{C})$, for all r in the domain of the plot there is $Z' \in \mathbf{C}$ such that $\mathrm{class}(z(r), Z(r)) = \mathrm{class}(1, Z')$. Thus $z(r) \in \mathbf{Z}_m \subset \mathbf{C}$, and since $\mathbf{Z}_m \subset \mathbf{C}$ is diffeologically discrete, $r \mapsto z(r)$ is locally constant, $z(r) = \varepsilon \in \mathbf{Z}_m$. Hence, $\zeta(r) =_{\mathrm{loc}} \mathrm{class}(1, \varepsilon Z(r))$, with $r \mapsto \varepsilon Z(r)$ smooth. Therefore, J is an induction.

Next, every S^1-orbit in $S^1 \times_{\mathbf{Z}_m} \mathbf{C}$ writes $S^1 \cdot \mathrm{class}(1, Z) = \{\mathrm{class}(z, Z) \mid z \in S^1\}$, for some $Z \in \mathbf{C}$. Its intersection with $J(\mathbf{C})$ is the set

$$\left(S^1 \cdot \mathrm{class}(1, Z)\right) \cap J(\mathbf{C}) = \{\mathrm{class}(1, \varepsilon Z) = J(\varepsilon Z) \mid \varepsilon \in \mathbf{Z}_m\}$$

Thus, there exists a natural bijection $j : \mathbf{C}/\mathbf{Z}_m \to \left(S^1 \times_{\mathbf{Z}_m} \mathbf{C}\right)/S^1$ mapping every $\mathbf{Z}_m$-orbit in $\mathbf{C}$ to the corresponding S^1-orbit in $S^1 \times_{\mathbf{Z}_m} \mathbf{C}$.

$$j(\mathrm{class}(Z)) = S^1 \cdot \mathrm{class}(1, Z).$$

Now, let $\mathrm{pr} : S^1 \times_{\mathbf{Z}_m} \mathbf{C} \to \left(S^1 \times_{\mathbf{Z}_m} \mathbf{C}\right)/S^1$ be the natural projection. Let us prove that the restriction $\mathrm{pr} \restriction J(\mathbf{C})$ is a still a subduction. Let $r \mapsto \zeta(r)$ be a plot in $\left(S^1 \times_{\mathbf{Z}_m} \mathbf{C}\right)/S^1$, locally $\zeta(r) = \mathrm{class}(z(r), Z(r))$, where $r \mapsto z(r)$ and $r \mapsto Z(r)$ are smooth. But, $\mathrm{class}(z(r), Z(r)) = \bar{z}(r) \cdot \mathrm{class}(1, z(r)Z(r))$, then $\mathrm{pr}(\mathrm{class}(z(r), Z(r))) = \mathrm{pr}(\mathrm{class}(1, z(r)Z(r)))$, and $\mathrm{class}(1, z(r)Z(r))$ belongs to $J(\mathbf{C})$. Thus, as claimed, $\mathrm{pr} \restriction J(\mathbf{C})$ is still a subduction.

Therefore, since $\mathrm{class} : \mathbf{C} \to \mathbf{C}/\mathbf{Z}_m$ and $\mathrm{pr} \restriction J(\mathbf{C}) : J(\mathbf{C}) \to \left(S^1 \times_{\mathbf{Z}_m} \mathbf{C}\right)/S^1$ are two subductions, since J is an induction, and the factorization $j : \mathbf{C}/\mathbf{Z}_m \to \left(S^1 \times_{\mathbf{Z}_m} \mathbf{C}\right)/S^1$ is a bijection, j is a diffeomorphism.

$$\begin{array}{ccccc}
\mathbf{C} & \xrightarrow{\ J\ } & S^1 \times_{\mathbf{Z}_m} \mathbf{C} & \xrightarrow{\ \varphi\ } & M \\
{\scriptstyle \mathrm{class}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \pi} \\
\mathbf{C}/\mathbf{Z}_m & \xrightarrow[j]{} & \left(S^1 \times_{\mathbf{Z}_m} \mathbf{C}\right)/S^1 & \xrightarrow[f]{} & M/S^1
\end{array}$$

Finally, the equivariant local diffeomorphism $\varphi : S^1 \times_{\mathbf{Z}_m} \mathbf{C} \to M$, given by the Smooth Linear Tube Theorem above, projects on a local diffeomorphism $f : \left(S^1 \times_{\mathbf{Z}_m} \mathbf{C}\right)/S^1 \to M/S^1$. The composite $f \circ j$ is then a local diffeomorphism from $\mathbf{C}/\mathbf{Z}_m$ into M/S^1. Therefore, M/S^1 is an orbifold according to [IKZ10, Definition 6]. The singular points are the images by $f \circ j$ of class(0) $\in \mathbf{C}/\mathbf{Z}_m$. They are clearly isolated and conic. If there is no exceptional orbit, then M/S^1 is obviously just a manifold (which is just a consequence of the Linear Tube Theorem). □

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